J. Math. Soc. Japan
Vol. 52, No. 4, 2000

Growth property and slowly increasing behaviour of singular solutions of linear partial differential equations in the complex domain

Dedicated to Professor Daisuke FUJIWARA on his sixtieth birthday

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(Received Apr. 21, 1998)
(Revised Mar. 12, 1999)

Abstract. Consider a linear partial differential equation in $\boldsymbol{C}^{d+1}$ $P(z,\partial)u(z)=f(z)$, where $u(z)$ and $f(z)$ admit singularities on the surface $\{z_0=0\}$. We assume that $|f(z)|\le A|z_0|^c$ in some sectorial region with respect to z_0. We can give an exponent $\gamma^*>0$ for each operator $P(z,\partial)$ and show for those satisfying some conditions that if $\forall\varepsilon>0\,\exists C_\varepsilon$ such that $|u(z)|\le C_\varepsilon\exp(\varepsilon|z_0|^{-\gamma^*})$ in the sectorial region, then $|u(z)|\le C|z_0|^{c'}$ for some constants c' and C.

§0. Introduction.

Let $P(z,\partial)$ be a linear partial differential operator with holomorphic coefficients in a neighbourhood Ω of $z=0$ in $\boldsymbol{C}^{d+1}$ and $K=\{z_0=0\}$. In the present paper we consider

$$P(z,\partial)u(z)=f(z),$$

where $f(z)$ is holomorphic except on K. The purpose of the present paper is to study behaviours of singular solutions near K. First we remark that for given $P(z,\partial)$ we can define an exponent $\gamma^*>0$ called minimal irregularity with respect to K and γ^* plays an important role in the present paper.

This paper follows Ōuchi [**11**]. In the present paper we treat a wider class of operators than in [**11**]. As stated in Abstract, the main result in this paper is the following.

If $u(z)$ grows at most some exponential order near $z_0=0$, that is, for any $\varepsilon>0$, $|u(z)|\le C_\varepsilon\exp(\varepsilon|z_0|^{-\gamma^*})$ and $f(z)$ is slowly increasing near K, that is, $|f(z)|\le A|z_0|^c$ in a sectorial region $\Omega(\theta)$, then the singularities of $u(z)$ are also slowly increasing.

The main Theorem in [**11**], where we considered a class of operators containing of the normal form with respect to $\partial/\partial z_0$ as a typical example, was the following.

If $u(z)$ grows at most some exponential order near $z_0=0$, that is, for any $\varepsilon>0$, $|u(z)|\le C_\varepsilon\exp(\varepsilon|z_0|^{-\gamma^*})$ and $f(z)$ has a Gevrey type asymptotic expansion $f(z)\sim\sum_{n=0}^{+\infty}f_n(z')z_0^n$ as $z_0\to 0$ in a sectorial region $\Omega(\theta)$, where $|f_n(z')|\le AB^n\Gamma(n/\gamma^*+1)$, then $u(z)$ has also an asymptotic expansion like $f(z)$ as z_0 tends to 0.

It was an extension of the main result of [**8**] and [**9**], where we had used an integral

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2000 *Mathematical Subject Classifications.* Primary 35A20; Secondary 35B05, 35B40.
Key Words and Phrases. Singular solutions, complex partial differential equations.

representation of solutions with singularities on K. We did not use it in [**11**] but we proved the main result there, by estimating the derivatives $(\partial/\partial z_0)^n u(z)$. The proof was much simpler and completely different from [**8**] and [**9**].

The class of operators considered here is wider than in [**11**]. So the main result in this paper becomes somewhat different from that in [**11**] as above and even if $f(z)$ has a Gevrey type asymptotic expansion, $u(z)$ does not necessarily have. We show the main result in this paper by constructing a parametrix. If $P(z,\partial)$ belongs to the class in [**11**], then we can show the results in [**11**] from those in the present paper, which will be discussed in the forthcoming paper.

Finally we comment about singular solutions. As for the existence of solutions with singularities on K, it was investigated in Hamada, Leray and Wagschal [**2**], Kashiwara and Schapira [**3**], Ōuchi [**7**], Persson [**12**] and other papers cited in those papers. The behaviours of singular solutions were also studied in Ōuchi [**4**] and [**5**] by using the integral representation. We considered some singular solutions in [**4**] and [**5**] and obtained results such as Stokes phenomenon, that is, they grow really with some exponential order as z tends to K in a region and behave mildly as z tends to K in another region.

§1. Notations and results.

In this section we give notations and definitions in order to state results more precisely. The coordinates of $\boldsymbol{C}^{d+1}$ are denoted by $z=(z_0,z_1,\dots,z_d)=(z_0,z')\in \boldsymbol{C}\times\boldsymbol{C}^d$. $|z|=\max\{|z_i|;0\le i\le d\}$ and $|z'|=\max\{|z_i|;1\le i\le d\}$. Its dual variables are $\xi=(\xi_0,\xi')=(\xi_0,\xi_1,\dots,\xi_d)$. $\boldsymbol{N}$ is the set of all nonnegative integers $\boldsymbol{N}=\{0,1,2,\dots\}$. The differentiation is denoted by $\partial_i=\partial/\partial z_i$, and $\partial=(\partial_0,\partial_1,\dots,\partial_d)=(\partial_0,\partial')$. For a multi-index $\alpha=(\alpha_0,\alpha')\in\boldsymbol{N}\times\boldsymbol{N}^d$, $|\alpha|=\alpha_0+|\alpha'|=\sum_{i=0}^d\alpha_i$. Define $\partial^\alpha=\prod_{i=0}^d\partial_i^{\alpha_i}$. We often denote $\partial'^{\alpha'}=\prod_{i=1}^d\partial_i^{\alpha_i}$ by $\partial^{\alpha'}$.

Let us define spaces of holomorphic functions in some regions. Let $\Omega=\Omega_0\times\Omega'$ be a polydisk with $\Omega_0=\{z_0\in\boldsymbol{C}^1;|z_0|<R\}$ and $\Omega'=\{z'\in\boldsymbol{C}^d;|z'|<R\}$ for some positive constant R. Put $\Omega_0(\theta)=\{z_0\in\Omega_0-\{0\};|\arg z_0|<\theta\}$ and $\Omega(\theta)=\Omega_0(\theta)\times\Omega'$. $\mathscr{O}(\Omega)(\mathscr{O}(\Omega'),\mathscr{O}(\Omega(\theta)))$ is the set of all holomorphic functions on Ω (*resp.* $\Omega',\Omega(\theta)$). $\mathscr{O}(\Omega(\theta))$ contains multi-valued functions, if $\theta>\pi$.

Now let $P(z,\partial)$ be an m-th order linear partial differential operator with coefficients in $\mathscr{O}(\Omega)$,

$$P(z,\partial)=\sum_{|\alpha|\le m}a_\alpha(z)\partial^\alpha. \tag{1.1}$$

Let j_α be the valuation of $a_\alpha(z)$ with respect to z_0. Hence $a_\alpha(z)=z_0^{j_\alpha}c_\alpha(z)$, where $c_\alpha(0,z')\not\equiv 0$ for $a_\alpha(z)\not\equiv 0$ and $j_\alpha=+\infty$ for $a_\alpha(z)\equiv 0$. Let us define some quantities for $P(z,\partial)$:

$$\begin{cases} e_*:=\min\{j_\alpha-\alpha_0;\alpha\in\boldsymbol{N}^{d+1}\}, \varDelta:=\{\alpha\in\boldsymbol{N}^{d+1};j_\alpha-\alpha_0=e_*\}\\ k^*:=\max\{|\alpha|;\alpha\in\varDelta\}.\end{cases} \tag{1.2}$$

We define minimal irregularity used in [**6**] and [**10**].

Definition 1.1.

$$\begin{cases} \gamma^* := \min\left\{\dfrac{j_\alpha - \alpha_0 - e_*}{|\alpha| - k^*};\ \alpha \in \boldsymbol{N}^{d+1} |\alpha| > k^*\right\} & \text{if } k^* < m, \\ \gamma^* := +\infty & \text{if } k^* = m. \end{cases} \tag{1.3}$$

We rewrite $P(z, \partial)$ in another form for later calculations. Put $\vartheta := z_0\partial_0$. It follows from the identity $z_0^{j_\alpha}\partial^\alpha = z_0^{j_\alpha - \alpha_0}(z_0^{\alpha_0}\partial^\alpha) = z_0^{j_\alpha - \alpha_0}\vartheta(\vartheta - 1)\cdots(\vartheta - j_\alpha + \alpha_0 - 1)\partial^{\alpha'}$ that we can write $P(z, \partial)$ in the following form

$$P(z, \partial) = \sum_{|\alpha| \le m} z_0^{e_\alpha} b_\alpha(z)\vartheta^{\alpha_0}\partial^{\alpha'}, \tag{1.4}$$

where $e_\alpha \in \boldsymbol{Z}$, $b_\alpha(z) = \sum_{h=0}^{+\infty} b_{\alpha,h}(z')z_0^h$ and $b_{\alpha,0}(z') \not\equiv 0$ for $b_\alpha(z) \not\equiv 0$. We put $e_\alpha = +\infty$ if $b_\alpha(z) \equiv 0$. We remark that the quantities defined by (1.2) do not depend on the representations, that is,

Lemma 1.2. *The following equalities hold*: $e_* = \min\{e_\alpha; \alpha \in \boldsymbol{N}^{d+1}\}$ *and* $\Delta = \{\alpha \in \boldsymbol{N}^{d+1}; e_\alpha = e_*\}$.

Lemma 1.3. *If* $k^* < m$,

$$\gamma^* = \min\left\{\frac{e_\alpha - e_*}{|\alpha| - k^*}; \alpha \in \boldsymbol{N}^{d+1},\ |\alpha| > k^*\right\}. \tag{1.5}$$

The proofs of Lemmas 1.2 and 1.3 are easy. So we omit them. Put

$$\mathfrak{P}(z, \partial) = \sum_{\alpha \in \Delta} z_0^{e_*} b_{\alpha,0}(z')\vartheta^{\alpha_0}\partial^{\alpha'}, \tag{1.6}$$

which plays an important role in this paper.

We introduce $\mathcal{O}_{(\kappa)}(\Omega(\theta))$ and $Asy_{\{\kappa\}}(\Omega(\theta))$, which are subspaces of $\mathcal{O}(\Omega(\theta))$ and are fundamental function spaces in this paper.

Definition 1.4. $\mathcal{O}_{(\kappa)}(\Omega(\theta))(0 < \kappa < +\infty)$ is the set of all $u(z) \in \mathcal{O}(\Omega(\theta))$ such that for any $\varepsilon > 0$ and any θ' with $0 < \theta' < \theta$

$$|u(z)| \le C\exp(\varepsilon|z_0|^{-\kappa}) \quad \text{for } z \in \Omega(\theta') \tag{1.7}$$

holds for a constant $C = C(\varepsilon, \theta')$. We put $\mathcal{O}_{(+\infty)}(\Omega(\theta)) = \mathcal{O}(\Omega(\theta))$ for $\kappa = +\infty$.

Definition 1.5. $\mathcal{O}_{reg,c}(\Omega(\theta))(c \in \boldsymbol{R})$ is the set of all $u(z) \in \mathcal{O}(\Omega(\theta))$ such that for any θ' with $0 < \theta' < \theta$

$$|u(z)| \le C|z_0|^c \quad z \in \Omega(\theta') \tag{1.8}$$

holds for a constant $C = C(\theta')$.

We say that $u(z) \in \mathcal{O}(\Omega(\theta))$ is regular singular or slowly increasing or tempered in $\Omega(\theta)$, if $u(z) \in \bigcup_{|c|<+\infty} \mathcal{O}_{reg,c}(\Omega(\theta))$. We proceed to introduce conditions on $P(z, \partial)$.

Condition 0. If $\alpha = (\alpha_0, \alpha') \in \Delta$, then $\alpha' = (0, 0, \ldots, 0)$.

Suppose that $P(z,\partial)$ satisfies Condition 0. Then $(k^*,0,0,\dots,0)\in\Delta$ and $\mathfrak{P}(z,\partial)$ is an ordinary differential operator

$$\mathfrak{P}(z,\partial)=\sum_{\alpha\in\Delta} z_0^{e_*}b_{\alpha,0}(z')\vartheta^{\alpha_0}. \tag{1.9}$$

Define the indicial polynomial $\chi_P(z',\lambda)$ of $\mathfrak{P}(z,\partial)$ by

$$\chi_P(z',\lambda):=\sum_{\alpha\in\Delta} b_{\alpha,0}(z')\lambda^{\alpha_0}. \tag{1.10}$$

The following condition is stronger than Condition 0.

Condition 1. $P(z,\partial)$ satisfies Condition 0 and $b_{(k^*,0,0,\dots,0)}(0)\neq 0$.

If $P(z,\partial)$ satisfies Condition 1, then $\chi_P(z',\lambda)$ is a polynomial in λ with degree k^*. So $\chi_P(z',\lambda)=0$ has k^* roots, by shrinking R, and we choose real numbers a_0,a_1 and b_0 so that all roots of the algebraic equation $\chi_P(z',\lambda)=0$ for $|z'|\leq R$ are contained in $\{\lambda\in\boldsymbol{C};a_0\leq\Re\lambda\leq a_1,|\Im\lambda|\leq b_0\}$. The main results are the following.

Theorem 1.6. *Suppose that $P(z,\partial)$ satisfies Condition* 1. *Let $u(z)\in\mathcal{O}_{(\gamma^*)}(\Omega(\theta))$ be a solution of $P(z,\partial)u(z)=f(z)$, where $f(z)\in\mathcal{O}_{reg,c}(\Omega(\theta))$. Then there is a polydisk U centered at $z=0$ such that for any $c'<\min\{c-e_*,a_0\}$, $u(z)\in\mathcal{O}_{reg,c'}(U(\theta))$.*

Theorem 1.7. *Suppose that $P(z,\partial)$ satisfies Condition* 0. *Let $u(z)\in\mathcal{O}_{(\gamma^*)}(\Omega(\theta))$ be a solution of $P(z,\partial)u(z)=f(z)$, where $f(z)\in\mathcal{O}_{reg,c}(\Omega(\theta))$. Then there is a polydisk U centered at $z=0$ and a constant c'' such that $u(z)\in\mathcal{O}_{reg,c''}(U(\theta))$.*

We give some examples satisfying Condition 1:

(a) Operators of normal type with respect to ∂_0,

$$\partial_0^{k^*}+\sum_{\alpha_0<k^*}a_\alpha(z)\partial^\alpha.$$

More concretely

$$\partial_0^{k^*}+z_0^j\partial_1^m\ (m>k^*)\quad \mathfrak{P}(z,\partial)=\partial_0^{k^*}\quad \gamma^*=(j+k^*)/(m-k^*),$$

$$\partial_0^3+\partial_1^4\partial_0^2+\partial_1^7\quad \mathfrak{P}(z,\partial)=\partial_0^3\quad \gamma^*=1/3.$$

(b) Operators of Fuchsian type defined in Baouendi-Goulaouic [**1**], where $\gamma^*=+\infty$.

(c) Other concrete examples are

$$\partial_1^3+z_0^2\partial_0^2+\partial_0\quad \mathfrak{P}(z,\partial)=\partial_0\quad \gamma^*=1/2,$$

and

$$\partial_1^3+z_0\partial_0^2+a(z)\partial_0+b(z)\quad \mathfrak{P}(z,\partial)=z_0\partial_0^2+a(0,z')\partial_0\quad \gamma^*=1.$$

§2. Construction of parametrix and proof of theorems.

In order to show Theorem 1.6 we construct a left parametrix of $P(z,\partial)$,

$$
(2.1)\qquad \begin{cases} P(z,\partial) := \sum_{|\alpha|\le m} z_0^{e_\alpha} b_\alpha(z)\vartheta^{\alpha_0}\partial^{\alpha'} \quad (\vartheta = z_0\partial_0), \\ b_\alpha(z) = \sum_{h=0}^{\infty} b_{\alpha,h}(z')z_0^h. \end{cases}
$$

We find the parametrix of the form for $0<\gamma^*<+\infty$:

$$
(2.2)\qquad G_\delta(z,w) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{z_0^{\lambda-e_*-\gamma^*}}{w_0^{\lambda+1}}\, d\lambda \int_0^\delta \exp(-\zeta z_0^{-\gamma^*})\, U(z,w',\lambda,\zeta)\, d\zeta,
$$

where $\delta>0$ is a small constant. When $\gamma^* = +\infty$, the form of the parametrix is slightly different. This section consists of 3 subsections. The form of the parametrix $G_\delta(z,w)$ implies that the construction of it is to determine $U(z,w',\zeta,\lambda)$, which is done in §2.1. By integrating it, we construct $G_\delta(z,w)$ in §2.2. We give the proofs of Theorems 1.6 and 1.7 in §2.3. The proofs of Propositions 2.1 and 2.2 and Theorem 2.3 are not given in this section but in §3 and §4.

§2.1. Construction of Parametrix 1.

We assume *Condition* 1 in this subsection. We need the transposed operator ${}^tP(z,\partial)$ to construct a left parametrix of $P(z,\partial)$,

$$
(2.3)\qquad {}^tP(z,\partial) = \sum_{|\alpha|\le m} (-1)^{|\alpha|}(\vartheta+1)^{\alpha_0} z_0^{e_\alpha}\partial^{\alpha'}(b_\alpha(z)\,\cdot).
$$

Recall

$$
(2.4)\qquad \chi_P(z',\lambda) = \sum_{\alpha\in\Delta} b_{\alpha,0}(z')\lambda^{\alpha_0}
$$

and we have

$$
(2.5)\qquad \chi_{{}^tP}(z',\lambda) = \sum_{\alpha\in\Delta} (-1)^{|\alpha_0|} b_{\alpha,0}(z')(\lambda+e_*+1)^{\alpha_0}.
$$

${}^tP(z,\partial)$ also satisfies *Condition* 1. We have chosen a_0, a_1 and b_0 so that all the roots of algebraic equation $\chi_P(z',\lambda)=0$ for $|z'|\le R$ are contained in $\{\lambda; a_0 \le \Re\lambda \le a_1, |\Im\lambda| \le b_0\}$. Hence it follows from this assumption that all the roots of $\chi_{{}^tP}(z',\lambda)=0$ for $|z'|\le R$ are contained in $\{\lambda\in \boldsymbol{C}; -a_1-e_*-1 \le \Re\lambda \le -a_0-e_*-1, |\Im\lambda|\le b_0\}$.

First suppose $\gamma^*<+\infty$. Let us introduce an integro-differential operator ${}^t\mathscr{P}(z,\lambda,\zeta,\partial_z,\partial_\zeta)$;

$$
(2.6)\qquad {}^t\mathscr{P}(z,\lambda,\zeta,\partial_z,\partial_\zeta) := \sum_{|\alpha|\le m} (-1)^{|\alpha|}(\gamma^*\zeta\partial_\zeta + z_0\partial_0 + \lambda + 1)^{\alpha_0}\partial_\zeta^{-(e_\alpha-e_*)/\gamma^*}\partial^{\alpha'}(b_\alpha(z)\,\cdot),
$$

where $\partial_\zeta = \partial/\partial\zeta$ and the operator ∂_ζ^{-s} $(s\ge 0)$ is defined as follows:

$$\partial_\zeta^{-s}\frac{\zeta^a}{\Gamma(a+1)}=\frac{\zeta^{a+s}}{\Gamma(a+s+1)}\quad\text{for } a\ge 0. \tag{2.7}$$

We determine $U(z,w',\lambda,\zeta)$ so as to satisfy

$$^t\mathscr{P}(z,\lambda,\zeta,\partial_z,\partial_\zeta)U(z,w',\lambda,\zeta)=\frac{1}{(2\pi i)^d\prod_{j=1}^d(z_j-w_j)}, \tag{2.8}$$

where $w'=(w_1,w_2,\dots,w_d)$. We try to find a solution of (2.8) of the form

$$U(z,w',\lambda,\zeta)=\sum_{l,n=0}^{\infty}\frac{z_0^l\zeta^{n/\gamma^*}}{\Gamma(l+1)\Gamma(n/\gamma^*+1)}u_{l,n}(z',w',\lambda). \tag{2.9}$$

By substituting $U(z,w',\lambda,\zeta)$ into (2.8), we have the following recursion formula,

$$\begin{aligned}&\left(\sum_{\{\alpha\in\varDelta\}}(-1)^{|\alpha|}b_{\alpha,0}(z')(\lambda+l+n+1)^{\alpha_0}\right)u_{l,n}(z',w',\lambda)\\&\quad+\sum_{\{(\alpha,h);\,e_\alpha-e_*+h>0\}}\frac{(-1)^{|\alpha|}(\lambda+l+n+1)^{\alpha_0}l!}{(l-h)!}\partial^{\alpha'}(b_{\alpha,h}(z')u_{l-h,n-e_\alpha+e_*}(z',w',\lambda))\\&=\begin{cases}\dfrac{1}{(2\pi i)^d\prod_{j=1}^d(z_j-w_j)} & (l,n)=(0,0),\\[2ex] 0 & (l,n)\neq(0,0).\end{cases}\end{aligned} \tag{2.10}$$

Thus we can determine $u_{l,n}(z',w',\lambda)$ successively by (2.10) and it is a rational function in λ, whose poles are in $\{\lambda;\prod_{i=0}^{l+n}\chi_{{}^tP}(z',\lambda+i-e_*)=0\}$, which are contained in $\{\lambda;-a_1-1-(l+n)\le\Re\lambda\le-a_0-1,|\Im\lambda|\le b_0\}$.

Suppose $\gamma^*=+\infty$. Then we put

$$\begin{aligned}&^t\mathscr{P}(z,\lambda,\partial_z)\\&\quad:=\sum_{|\alpha|\le m}(-1)^{|\alpha|}(z_0\partial_0+\lambda+1)^{\alpha_0}z_0^{e_\alpha-e_*}\partial^{\alpha'}(b_\alpha(z)\,\cdot),\end{aligned} \tag{2.11}$$

which is independent of ζ and ∂_ζ. We determine $U(z,w',\lambda)$ so as to satisfy

$$^t\mathscr{P}(z,\lambda,\partial_z)U(z,w',\lambda)=\frac{1}{(2\pi i)^d\prod_{j=1}^d(z_j-w_j)} \tag{2.12}$$

and find a solution of (2.12) of the form

$$U(z,w',\lambda)=\sum_{l=0}^{\infty}\frac{z_0^l}{\Gamma(l+1)}u_l(z',w',\lambda). \tag{2.13}$$

We have

$$\left(\sum_{\{\alpha\in\Delta\}}(-1)^{|\alpha|}b_{\alpha,0}(z')(\lambda+l+1)^{\alpha_0}\right)u_l(z',w',\lambda) \tag{2.14}$$
$$+\sum_{\{(\alpha,h);\,e_\alpha-e_*+h>0\}}\frac{(-1)^{|\alpha|}(\lambda+l+1)^{\alpha_0}l!}{(l-e_\alpha+e_*-h)!}\partial^{\alpha'}(b_{\alpha,h}(z')u_{l-e_\alpha+e_*-h}(z',w',\lambda))$$
$$=\frac{\delta_{l,0}}{(2\pi i)^d\prod_{j=1}^d(z_j-w_j)}$$

and can determine $u_l(z',w',\lambda)$ successively by (2.14) and it is a rational function in λ, whose poles are in $\{\lambda;\prod_{i=0}^{l}\chi_{\prime P}(z',\lambda+i-e_*)=0\}$, which are contained in $\{\lambda;-a_1-1-l\le\Re\lambda\le-a_0-1,|\Im\lambda|\le b_0\}$.

We need the estimate of $u_{l,n}(z',w',\lambda)$ ($u_l(z',w',\lambda)$) to show the convergence of $U(z,w',\lambda,\zeta)$ (*resp.* $U(z,w',\lambda)$). For this purpose we introduce notations of regions. Let $0<r_1<r_2<r_3\le R$. Define regions $X'(r_1,r_2,r_3)$ in (z',w')-space and $\Lambda(a,b)$ in λ-space,

$$X'(r_1,r_2,r_3)=\{(z',w');|w_i|<r_1,r_2<|z_i|<r_3\ \text{for}\ 1\le i\le d\}, \tag{2.15}$$

$$\Lambda(a,b)=\{\lambda;\Re\lambda\le a,|\Im\lambda|\le b\}. \tag{2.16}$$

Proposition 2.1. *Let $a'<a_0$ and $b_0<b'$. Then there are positive constants $A=A(a',b')$, $B=B(b')$ and small constants $0<r_1<r_2<r_3$ such that the following estimates hold for $(z',w')\in X'(r_1,r_2,r_3)$ and $\lambda\notin\Lambda(-a'-1,b')$: if $\gamma^*<+\infty$,*

$$|u_{l,n}(z',w',\lambda)|\le AB^{l+n}(|\lambda|^l+\Gamma(l+1))(|\lambda|^{n/\gamma^*}+\Gamma(n/\gamma^*+1)) \tag{2.17}$$

for all $l,n\in\boldsymbol{N}$ and if $\gamma^*=+\infty$,

$$|u_l(z',w',\lambda)|\le AB^l(|\lambda|^l+\Gamma(l+1)) \tag{2.18}$$

for all $l\in\boldsymbol{N}$.

We have the following proposition from Proposition 2.1.

Proposition 2.2. *Let $a'<a_0$ and $b_0<b'$ and suppose that $(z',w')\in X'(r_1,r_2,r_3)$ and $\lambda\notin\Lambda(-a'-1,b')$.*

(1) *If $\gamma^*<+\infty$, then there exist positive constants r_0 and δ_0 such that the series $U(z,w',\lambda,\zeta)$ defined by* (2.9) *converges for $|z_0|<r_0$ and $|\zeta|<\delta_0$, and*

$$|U(z,w',\lambda,\zeta)|\le A\exp(c(|z_0|+|\zeta|)|\lambda|) \tag{2.19}$$

for some constants $A=A(a',b')$ and $c=c(b')$.

(2) *If $\gamma^*=+\infty$, then there exists a positive constant r_0 such that the series $U(z,w',\lambda)$ defined by* (2.13) *converges for $|z_0|<r_0$ and*

$$|U(z,w',\lambda)|\le A\exp(c|z_0|\,|\lambda|) \tag{2.20}$$

for some constants $A=A(a',b')$ and $c=c(b')$.

The proofs of Propositions 2.1 and 2.2 are given in §3.

§2.2. Construction of parametrix II.

In this subsection we assume that $P(z,\partial)$ satisfies *Condition* 1. The constants $c = c(b')$, δ_0 and r_i $(0 \le i \le 3)$ are those in Propositions 2.1 and 2.2 in §2.1. Now suppose $\gamma^* < +\infty$. Let us construct a parametrix $G_\delta(z,w)$ with a parameter δ, by integrating $U(z,w',\lambda,\zeta)$. Let δ be a constant with $0 < \delta < \delta_0$. Define

$$K_\delta(z,w',\lambda) = z_0^{-e_*-\gamma^*}\int_0^\delta \exp(-\zeta z_0^{-\gamma^*})U(z,w',\lambda,\zeta)\,d\zeta \tag{2.21}$$

and a region $X_\theta(r_0,r_1,r_2,r_3)$ in (z,w')-space

$$X_\theta(r_0,r_1,r_2,r_3) := \{0 < |z_0| < r_0, |\arg z_0| < \theta\} \times X'(r_1,r_2,r_3). \tag{2.22}$$

We have

Theorem 2.3. *Suppose $\gamma^* < +\infty$. Let a' and b' be constants with $a' < a_0$ and $b' > b_0$. Then*

(1) *$K_\delta(z,w',\lambda)$ is holomorphic on $X_\infty(r_0,r_1,r_2,r_3) \times (\boldsymbol{C} - \Lambda(-a'-1,b'))$.*

(2) *For any $0 < \theta < \pi/(2\gamma^*)$ there exist constants $c_0 = c_0(b') \ge c(b')$ and $A = A(a',b',\theta)$ such that*

$$|K_\delta(z,w',\lambda)| \le A|z_0|^{-e_*}\exp(c_0(|z_0|+\delta)|\lambda|) \tag{2.23}$$

for $(z,w') \in X_\theta(r_0,r_1,r_2,r_3)$ and $\lambda \notin \Lambda(-a'-1,b')$.

(3) *It holds that*

$${}^tP(z,\partial)(z_0^\lambda K_\delta(z,w',\lambda)) = \frac{z_0^\lambda}{(2\pi i)^d\prod_{j=1}^d (z_j - w_j)} + z_0^\lambda K_\delta^R(z,w',\lambda), \tag{2.24}$$

where $K_\delta^R(z,w',\lambda)$ is holomorphic in $(z,w') \in X_\infty(r_0,r_1,r_2,r_3)$ and $\lambda \notin \Lambda(-a'-1,b')$. Moreover for any $0 < \theta < \pi/(2\gamma^*)$ there are positive constants $c_0 = c_0(b')$, $c_1 = c_1(\theta,\delta)$ and $A = A(a',b',\theta)$ such that

$$|K_\delta^R(z,w',\lambda)| \le A(1+|\lambda|)^m \exp(-c_1|z_0|^{-\gamma^*} + c_0(|z_0|+\delta)|\lambda|) \tag{2.25}$$

for $(z,w') \in X_\theta(r_0,r_1,r_2,r_3)$ and $\lambda \notin \Lambda(-a'-1,b')$.

The constant c_0 in (2) *and* (3) *is independent of θ and δ.*

The proof of Theorem 2.3 is given in §4. Define

$$G_\delta(z,w) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{z_0^\lambda}{w_0^{\lambda+1}} K_\delta(z,w',\lambda)\,d\lambda, \tag{2.26}$$

where $\mathscr{C} := \mathscr{C}(\phi)$ $(|\phi| < \pi)$ is an infinite path in $\boldsymbol{C} - \Lambda(-a'-1,b')$ starting at $\lambda = \infty\exp(i\phi)$ and ends at $\lambda = -a-1 (a < a' < a_0)$. More precisely if $|\phi| \le \pi/2$, $\mathscr{C}(\phi)$ is a half line connecting $\infty\exp(i\phi)$ with $-a-1$, if $\pi/2 < \phi < \pi$ $(-\pi < \phi < -\pi/2)$, $\mathscr{C}(\phi)$ is a broken line through $\infty\exp(i\phi)$, $-a-1+bi$ (*resp.* $-a-1-bi$) and $-a-1$, where $b_0 < b' < b$. Define

$$R_\delta(z,w) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{{z_0}^{\lambda}}{w_0^{\lambda+1}} K_\delta^R(z,w',\lambda)\,d\lambda. \tag{2.27}$$

Now let us proceed to study $G_\delta(z,w)$ and $R_\delta(z,w)$. By putting

$$\begin{aligned} \tilde{G}_\delta(t,z,w') &:= \frac{1}{2\pi i}\int_{\mathscr{C}} \exp(\lambda t) K_\delta(z,w',\lambda)\,d\lambda, \\ \tilde{R}_\delta(t,z,w') &:= \frac{1}{2\pi i}\int_{\mathscr{C}} \exp(\lambda t) K_\delta^R(z,w',\lambda)\,d\lambda, \end{aligned} \tag{2.28}$$

we have

$$\begin{aligned} G_\delta(z,w) &= \tilde{G}_\delta(\log z_0 - \log w_0, z, w')/w_0, \\ R_\delta(z,w) &= \tilde{R}_\delta(\log z_0 - \log w_0, z, w')/w_0, \end{aligned} \tag{2.29}$$

where $\log z_0$ and $\log w_0$ are principal valued.

Proposition 2.4. *Let $0 < \theta < \pi/(2\gamma^*)$. For any $0 < \varepsilon_0 \le \pi/2$ $\tilde{G}_\delta(t,z,w')$ and $\tilde{R}_\delta(t,z,w')$ are holomorphic in $\{t; -\pi/2+\varepsilon_0 < \arg t < 5\pi/2 - \varepsilon_0,\ |t|\sin\varepsilon_0 - c_0(|z_0|+\delta) > 0\} \times X_\theta(r_0,r_1,r_2,r_3)$ and there are constants $A = A(\theta,\delta)$ and $c_1 := c_1(\theta,\delta)$ such that*

$$|\tilde{G}_\delta(t,z,w')| \le \frac{A\exp(-(a+1)\Re t + b|\Im t|)}{|z_0|^{e_*}(|t|\sin\varepsilon_0 - c_0(|z_0|+\delta))}, \tag{2.30}$$

$$|\tilde{R}_\delta(t,z,w')| \le \frac{A\exp(-(a+1)\Re t + b|\Im t| - c_1|z_0|^{-\gamma^*})}{(|t|\sin\varepsilon_0 - c_0(|z_0|+\delta))^{m+1}}. \tag{2.31}$$

Proof. Let $(z,w') \in X_\theta(r_0,r_1,r_2,r_3)$ and $-\pi/2+\varepsilon_0 < \arg t < 5\pi/2 - \varepsilon_0$. Then we can choose λ so that $|\arg\lambda| < \pi$ and $|\arg\lambda + \arg t - \pi| < \pi/2 - \varepsilon_0$. Hence by Theorem 2.3

$$\begin{aligned} |\tilde{G}_\delta(t,z,w')| &\le A|e^{-(a+1)\Re t + b|\Im t|} z_0^{-e_*}| \int_0^{+\infty} e^{(|t|\cos(\arg\lambda+\arg t) + c_0(|z_0|+\delta))|\lambda|}\,d|\lambda| \\ &\le A|e^{-(a+1)\Re t + b|\Im t|} z_0^{-e_*}|/(|t|\sin\varepsilon_0 - c_0(|z_0|+\delta)). \end{aligned}$$

By the same method we have the holomorphy and the estimate of $\tilde{R}_\delta(t,z,w')$.

Define a region $Y_{\theta,\theta'}(r_0,\varepsilon)$ in (z_0,w_0)-space,

$$\begin{aligned} Y_{\theta,\theta'}(r_0,\varepsilon) = \Big\{ &(z_0,w_0); 0 < |z_0| < r_0, |\arg z_0| < \theta, w_0 \neq 0, |\arg w_0| < \theta', \\ &|\arg(\log z_0 - \log w_0) - \pi| < \frac{3\pi}{2} - \varepsilon, |\log z_0 - \log w_0|\sin\varepsilon - c_0(|z_0|+\delta) > 0 \Big\}. \end{aligned}$$

Then we have from (2.29)

Proposition 2.5. *Let $0 < \theta < \pi/(2\gamma^*)$. For any $0 < \varepsilon_0 \le \pi/2$, $G_\delta(z,w)$ and $R_\delta(z,w)$ are holomorphic in $Y_{\theta,\theta'}(r_0,\varepsilon_0) \times X'(r_1,r_2,r_3)$ and satisfy*

$$ {}^tP(z,\partial)G_\delta(z,w) = \delta(z,w) + R_\delta(z,w), $$

(2.32)
$$ \delta(z,w) = \frac{1}{(2\pi i)^{d+1}} \times \frac{w_0^a}{z_0^{a+1}(\log z_0 - \log w_0)\prod_{j=1}^d (z_j - w_j)} $$

and there are constants $A = A(\theta,\theta',\delta)$ *and* $c_1 = c_1(\theta,\delta)$ *such that*

$$ |G_\delta(z,w)| \le \frac{A|w_0|^a/|z_0|^{a+e_*+1}}{|\log z_0 - \log w_0|\sin\varepsilon_0 - c_0(|z_0|+\delta)} \tag{2.33} $$

and

$$ |R_\delta(z,w)| \le \frac{A|w_0|^a e^{-c_1|z_0|^{-\gamma^*}}}{(|\log z_0 - \log w_0|\sin\varepsilon_0 - c_0(|z_0|+\delta))^{m+1}}. \tag{2.34} $$

Proof. It follows from Theorem 2.3 that

$$ {}^tP(z,\partial)G_\delta(z,w) = \delta(z,w) + R_\delta(z,w), $$

where

$$ \begin{aligned} \delta(z,w) &= \frac{1}{(2\pi i)^{d+1}} \int_{\mathscr{C}} \frac{z_0^\lambda}{w_0^{\lambda+1}}\, d\lambda \prod_{j=1}^d (z_j - w_j)^{-1} \\ &= \frac{1}{(2\pi i)^{d+1}} \times \frac{w_0^a}{z_0^{a+1}(\log z_0 - \log w_0)\prod_{j=1}^d (z_j - w_j)}. \end{aligned} $$

The estimates (2.33) and (2.34) follow from Proposition 2.4.

Let us define integral operators with kernel $G_\delta(z,w)$ and $R_\delta(z,w)$ respectively. Put

$$ \begin{aligned} W(\theta') &= \{w_0; 0 < |w_0| < \hat{r}_0/2, |\arg w_0| < \theta'\} \times \{w' \in \boldsymbol{C}^d; |w_i| < r_1 \quad (1 \le i \le d)\}, \\ Z(\theta) &= \{z_0; 0 < |z_0| < 2\hat{r}_0, |\arg z_0| < \theta\} \times \{z' \in \boldsymbol{C}^d; |z_i| < r_3 \quad (1 \le i \le d)\}, \end{aligned} \tag{2.35} $$

where $0 < \theta' < \theta$, $0 < 2\hat{r}_0 < r_0$ and $\hat{r}_0$ will be chosen so small. Define a chain $S(w_0)$ in z-space. Let w_0 with $0 < |w_0| < \hat{r}_0/2$ and $|\arg w_0| < \theta'$. Put for small $0 < \varepsilon < \theta - \theta'$

$$ \begin{cases} S_{0,1}(w_0) = \{z_0 = \hat{r}_0 e^{is\arg w_0}; 0 \le s \le 1\} \\ S_{0,2}(w_0) = \{z_0 = (1-s)\hat{r}_0 e^{i\arg w_0} + s(w_0 + \varepsilon|w_0|e^{i\arg w_0}); 0 \le s \le 1\} \\ S_{0,3}(w_0) = \{z_0 = w_0 + (\varepsilon|w_0|e^{i\arg w_0})e^{is}; 0 \le s \le 2\pi\} \\ S_{0,4}(w_0) = \{z_0 = (1-s)(w_0 + \varepsilon|w_0|e^{i\arg w_0}) + s\hat{r}_0 e^{i\arg w_0}; 0 \le s \le 1\} \\ S_{0,5}(w_0) = \{z_0 = \hat{r}_0 e^{i(1-s)\arg w_0}; 0 \le s \le 1\}. \end{cases} \tag{2.36} $$

Put $S_0(w_0) := S_{0,1}(w_0) + S_{0,2}(w_0) + S_{0,3}(w_0) + S_{0,4}(w_0) + S_{0,5}(w_0)$. $S_0(w_0)$ is a path in $\{z_0; 0 < |z_0| < r, |\arg z_0| < \theta\}$, which starts at $\hat{r}_0$, encloses w_0 once on the circle $|z_0 - w_0| = \varepsilon|w_0|$ and ends at $\hat{r}_0$. For $1 \le i \le d$ put $S_i = \{z_i = (r_2 + r_3)e^{is}/2;\ 0 \le s \le 2\pi\}$ and $S(w_0) = S_0(w_0) \times \prod_{i=1}^d S_i$.

Let $0 < \theta' < \theta < \pi/(2\gamma^*)$. Let $f(z) \in \mathcal{O}(Z(\theta))$. Define

$$(G_\delta f)(w) := \int_{S(w_0)} f(z) G_\delta(z,w)\,dz, \tag{2.37}$$

$$(R_\delta f)(w) := \int_{S(w_0)} f(z) R_\delta(z,w)\,dz. \tag{2.38}$$

Now we can choose a small $\varepsilon_0 := \varepsilon_0(\theta') > 0$, which is independent of $\hat{r}_0$ such that for $z_0 \in S_0(w_0)$

$$|\arg(\log z_0 - \log w_0) - \pi| < 3\pi/2 - \varepsilon_0$$

and fix it. We note for $z_0 \in S_0(w_0)$

$$|\log z_0 - \log w_0| \geq \inf_{0 \leq s \leq 2\pi} |\log(w_0 + \varepsilon|w_0|e^{i(s+\arg w_0)}) - \log w_0| \geq 3\varepsilon/4.$$

Hence for $z_0 \in S(w_0)$ and small $\varepsilon, \varepsilon_0 > 0$

$$|\log z_0 - \log w_0| \sin\varepsilon_0 - c_0(|z_0| + \delta) \geq (3\varepsilon\sin\varepsilon_0)/4 - c_0(\hat{r}_0 + \delta).$$

Finally we choose $\hat{r}_0$ and $\delta > 0$ so small that $0 < \hat{r}_0, \delta < (\varepsilon\sin\varepsilon_0)/4c_0$ and

$$|\log z_0 - \log w_0| \sin\varepsilon_0 - c_0(|z_0| + \delta) \geq (\varepsilon\sin\varepsilon_0)/4.$$

Hence on $z \in S(w_0)$

$$\begin{aligned} |G_\delta(z,w)| &\leq \frac{A|w_0|^a}{(\varepsilon\sin\varepsilon_0)|z_0|^{a+e_*+1}}, \\ |R_\delta(z,w)| &\leq \frac{A|w_0|^a \exp(-c_1|z_0|^{-\gamma^*})}{(\varepsilon\sin\varepsilon_0)^{m+1}}. \end{aligned} \tag{2.39}$$

We have chosen positive constants $\varepsilon, \varepsilon_0, \hat{r}_0$ and δ and fix these constants and omit the suffix δ. Thus we can construct integral operators $(Gf)(w)$ and $(Rf)(w)$ and these operators have the following properties.

Theorem 2.6. *Suppose $\gamma^* < +\infty$. Let $a < a_0$ and $0 < \theta' < \theta < \pi/(2\gamma^*)$. Let $Z(\theta)$ and $W(\theta')$ be sectorial domains defined by (2.35) and $f(z) \in \mathcal{O}(Z(\theta))$. Then*

(1) *$(Gf)(w)$ and $(Rf)(w)$ are holomorphic in $W(\theta')$*

(2) *$f(z) \in \mathcal{O}_{reg,c}(\Omega(\theta))$, then $(Gf)(w) \in \mathcal{O}_{reg,c'}(W(\theta'))$, where $c' = \min\{c - e_*, a\}$.*

(3) *If $f(z) \in \mathcal{O}_{(\gamma^*)}(\Omega(\theta))$, then $|(Rf)(w)| \leq A|w_0|^a$ in $W(\theta')$.*

(4) *Let $u(z) \in \mathcal{O}(\Omega(\theta))$ and $P(z,\partial)u(z) = f(z)$. Then $(Gf)(w) = u(w) + (Ru)(w) + (Iu)(w)$ and $|(Iu)(w)| \leq A|w_0|^a$ in $W(\theta')$.*

Proof. Though the integral path $S(w_0)$ depends on w_0, we can take locally a fixed path. So $(Gf)(w)$ and $(Rf)(w)$ are holomorphic in $W(\theta')$. Let us show (2) and (3). We have

$$|(Gf)(w)| \leq \int_{S(w_0)} |f(z)G(z,w)|\,|dz| \leq \frac{A|w_0|^a}{(\varepsilon\sin\varepsilon_0)} \int_{S_0(w_0)} \frac{|z_0|^c}{|z_0|^{a+e_*+1}} |dz_0|.$$

Since

$$\int_{S_0(w_0)} \frac{|dz_0|}{|z_0|^{a+e_*-c+1}} \le C(1+|w_0|^{-(a+e_*-c)}),$$

we have $|(Gf)(w)| \le A(\varepsilon, \varepsilon_0)|w_0|^{c'}$, $c' = \min\{a, c - e_*\}$. Similarly we have

$$|(Rf)(w)| \le \int_{S(w_0)} |f(z)R(z,w)|\,|dz| \le \frac{A|w_0|^a}{(\varepsilon \sin \varepsilon_0)^{m+1}} \int_{S_0(w_0)} \exp(-c_1|z_0|^{-\gamma^*}/2)|dz_0| \le C|w_0|^a.$$

Finally we show (4). Let $P(z,\partial)u(z) = f(z)$. Then

$$\begin{aligned}(Gf)(w) &= \int_{S(w_0)} (P(z,\partial)u(z))G(z,w)\,dz \\ &= \int_{S(w_0)} u(z)\,{}^tP(z,\partial)G(z,w)\,dz + I(w) \quad \text{(by intergrations by parts)} \\ &= \int_{S(w_0)} u(z)\delta(z,w)\,dz + \int_{S(w_0)} u(z)R(z,w)\,dz + I(w) \\ &= u(w) + (Ru)(w) + I(w),\end{aligned}$$

where $I(w)$ is determined by the values of $\partial^\alpha u(z)$ and $\partial_z^\alpha G(z,w)(|\alpha| \le m)$ at $\{(\hat{r}, z');\ z' \in \prod_{i=1}^d S_i\}$. Hence $|I(w)| \le A|w_0|^a$.

Let us consider the case $\gamma^* = +\infty$. Define

$$K(z,w',\lambda) = z_0^{-e_*}U(z,w',\lambda), \tag{2.40}$$

which does not contain a parameter δ. We have, instead of Theorem 2.3,

Theorem 2.7. *Let a' and b' be constants with $a' < a_0$ and $b' > b_0$. Then*

(1) *$K(z,w',\lambda)$ is holomorphic on $X_\infty(r_0,r_1,r_2,r_3) \times (\boldsymbol{C} - \Lambda(-a'-1,b'))$ and there exist constants $c_0 = c(b')$ and $A = A(a',b')$ such that*

$$|K(z,w',\lambda)| \le A|z_0|^{-e_*}\exp(c_0|z_0|\,|\lambda|). \tag{2.41}$$

(2) *It holds that*

$${}^tP(z,\partial)(z_0^\lambda K(z,w',\lambda)) = \frac{z_0^\lambda}{(2\pi i)^d \prod_{j=1}^d (z_j - w_j)}. \tag{2.42}$$

Proof. The statement (1) follows from Proposition 2.2. We have

$$\begin{aligned}&{}^tP(z,\partial)(z_0^\lambda K(z,w',\lambda)) \\ &\quad = z_0^\lambda \sum_{|\alpha| \le m} (-1)^{|\alpha|}(\vartheta + \lambda + 1)^{\alpha_0} z_0^{e_\alpha}\partial^{\alpha'}(b_\alpha(z)z_0^{-e^*}U(z,w',\lambda)) \\ &\quad = z_0^\lambda\,{}^t\mathscr{P}(\lambda;z,\partial)U(z,w',\lambda) = \frac{z_0^\lambda}{(2\pi i)^d \prod_{j=1}^d (z_j - w_j)}.\end{aligned}$$

Define

$$G(z,w)=\frac{1}{2\pi i}\int_{\mathscr{C}}\frac{z_0^{\lambda}}{w_0^{\lambda+1}}K(z,w',\lambda)\,d\lambda, \tag{2.43}$$

where $\mathscr{C}:=\mathscr{C}(\phi)(|\phi|<\pi)$ is an infinite path in $\boldsymbol{C}-\Lambda(-a'-1,b')$ that is the same as in (2.26), and an integral operator with kernel $G(z,w)$ for $f(z)\in\mathcal{O}(Z(\theta))$,

$$(Gf)(w):=\int_{S(w_0)}f(z)G(z,w)\,dz, \tag{2.44}$$

where $S(w_0)$ is the same path as in (2.37). By repeating the similar method to the case $\gamma^*<\infty$, we have

Theorem 2.8. *Suppose* $\gamma^*=+\infty$. *Let* $a<a_0$ *and* $0<\theta'<\theta$. *Let* $Z(\theta)$ *and* $W(\theta')$ *be sectorial domains defined by* (2.35) *and* $f(z)\in\mathcal{O}(Z(\theta))$. *Then*

(1) $(Gf)(w)$ *is holomorphic in* $W(\theta')$.

(2) $f(z)\in\mathcal{O}_{reg,c}(\Omega(\theta))$, *then* $(Gf)(w)\in\mathcal{O}_{reg,c'}(W(\theta'))$, *where* $c'=\min\{c-e_*,a\}$.

(3) *Let* $u(z)\in\mathcal{O}(\Omega(\theta))$ *and* $P(z,\partial)u(z)=f(z)$. *Then* $(Gf)(w)=u(w)+(Iu)(w)$ *and* $|(Iu)(w)|\le A|w_0|^a$ *in* $W(\theta')$.

§2.3. Proof of Theorems 1.6 and 1.7.

Proof of Theorem 1.6. Suppose $\gamma^*<+\infty$. Theorem 2.6 is valid for any $0<\theta'<\theta<\pi/(2\gamma^*)$ and we note that we can choose $a<a_0$ as close to a_0 as possible. So if $\theta<\pi/(2\gamma^*)$, then we have Theorem 1.6 easily from Theorem 2.6. Otherwise let $-\theta=\theta_0<\theta_1<\cdots<\theta_n=\theta$ such that $\theta_i-\theta_{i-1}<\pi/\gamma^*$ for $i=1,2,\dots,n$. Put $\varphi_i=(\theta_i+\theta_{i-1})/2$. By rotating the variable z_0, let us consider $u_i(z)=u(z_0e^{i\varphi_i},z')$ $(i=1,2,\dots,n)$. Then, by applying Theorem 2.6 to $u_i(z)$, we have Theorem 1.6 for any θ. If $\gamma^*=+\infty$, we have Theorem 1.6 from Theorem 2.8.

Proof of Theorem 1.7. Put $b(z'):=b_{k^*,0,\dots,0}(z')$. *Condition* 0 means $b(z')\not\equiv 0$. So there is a polycircle $M=\prod_{i=1}^{d}\{z_i;|z_i|=\rho_i\}$ such that $b(z')\neq 0$ on M. For $\hat{z}'\in M$ there is a neighbourhood of $U_{\hat{z}'}$ of $(0,\hat{z}')$ such that $|u(z)|\le C_{\hat{z}',\theta'}|z_0|^{c(\hat{z}')}$ in $U_{\hat{z}'}(\theta')$ for any $0<\theta'<\theta$. Since M is compact, there are a constant c'' and a neighbourhood U_M of $\{0\}\times M$ such that $|u(z)|\le C|z_0|^{c''}$ in $\{z\in U_M;|\arg z_0|<\theta'\}$. Hence it follows from the maximal principle of holomorphic functions that there is a neighbourhood U of $z=0$ such that $|u(z)|\le C_{\theta'}|z_0|^{c''}$ in $U(\theta')$. This means $u(z)\in\mathcal{O}_{reg,c''}(U(\theta))$.

§3. Estimate of $u_{l,n}(z',w',\lambda)$ and $u_l(z',w',\lambda)$.

In this section we give the proofs of Propositions 2.1 and 2.2. First let $\gamma^*<+\infty$ and let us estimate the coefficient $u_{l,n}(z',w',\lambda)$'s of

$$U(z',w',\lambda,\zeta)=\sum_{l,n=0}^{\infty}\frac{z_0^l\zeta^{n/\gamma^*}}{\Gamma(l+1)\Gamma(n/\gamma^*+1)}u_{l,n}(z',w',\lambda). \tag{3.1}$$

Recall what we need in this section. We assume $P(z,\partial)$ satisfies *Condition* 1. Choose $R>0$ so that $b_{\hat{\alpha}}(0,z')\neq 0$, $\hat{\alpha}=(k^*,0,0,\dots,0)$ on $\{z'\in\boldsymbol{C}^d;|z'|\le R\}$. Let a_0, a_1 and

b_0 be real numbers such that all the roots of algebraic equation $\chi_P(z', \lambda) = 0$ for $\{z'; |z'| \le R\}$ are contained in $\{\lambda; a_0 \le \Re\lambda \le a_1, |\Im\lambda| \le b_0\}$. So all the roots of $\chi_{{}^tP}(z', \lambda - e_*) = 0$ for $\{|z'| \le R\}$ are contained in Λ_0,

$$\Lambda_0 = \{\lambda \in \boldsymbol{C}; -a_1 - 1 \le \Re\lambda \le -a_0 - 1, |\Im\lambda| \le b_0\}, \tag{3.2}$$

by the relation $\chi_{{}^tP}(z', \lambda) = \chi_P(z', -\lambda - e_* - 1)$.

The recursion formula of $u_{l,n}(z', w', \lambda)$ is

$$\begin{aligned} &\chi_{{}^tP}(z', \lambda + l + n - e^*) u_{l,n}(z', w', \lambda) \\ &\qquad + \sum_{\substack{(\alpha, h) \\ e_\alpha - e_* + h > 0}} \frac{(-1)^{|\alpha|}(\lambda + l + n + 1)^{\alpha_0} l!}{(l-h)!} \partial^{\alpha'}(b_{\alpha,h}(z') u_{l-h, n-e_\alpha+e_*}(z', w', \lambda)) \\ &\quad = \begin{cases} \dfrac{1}{(2\pi i)^d \prod_{j=1}^d (z_j - w_j)} & (l,n) = (0,0), \\ 0 & (l,n) \ne (0,0). \end{cases} \end{aligned} \tag{3.3}$$

By the above formula, we can determine $u_{l,n}(z', w', \lambda)$ successively and it is a rational function in λ, whose poles are in $\{\lambda; \prod_{i=0}^{l+n} \chi_{{}^tP}(z', \lambda + i - e_*) = 0\}$.

We have introduced regions $X'(r_1, r_2, r_3)$ in (z', w')-space and $\Lambda(a,b)$ in λ-space. Let $0 < r_1 < r_2 < r_3 < R$ and $a, b \in \boldsymbol{R}$. Then they were defined by

$$X'(r_1, r_2, r_3) = \{(z', w'); |w_i| < r_1, r_2 < |z_i| < r_3 \text{ for } 1 \le i \le d\}, \tag{3.4}$$

$$\Lambda(a,b) = \{\lambda; \Re\lambda \le a, |\Im\lambda| \le b\}. \tag{3.5}$$

For our purpose, to estimate functions, the method of majorant functions is available. Let $A(x) = \sum_\alpha A_\alpha (x - \hat{x})^\alpha$ and $B(x) = \sum_\alpha B_\alpha (x - \hat{x})^\alpha$ be formal power series of N-variables x centered at $x = \hat{x}$. Then $A(x) \gg 0$ means $A_\alpha \ge 0$ for all α and $A(x) \ll B(x)$ means $|A_\alpha| \le B_\alpha$ for all α. We give elementary properties of majorant power series.

Lemma 3.1. *Let $\theta(t)$ be a formal power series of one variable t centered at $t = 0$ such that $\theta(t) \gg 0$ and $(r - t)\theta(t) \gg 0$. Then for the derivatives $\theta^{(j)}(t) = (d/dt)^j \theta(t)$, $j = 0, 1, \dots,$ we have*

$$\begin{aligned} &(r - t)\theta^{(j)}(t) \gg 0, \quad r\theta^{(j+1)}(t) \gg \theta^{(j)}(t), \\ &(r' - t)^{-1}\theta^{(j)}(t) \ll (r' - r)^{-1}\theta^{(j)}(t) \quad \text{for } r' > r. \end{aligned} \tag{3.6}$$

For the proof of Lemma 3.1 we refer to [**13**]. Let $\theta(t) = 1/(r - t)$. Then $\theta^{(j)}(t) = (\Gamma(j+1))/(r-t)^{j+1}$ for $j \ge 0$. We have

Lemma 3.2. *Suppose $0 < r < 1$. Then*

$$\frac{\theta^{(n_1)}(t)}{n_1!} \ll \frac{\theta^{(n_2)}(t)}{n_2!} \quad \text{for } n_1 \le n_2 \tag{3.7}$$

and if $0 \le m_0 \le m$, $0 \le N_0 < N$ and $N' \ge 0$,

$$\frac{\theta^{(mN_0+N'+m_0)}(t)}{(mN_0)!} \ll (m(N_0+1))^{m_0}\frac{\theta^{(mN+N')}}{(mN)!}. \tag{3.8}$$

PROOF. The first inequality is obvious. Since

$$\begin{aligned}&\frac{\Gamma(mN_0+N'+m_0+1)}{\Gamma(mN+N'+1)}\frac{(mN)!}{(mN_0)!}\\ &\quad\le \frac{(mN)(mN-1)\cdots(mN_0+1)}{(mN+N')(mN+N'-1)\cdots(mN_0+N'+m_0+1)}\\ &\quad\le (mN_0+m_0)\cdots(mN_0+1)\le (m(N_0+1))^{m_0},\end{aligned}$$

we have the second.

Now let us proceed to estimate $u_{l,n}(z',w',\lambda)$. Let $\hat{z}'=(\hat{z}_1',\hat{z}_2',\dots,\hat{z}_d')$ be a point with $|\hat{z}_i{}'|=(r_2+r_3)/2$ and $r'=(r_3-r_2)/3$. Put $t=\sum_{j=1}^d(z_j-\hat{z}_j)$ and $A(z')\ll_{z'-\hat{z}'}B(z')$ means as formal power series of $(z'-\hat{z}')$. Let $d(\lambda,\Lambda_0)$ be the distance of λ and the set Λ_0 (see (3.2)). First we have from the location of the zeros of $\chi_{{}^tP}(z',\lambda-e_*)$

LEMMA 3.3. *Let $a'<a_0$ and $b_0<b'$ and $(z',w')\in X'(r_1,r_2,r_3)$. Then the following inequalities hold.*

(1) $d(\lambda+i,\Lambda_0)\le|\lambda|+|a_0+1|+i$ *for* $i\ge 0$.

(2) *There is a positive constant C such that*

$$\chi_{{}^tP}(z',\lambda+i-e^*)^{-1}\underset{z'-\hat{z}'}{\ll}\frac{C}{d(\lambda+i,\Lambda_0)^{k^*}}\frac{1}{r'-t}. \tag{3.9}$$

(3) *There is a positive constant C such that for $\lambda\notin\Lambda(-a'-1,b')$*

$$\prod_{i=1}^N\left(\frac{|\lambda|+|a_0+1|+i}{d(\lambda+i,\Lambda_0)}\right)^{k^*}\le C^{N+1}. \tag{3.10}$$

PROOF. We have $d(\lambda+i,\Lambda_0)\le|\lambda+i-a_0-1|\le|\lambda|+|a_0+1|+i$, which means (1). Let $\lambda_i(z')(1\le i\le k^*)$ be roots of $\chi_P(z',\lambda)=0$ such that $a_0\le\Re\lambda_1(z')\le\Re\lambda_2(z')\le\cdots\le\Re\lambda_{k^*}(z')\le a_1$. Then $\chi_P(z',\lambda)=b_{\hat{\alpha}}(0,z')\prod_{i=1}^{k^*}(\lambda-\lambda_i(z'))$ and by $\chi_{{}^tP}(z',\lambda)=\chi_P(z',-\lambda-e_*-1)$ there exists $B_0>0$ such that

$$\begin{aligned}|\chi_{{}^tP}(z',\lambda+s-e^*)|&=|b_{\hat{\alpha}}(0,z')|\prod_{i=1}^{k^*}|\lambda+\lambda_i(z')+s+1|\\ &\ge B_0 d(\lambda+s,\Lambda_0)^{k^*},\end{aligned}$$

from which the estimate (3.9) follows.

Let us show (3). First we note that $\Lambda_0\subset\Lambda(-a'-1,b')\subset\Lambda(-a',b')$. If $\lambda'\notin\Lambda(-a',b')$, then $d(\lambda',\Lambda_0)\ge\min\{b'-b_0,1\}$. So there is a constant $B_1=B_1(b')$ such that $(|\lambda'|+1)/d(\lambda',\Lambda_0)\le B_1$. Hence if $\lambda\notin\Lambda(-a'-1,b')$ and $i\ge1$, we have $\lambda+i\notin\Lambda(-a',b')$ and $|\lambda+i|+1\le B_1d(\lambda+i,\Lambda_0)$. So for $\lambda\notin\Lambda(-a'-1,b')$

$$(3.11)\qquad \prod_{i=1}^{N}\left(\frac{|\lambda|+|a_0+1|+i}{d(\lambda+i,\Lambda_0)}\right)^{k^*} \le \left(B_1^N\prod_{i=1}^{N}\frac{|\lambda|+|a_0+1|+i}{|\lambda+i|+1}\right)^{k^*}.$$

We note the inequality,

$$\prod_{i=1}^{N}\frac{|\lambda|+|a_0+1|+i}{|\lambda+i|+1} \le \prod_{i=1}^{N}\frac{|\lambda|+|a_0+1|+i}{||\lambda|-i|+1} = \varphi(|\lambda|) \quad \text{for all } \lambda\in \boldsymbol{C}.$$

Hence we only have to show that there is a constant B such that $\varphi(s)\le B^N$ for all $s\ge 0$. Let $s\ge 2N$. Then there is a constant B_2 such that for $1\le i\le N$

$$\frac{s+|a_0+1|+i}{s-i+1} \le \frac{s+|a_0+1|+N}{s-N+1} \le \frac{1+(|a_0+1|+N)/s}{1+(1-N)/s} \le B_2$$

and $\varphi(s)\le B_2^N$. For $N\le s<2N$ we have $\prod_{i=1}^{N}(|s-i|+1)=\prod_{i=1}^{N}(s-i+1)\ge \prod_{i=1}^{N}(N-i+1)=N!$ and for $0\le s<N$ there exists $0<B_3<1$ such that $\prod_{i=1}^{N}(|s-i|+1) = \prod_{i=1}^{[s]}(s-i+1)\prod_{i=[s]+1}^{N}(i-s+1) \ge \prod_{i=1}^{[s]}([s]-i+1)\prod_{i=[s]+1}^{N}(i-[s]) = [s]!(N-[s])!\ge B_3^N N!$. Hence if $0\le s<2N$,

$$\varphi(s)=\prod_{i=1}^{N}\frac{(s+|a_0+1|+i)}{|s-i|+1} \le \frac{\prod_{i=1}^{N}(2N+1+|a_0|+i)}{B_3^N N!} \le B_4^N.$$

Thus we have the desired inequality.

Let $\theta(t)=(r-t)^{-1}$ with $0<r<\min\{r',1\}$, where $r'=(r_3-r_2)/3$. We have

Proposition 3.4. *Let $a'<a_0$, $b'>b_0$ and $(z',w')\in X'(r_1,r_2,r_3)$. Then there are positive constants A and B such that for $\lambda\notin\Lambda(-a'-1,b')$*

$$(3.12)\qquad u_{l,n}(z',w',\lambda) \underset{z'-\hat{z}'}{\ll} AB^{l+n}\frac{\theta^{(m(l+n))}(t)}{(m(l+n))!}\,\frac{\Gamma(|\lambda|+l+n/\gamma^*+1)}{\Gamma(|\lambda|+1)\,d(\lambda,\Lambda_0)^{k^*}}\prod_{i=1}^{l+n}\left(\frac{|\lambda|+|a_0+1|+i}{d(\lambda+i,\Lambda_0)}\right)^{k^*}$$

for all $l,n\in\boldsymbol{N}$, where $t=\sum_{j=1}^{d}(z_j-\hat{z}_j)$.

Proof. We show the estimate by induction on $N=l+n$. We have $u_{0,0}(z',w',\lambda) \ll_{z'-\hat{z}'} Ad(\lambda,\Lambda_0)^{-k^*}\theta(t)$ for some constant $A=A(a',b')$. Assume the estimate (3.12) holds for $l+n<N$. It follows from Lemmas 3.1 and 3.2 that

$$\begin{aligned}
&\frac{\partial^{\alpha'}(b_{\alpha,h}(z')\theta^{(m(l+n-e_\alpha+e_*-h))}(t))}{(m(l+n-e_\alpha+e_*-h))!}\\
&\qquad \underset{z'-\hat{z}'}{\ll} C_0^{h+1}\frac{\theta^{(m(l+n-e_\alpha+e_*-h)+|\alpha'|)}(t)}{(m(l+n-e_\alpha+e_*-h))!}\\
&\qquad \underset{z'-\hat{z}'}{\ll} C_1^{h+1}(l+n-e_\alpha+e_*-h+1)^{|\alpha'|}\frac{\theta^{(m(l+n))}(t)}{(m(l+n))!}.
\end{aligned}$$

Hence

$$\frac{l!}{(l-h)!}\partial^{\alpha'}(b_{\alpha,h}(z')u_{l-h,n-e_\alpha+e_*}(z',w',\lambda))$$

$$\underset{z'-\hat{z}'}{\ll} AB^{l+n-e_\alpha+e_*-h}C_1^{h+1}\frac{\theta^{(m(l+n))}(t)l!}{(m(l+n))!(l-h)!}(l+n-e_\alpha+e_*-h+1)^{|\alpha'|}$$

$$\times\frac{\Gamma(|\lambda|+l-h+(n-e_\alpha+e_*)/\gamma^*+1)}{\Gamma(|\lambda|+1)\,d(\lambda,\Lambda_0)^{k^*}}\prod_{i=1}^{l+n-e_\alpha+e_*-h}\left(\frac{|\lambda|+|a_0+1|+i}{d(\lambda+i,\Lambda_0)}\right)^{k^*}$$

$$\underset{z'-\hat{z}'}{\ll} AB^{l+n-e_\alpha+e_*-h}C_1^{h+1}\frac{\theta^{(m(l+n))}(t)}{(m(l+n))!}(l+n+1)^{|\alpha'|}$$

$$\times\frac{\Gamma(|\lambda|+l+(n-e_\alpha+e_*)/\gamma^*+1)}{\Gamma(|\lambda|+1)\,d(\lambda,\Lambda_0)^{k^*}}\prod_{i=1}^{l+n-e_\alpha+e_*-h}\left(\frac{|\lambda|+|a_0+1|+i}{d(\lambda+i,\Lambda_0)}\right)^{k^*}.$$

We have, by using $(e_\alpha-e_*)/\gamma^*\ge|\alpha|-k^*$,

$$\begin{aligned}
&(l+n+1)^{|\alpha'|}(|\lambda|+l+n+1)^{\alpha_0}\Gamma(|\lambda|+l+(n-e_\alpha+e_*)/\gamma^*+1)\\
&\quad\le C_2(|\lambda|+l+n)^{k^*}(|\lambda|+l+n+1)^{|\alpha|-k^*}\Gamma(|\lambda|+l+(n-e_\alpha+e_*)/\gamma^*+1)\\
&\quad\le C_3(|\lambda|+l+n)^{k^*}\Gamma(|\lambda|+l+(n-e_\alpha+e_*)/\gamma^*+|\alpha|-k^*+1)\\
&\quad\le C_3(|\lambda|+l+n)^{k^*}\Gamma(|\lambda|+l+n/\gamma^*+1)
\end{aligned}$$

and by Lemmas 3.2 and 3.3,

$$(3.13)\qquad \frac{(\lambda+l+n+1)^{\alpha_0}l!\partial^{\alpha'}(b_{\alpha,h}(z')u_{l-h,n-e_\alpha+e_*}(z',w',\lambda))}{(l-h)!\chi_{{}^tP}(z',\lambda+l+n-e_*)}$$

$$\underset{z'-\hat{z}'}{\ll} AB^{l+n-e_\alpha+e_*-h}C^{h+1}\frac{\theta^{(m(l+n))}(t)\Gamma(|\lambda|+l+n/\gamma^*+1)M(l,n,\alpha,\lambda)}{(m(l+n))!\Gamma(|\lambda|+1)\,d(\lambda,\Lambda_0)^{k^*}},$$

where

$$M(l,n,\alpha,\lambda)=\left(\frac{|\lambda|+l+n}{d(\lambda+l+n,\Lambda_0)}\right)^{k^*}\prod_{i=1}^{l+n-e_\alpha+e_*-h}\left(\frac{|\lambda|+|a_0+1|+i}{d(\lambda+i,\Lambda_0)}\right)^{k^*}$$

$$\le\prod_{i=1}^{l+n}\left(\frac{|\lambda|+|a_0+1|+i}{d(\lambda+i,\Lambda_0)}\right)^{k^*}.$$

Thus we have

$$u_{l,n}(z',w',\lambda)=\sum_{\substack{(\alpha,h)\\ e_\alpha-e_*+h>0}}\frac{(-1)^{|\alpha|}(\lambda+l+n+1)^{\alpha_0}l!\partial^{\alpha'}(b_{\alpha,h}(z')u_{l-h,n-e_\alpha+e_*}(z',w',\lambda))}{(l-h)!\chi_{{}^tP}(z',\lambda+l+n+e_*)}$$

$$\underset{z'-\hat{z}'}{\ll}\frac{AB^{l+n-1}\theta^{(m(l+n))}(t)\Gamma(|\lambda|+l+n/\gamma^*+1)}{(m(l+n))!d(\lambda,\Lambda_0))^{k^*}\Gamma(|\lambda|+1)}\left(\prod_{i=1}^{l+n}\frac{|\lambda|+|a_0+1|+1}{d(\lambda+i,\Lambda_0)}\right)^{k^*}N(l,n),$$

where $N(l,n) = \sum_{\substack{(\alpha,h)\\ e_\alpha - e_* + h > 0}} B^{-e_\alpha + e_* - h + 1} C^{h+1}$. We may assume that $C \geq 1$ and $B \geq 2C$. Then

$$N(l,n) \leq C^2 \sum_{\substack{(\alpha,h)\\ e_\alpha - e_* + h > 0}} (B/C)^{-e_\alpha + e_* - h + 1} \leq C^2 \sum_{\substack{(\alpha,h)\\ e_\alpha - e_* + h > 0}} \left(\frac{1}{2}\right)^{e_\alpha - e_* + h - 1} \leq C'.$$

Thus, by choosing $B > C'$, the estimate (3.12) holds for (l,n) with $l + n = N$.

Now we give the proofs of Propositions 2.1 and 2.2. First let $\gamma^* < +\infty$.

Proof of Proposition 2.1. Let $a' < a_0$, $b_0 < b'$, $\lambda \notin \Lambda(-a' - 1, b')$ and $r'' < r < r' = (r_3 - r_2)/3$. Suppose $\sum_{j=1}^{d} |z_j - \hat{z}_j| < r''$. Then $|\theta^{(N)}(t)/N!| \leq C_0^{N+1}$ for $N \in \boldsymbol{N}$. It follows from (3) in Lemma 3.3 that there exist constants $A = A(a', b')$ and $B = B(b')$ such that

$$|u_{l,n}(z', w', \lambda)| \leq AB^{l+n} \frac{\Gamma(|\lambda| + l + n/\gamma^* + 1)}{\Gamma(|\lambda| + 1)}. \tag{3.14}$$

The above estimate holds for all $\hat{z}' = (\hat{z}_1, \ldots, \hat{z}_d)$ with $|\hat{z}_i| = (r_2 + r_3)/2$. Therefore there are r_2' and r_3' with $r_2' < (r_2 + r_3)/2 < r_3'$ such that the estimate (3.14) holds on $\{z' = (z_1, z_2, \ldots, z_d) \in \boldsymbol{C}^d; r_2' \leq |z_i'| \leq r_3'$ *for* $i = 1, 2, \ldots, d\}$. By the formula $\Gamma(z+1) = z\Gamma(z)$ and the Stirling's formula, there is a constant C_0 such that

$$\begin{aligned}\frac{\Gamma(|\lambda| + l + n/\gamma^* + 1)}{\Gamma(|\lambda| + 1)} &= \left(\prod_{i=0}^{l+[n/\gamma^*]-1} (|\lambda| + l + n/\gamma^* - i)\right) \frac{\Gamma(|\lambda| + n/\gamma^* - [n/\gamma^*] + 1)}{\Gamma(|\lambda| + 1)} \\ &\leq C_0 \left(\prod_{i=0}^{l+[n/\gamma^*]-1} (|\lambda| + l + n/\gamma^* - i)\right) (|\lambda| + n/\gamma^* - [n/\gamma^*])^{n/\gamma^* - [n/\gamma^*]}.\end{aligned}$$

Hence there are constants C_1 and C_2 such that

$$\begin{aligned}&\prod_{i=0}^{l+[n/\gamma^*]-1} (|\lambda| + l + n/\gamma^* - i)(|\lambda| + n/\gamma^* - [n/\gamma^*])^{n/\gamma^* - [n/\gamma^*]} \\ &\qquad \leq \begin{cases} C_1^{l+n/\gamma^*+1} \Gamma(l + n/\gamma^* + 1) & \text{for } |\lambda| \leq l + n/\gamma^*, \\ C_1^{l+n/\gamma^*+1} |\lambda|^{l+n/\gamma^*} & \text{for } |\lambda| \geq l + n/\gamma^* \end{cases}\end{aligned}$$

and

$$\begin{aligned}\frac{\Gamma(|\lambda| + l + n/\gamma^* + 1)}{\Gamma(|\lambda| + 1)} &\leq C_1^{l+n/\gamma^*+1} (\Gamma(l + n/\gamma^* + 1) + |\lambda|^{l+n/\gamma^*}) \\ &\leq C_2^{l+n/\gamma^*+1} (\Gamma(l + 1) + |\lambda|^l)(\Gamma(n/\gamma^* + 1) + |\lambda|^{n/\gamma^*}).\end{aligned}$$

Thus there are constants $A = A(a', b')$ and $B = B(b')$ such that

$$|u_{l,n}(z', w', \lambda)| \leq AB^{l+n} (\Gamma(l + 1) + |\lambda|^l)(\Gamma(n/\gamma^* + 1) + |\lambda|^{n/\gamma^*}).$$

Proof of Proposition 2.2. We have by Proposition 2.1

$$|U(z',w',\lambda,\zeta)| \le \sum_{l,n=0}^{\infty} \frac{|z_0|^l|\zeta|^{n/\gamma^*}}{\Gamma(n/\gamma^*+1)\Gamma(l+1)}|u_{l,n}(z',w',\lambda)|$$

$$\le A\sum_{l,n=0}^{\infty} B^{l+n}|z_0|^l|\zeta|^{n/\gamma^*}\left(1+\frac{|\lambda|^l}{\Gamma(l+1)}\right)\left(1+\frac{|\lambda|^{n/\gamma^*}}{\Gamma(n/\gamma^*+1)}\right).$$

Let $|z_0| < r_0 < B^{-1}$ and $|\zeta| < \delta_0 < B^{-\gamma^*}$. The constant $B = B(b')$ depends only on b'. So there is a constant $c = c(b') > 0$ such that

$$|U(z',w',\lambda,\zeta)| \le C\exp(c(|z_0|+|\zeta|)|\lambda|).$$

Secondly suppose $\gamma^* = +\infty$ and consider

$$U(z',w',\lambda) = \sum_{l=0}^{+\infty} \frac{z_0^l}{\Gamma(l+1)}u_l(z',w',\lambda). \tag{3.15}$$

By repeating the preceding arguments, we have

$$u_l(z',w',\lambda) \underset{z'-\hat{z}'}{\ll} AB^l\frac{\theta^{(ml)}(t)}{(ml)!}\,\frac{\Gamma(|\lambda|+l+1)}{\Gamma(|\lambda|+1)\,d(\lambda,\Lambda_0)^{k^*}}\prod_{i=1}^{l}\left(\frac{|\lambda|+|a_0+1|+i}{d(\lambda+i,\Lambda_0)}\right)^{k^*} \tag{3.16}$$

and

$$|u_l(z',w',\lambda)| \le AB^l\frac{\Gamma(|\lambda|+l+1)}{\Gamma(|\lambda|+1)} \tag{3.17}$$

and we can show Propositions 2.1 and 2.2 for $\gamma^* = +\infty$.

§4. Proof of Theorem 2.3.

In this section we give the proof of Theorem 2.3. First we prepare lemmas.

Lemma 4.1. *Let* $V(z_0,\lambda,\zeta) = \sum_{n,l=0}^{+\infty}\dfrac{v_{l,n}(\lambda)z_0^l\zeta^{n/\kappa}}{\Gamma(l+1)\Gamma(n/\kappa+1)}$, *where* $v_{l,n}(\lambda)$*'s are functions in* λ *defined on a domain* $\hat{\Lambda}\subset \boldsymbol{C}$ *such that*

$$|v_{l,n}(\lambda)| \le AB^{l+n}(\Gamma(l+1)+|\lambda|^l)(\Gamma(n/\kappa+1)+|\lambda|^{n/\kappa}). \tag{4.1}$$

Let $|z_0| < r_0 < B^{-1}$ *and* $v_n(z_0,\lambda) = \sum_{l=0}^{+\infty}\dfrac{v_{l,n}(\lambda)z_0^l}{\Gamma(l+1)}$. *Put* $V_N^1(z_0,\lambda,\zeta) = \sum_{n=0}^{N}\dfrac{v_n(z_0,\lambda)\zeta^{n/\kappa}}{\Gamma(n/\kappa+1)}$ *and* $V_N^2(z_0,\lambda,\zeta) = V(z_0,\lambda,\zeta) - V_N^1(z_0,\lambda,\zeta)$. *Let* $0 < d < d_0 < B^{-\kappa}$. *Then there exist constants* $c = c(B)$ *and* $A_1 = A_1(d_0)$ *such that if* $|\zeta| \ge d$,

$$\begin{cases} |V_N^1(z_0,\lambda,\zeta)| \le A_1\exp(c(|z_0|+d)|\lambda|)\,d^{-N/\kappa}|\zeta|^{N/\kappa}, \\ |\partial_\zeta^{-h/\kappa}V_N^1(z_0,\lambda,\zeta)| \le \dfrac{A_1\exp(c(|z_0|+d)|\lambda|)\,d^{-N/\kappa}|\zeta|^{(N+h)/\kappa}}{\Gamma(h/\kappa+1)}, \end{cases} \tag{4.2}$$

and if $|\zeta| \le d$,

$$
(4.3)\qquad \begin{cases} |V_N^2(z_0,\lambda,\zeta)| \le A_1 \exp(c(|z_0|+d)|\lambda|)\, d^{-N/\kappa}|\zeta|^{N/\kappa}, \\ |\partial_\zeta^{-h/\kappa} V_N^2(z_0,\lambda,\zeta)| \le \dfrac{A_1 \exp(c(|z_0|+d)|\lambda|)\, d^{-N/\kappa}|\zeta|^{(N+h)/\kappa}}{\Gamma(h/\kappa+1)}, \\ |\partial_\zeta^{-h/\kappa} V(z_0,\lambda,\zeta)| \le \dfrac{A_1 \exp(c(|z_0|+d)|\lambda|)|\zeta|^{h/\kappa}}{\Gamma(h/\kappa+1)}, \end{cases}
$$

and there are constants $c'(B,p,q)$ and $A_2(d_0,p,q)$ such that for $|\zeta| \le d$

$$
(4.4)\qquad |(z_0\partial_0)^p(\kappa\zeta\partial_\zeta)^q(\partial_\zeta^{-h/\kappa}V(z_0,\lambda,\zeta))| \le A_2 \exp(c'(|z_0|+d)|\lambda|).
$$

Proof. It holds that

$$
\begin{aligned}
\frac{|v_n(z_0,\lambda)|}{\Gamma(n/\kappa+1)} &\le AB^n\left(1+\frac{|\lambda|^{n/\kappa}}{\Gamma(n/\kappa+1)}\right)\left(\sum_{l=0}^{+\infty} B^l|z_0|^l\left(1+\frac{|\lambda|^l}{\Gamma(l+1)}\right)\right) \\
&\le A' \exp(c|z_0|\,|\lambda|)B^n\left(1+\frac{|\lambda|^{n/\kappa}}{\Gamma(n/\kappa+1)}\right).
\end{aligned}
$$

Let $|\zeta| \ge d$. Then we have

$$
\begin{aligned}
|V_N^1(z_0,\lambda,\zeta)| &\le A' \exp(c|z_0|\,|\lambda|)|\zeta|^{N/\kappa}\sum_{n=0}^{N} B^n|\zeta|^{(n-N)/\kappa}\left(1+\frac{|\lambda|^{n/\kappa}}{\Gamma(n/\kappa+1)}\right) \\
&\le A' \exp(c|z_0|\,|\lambda|)\, d^{-N/\kappa}|\zeta|^{N/\kappa}\sum_{n=0}^{N}(d^{1/\kappa}B)^n\left|\frac{\zeta}{d}\right|^{(n-N)/\kappa}\left(1+\frac{|\lambda|^{n/\kappa}}{\Gamma(n/\kappa+1)}\right) \\
&\le A' \exp(c|z_0|\,|\lambda|)\, d^{-N/\kappa}|\zeta|^{N/\kappa}\sum_{n=0}^{N}(d^{1/\kappa}B)^n\left(1+\frac{|\lambda|^{n/\kappa}}{\Gamma(n/\kappa+1)}\right) \\
&\le A_1 \exp(c(|z_0|+d)|\lambda|)\, d^{-N/\kappa}|\zeta|^{N/\kappa}.
\end{aligned}
$$

Let $|\zeta| \le d$. Then

$$
\begin{aligned}
|V_N^2(z_0,\lambda,\zeta)| &\le A' \exp(c|z_0|\,|\lambda|)\sum_{n=N+1}^{+\infty} B^n|\zeta|^{n/\kappa}\left(1+\frac{|\lambda|^{n/\kappa}}{\Gamma(n/\kappa+1)}\right) \\
&\le A' \exp(c|z_0|\,|\lambda|)|\zeta|^{(N+1)/\kappa}\sum_{n=N+1}^{+\infty} B^n|\zeta|^{(n-N-1)/\kappa}\left(1+\frac{|\lambda|^{n/\kappa}}{\Gamma(n/\kappa+1)}\right) \\
&\le A' \exp(c|z_0|\,|\lambda|)\left|\frac{\zeta}{d}\right|^{(N+1)/\kappa}\sum_{n=N+1}^{+\infty}(d^{1/\kappa}B)^n\left|\frac{\zeta}{d}\right|^{(n-N-1)/\kappa}\left(1+\frac{|\lambda|^{n/\kappa}}{\Gamma(n/\kappa+1)}\right) \\
&\le A' \exp(c|z_0|\,|\lambda|)\left|\frac{\zeta}{d}\right|^{(N+1)/\kappa}\sum_{n=N+1}^{+\infty}(d^{1/\kappa}B)^n\left(1+\frac{|\lambda|^{n/\kappa}}{\Gamma(n/\kappa+1)}\right) \\
&\le A_1 \exp(c(|z_0|+d)|\lambda|)\, d^{-N/\kappa}|\zeta|^{N/\kappa}.
\end{aligned}
$$

We have

$$\partial_\zeta^{-h/\kappa} V(z_0, \lambda, \zeta) = \sum_{n=0}^{+\infty} v_n(z_0, \lambda)\zeta^{(n+h)/\kappa}/\Gamma((n+h)/\kappa + 1). \tag{4.5}$$

Hence we have for $|\zeta| \geq d$

$$\begin{aligned}
|\partial_\zeta^{-h/\kappa} V_N^1(z_0, \lambda, \zeta)| &\leq A' \exp(c|z_0|\,|\lambda|)|\zeta|^{(N+h)/\kappa} \\
&\quad \times \sum_{n=0}^{N} B^n |\zeta|^{(n-N)/\kappa} \left(\frac{\Gamma(n/\kappa + 1)}{\Gamma((n+h)/\kappa + 1)} + \frac{|\lambda|^{n/\kappa}}{\Gamma((n+h)/\kappa + 1)} \right) \\
&\leq \frac{A' \exp(c|z_0|\,|\lambda|)|\zeta|^{(N+h)/\kappa}}{\Gamma(h/\kappa + 1)} \sum_{n=0}^{N} B^n |\zeta|^{(n-N)/\kappa} \left(1 + \frac{|\lambda|^{n/\kappa}}{\Gamma(n/\kappa + 1)} \right) \\
&\leq \frac{A_1 \exp(c(|z_0| + d)|\lambda|)\, d^{-N/\kappa} |\zeta|^{(N+h)/\kappa}}{\Gamma(h/\kappa + 1)}
\end{aligned}$$

and for $|\zeta| \leq d$

$$\begin{aligned}
|\partial_\zeta^{-h/\kappa} V_N^2(z_0, \lambda, \zeta)| &\leq A' \exp(c|z_0|\,|\lambda|)|\zeta|^{(N+h+1)/\kappa} \\
&\quad \times \sum_{n=N+1}^{+\infty} B^n |\zeta|^{(n-N-1)/\kappa} \left(\frac{\Gamma(n/\kappa + 1)}{\Gamma((n+h)/\kappa + 1)} + \frac{|\lambda|^{n/\kappa}}{\Gamma((n+h)/\kappa + 1)} \right) \\
&\leq \frac{A' \exp(c|z_0|\,|\lambda|)|\zeta|^{(N+h+1)/\kappa}}{\Gamma(h/\kappa + 1)} \left(\sum_{n=N+1}^{+\infty} B^n |\zeta|^{(n-N-1)/\kappa} \left(1 + \frac{|\lambda|^{n/\kappa}}{\Gamma(n/\kappa + 1)} \right) \right) \\
&\leq \frac{A_1 \exp(c(|z_0| + d)|\lambda|)|\zeta|^{(N+h)/\kappa}}{d^{N/\kappa}\Gamma(h/\kappa + 1)}
\end{aligned}$$

and similarly

$$|\partial_\zeta^{-h/\kappa} V(z_0, \lambda, \zeta)| \leq \frac{A_1 \exp(c(|z_0| + d)|\lambda|)|\zeta|^{h/\kappa}}{\Gamma(h/\kappa + 1)}.$$

We have

$$(z_0\partial_0)^p (\kappa\zeta\partial_\zeta)^q (\partial_\zeta^{-h/\kappa} V(z_0, \lambda, \zeta)) = \sum_{n,l=0}^{+\infty} \frac{l^p (n+h)^q v_{l,n}(\lambda) z_0^l \zeta^{(n+h)/\kappa}}{\Gamma(l+1)\Gamma((n+h)/\kappa + 1)} \tag{4.6}$$

and by the same way as above

$$\begin{aligned}
&|(z_0\partial_0)^p (\kappa\zeta\partial_\zeta)^q (\partial_\zeta^{-h/\kappa} V(z_0, \lambda, \zeta))| \\
&\qquad \leq \sum_{n,l=0}^{+\infty} \frac{l^p (n+h)^q |v_{l,n}(\lambda)| |z_0|^l |\zeta|^{(n+h)/\kappa}}{\Gamma(l+1)\Gamma((n+h)/\kappa + 1)}
\end{aligned}$$

$$\leq A\sum_{n,l=0}^{+\infty} B^{l+n}\frac{l^p(n+h)^q(\Gamma(l+1)+|\lambda|^l)(\Gamma(n/\kappa+1)+|\lambda|^{n/\kappa})}{\Gamma(l+1)\Gamma((n+h)/\kappa+1)}|z_0|^l|\zeta|^{(n+h)/\kappa}$$

$$\leq A\left(\sum_{l=0}^{+\infty} B^l l^p|z_0|^l\left(1+\frac{|\lambda|^l}{\Gamma(l+1)}\right)\right)\left(\sum_{n=0}^{+\infty} B^n(n+h)^q|\zeta|^{(n+h)/\kappa}\left(1+\frac{|\lambda|^{n/\kappa}}{\Gamma((n+h)/\kappa+1)}\right)\right)$$

$$\leq A_2\exp(c'(|z_0|+d)|\lambda|).$$

Lemma 4.2. *Let* $V(z_0,\lambda,\zeta)=\sum_{n,l=0}^{+\infty}\frac{v_{l,n}(\lambda)z_0^l\zeta^{n/\kappa}}{\Gamma(l+1)\Gamma(n/\kappa+1)}$ *be the same as in Lemma 4.1. Let* $0<d_0<B^{-\kappa}$ *and* $\hat{e}\in\boldsymbol{Z}$ *and define for* $0<d<d_0$

$$\hat{V}_d(z_0,\lambda)=z_0^{-\hat{e}-\kappa}\int_0^d \exp(-\zeta z_0^{-\kappa})V(z_0,\lambda,\zeta)\,d\zeta. \tag{4.7}$$

Suppose $h\in\boldsymbol{Z}$ *with* $h\geq\hat{e}$, $l\in\boldsymbol{N}$ *and* $0<\theta<\pi/(2\kappa)$. *Let* $z_0\in\{|z_0|<r_0;|\arg z_0|<\theta\}$,

$$\begin{aligned}\hat{V}_{l,h}(d;z_0,\lambda) &:= (\vartheta+\lambda+1)^l z_0^h\hat{V}_d(z_0,\lambda)\\ &\quad - z_0^{-\kappa}\int_0^d \exp(-\zeta z_0^{-\kappa})(\kappa\zeta\partial_\zeta+\vartheta+\lambda+1)^l\partial_\zeta^{-(h-\hat{e})/\kappa}V(z_0,\lambda,\zeta)\,d\zeta\end{aligned} \tag{4.8}$$

and L be a fixed positive integer. Then there are positive constants $0<k_0<1$, c_0 *which are independent of* d,θ *and* h *and* $A'=A'(\theta,h)$ *such that for* $0\leq l\leq L$

$$|\hat{V}_{l,h}(d;z_0,\lambda)|\leq A'(1+|\lambda|)^l\exp(-k_0 d\cos(\kappa\theta)|z_0|^{-\kappa}+c_0(|z_0|+d)|\lambda|). \tag{4.9}$$

Proof. By replacing $h-\hat{e}$ by h, we may assume $\hat{e}=0$. Let $V_N^1(z_0,\lambda,\zeta)$ and $V_N^2(z_0,\lambda,\zeta)$ be those in the proof of Lemma 4.1. We show (4.9) by induction on l. We note

$$\int_0^{+\infty}\exp(-\zeta z_0^{-\kappa})(z_0^h V_N^1(z_0,\lambda,\zeta)-\partial_\zeta^{-h/\kappa}V_N^1(z_0,\lambda,\zeta))\,d\zeta=0.$$

Hence

$$\begin{aligned}&\hat{V}_{0,h}(d;z_0,\lambda)\\ &:= z_0^{h-\kappa}\int_0^d \exp(-\zeta z_0^{-\kappa})V(z_0,\lambda,\zeta)\,d\zeta - z_0^{-\kappa}\int_0^d\exp(-\zeta z_0^{-\kappa})\partial_\zeta^{-h/\kappa}V(z_0,\lambda,\zeta)\,d\zeta\\ &= -\overbrace{z_0^{h-\kappa}\int_d^{+\infty}\exp(-\zeta z_0^{-\kappa})V_N^1(z_0,\lambda,\zeta)\,d\zeta}^{I_1(z_0,\lambda)}+\overbrace{z_0^{-\kappa}\int_d^{+\infty}\exp(-\zeta z_0^{-\kappa})\partial_\zeta^{-h/\kappa}V_N^1(z_0,\lambda,\zeta)\,d\zeta}^{I_2(z_0,\lambda)}\\ &\quad+\underbrace{z_0^{h-\kappa}\int_0^d\exp(-\zeta z_0^{-\kappa})V_N^2(z_0,\lambda,\zeta)\,d\zeta}_{I_3(z_0,\lambda)}-\underbrace{z_0^{-\kappa}\int_0^d\exp(-\zeta z_0^{-\kappa})\partial_\zeta^{-h/\kappa}V_N^2(z_0,\lambda,\zeta)\,d\zeta}_{I_4(z_0,\lambda)}.\end{aligned}$$

We estimate $I_i(z_0,\lambda)$. Put $B_1=(d\cos\kappa\theta)^{1/\kappa}$, $A_2=A_2(\theta,h)$ means various constants depending on θ and h, and C means various absolute constants. We have from Lemma 4.1

$$|I_1(z_0,\lambda)| \le A_1 d^{-N/\kappa}|z_0|^{h-\kappa}\exp(c(|z_0|+d)|\lambda|)\int_d^{+\infty}\exp(-\cos(\kappa\theta)|z_0|^{-\kappa}\zeta)\zeta^{N/\kappa}\,d\zeta$$
$$\le A_2 B_1^{-N}\exp(c(|z_0|+d)|\lambda|)|z_0|^{h+N}\Gamma(N/\kappa+1).$$

By the same method we have $|I_2(z_0,\lambda)| \le A_2 B_1^{-N}\exp(c(|z_0|+d)|\lambda|)|z_0|^{h+N}\cdot \Gamma((N+h)/\kappa+1)/\Gamma(h/\kappa+1)$. As for $I_3(z_0,\lambda)$ and $I_4(z_0,\lambda)$ we have

$$|I_3(z_0,\lambda)| \le A_1 d^{-N/\kappa}\exp(c(|z_0|+d)|\lambda|)|z_0|^{h-\kappa}\int_0^d \exp(-\cos(\kappa\theta)|z_0|^{-\kappa}\zeta)\zeta^{N/\kappa}\,d\zeta$$
$$\le A_2 B_1^{-N}|z_0|^{h+N}\exp(c(|z_0|+d)|\lambda|)\Gamma(N/\kappa+1)$$

and $|I_4(z_0,\lambda)| \le A_2 B_1^{-N}\exp(c(|z_0|+d)|\lambda|)|z_0|^{h+N}\Gamma((N+h)/\kappa+1)/\Gamma(h/\kappa+1)$.

Hence

$$|\hat{V}_{0,h}(d;z_0,\lambda)| \le \sum_{i=1}^{4}|I_i(z_0,\lambda)|$$
$$\le 4A_2|z_0|^h\left|\frac{z_0}{B_1}\right|^N\exp(c(|z_0|+d)|\lambda|)\frac{\Gamma((N+h)/\kappa+1)}{\Gamma(h/\kappa+1)}$$
$$\le 4A_2|z_0|^h C^{h+N}\left|\frac{z_0}{B_1}\right|^N\exp(c(|z_0|+d)|\lambda|)\Gamma(N/\kappa+1)$$

holds for all $N\in\boldsymbol{N}$. This implies

$$\left|\frac{\hat{V}_{0,h}(d;z_0,\lambda)}{(Cz_0)^h\exp(c(|z_0|+d)|\lambda|)}\right| \le 4A_2\left|\frac{Cz_0}{B_1}\right|^N N^{N/\kappa} \tag{4.10}$$

for all $N\in\boldsymbol{N}$. The left hand side of (4.10) does not depend on N. Let $e^{-1}(1+N)^{-1/\kappa}\le C|z_0|/B_1\le e^{-1}N^{-1/\kappa}$, where recall $B_1=(d\cos\kappa\theta)^{1/\kappa}$. Then there is a constant $0<k'<1$ such that

$$\left(\frac{C|z_0|}{B_1}\right)^N N^{N/\kappa}\le e^{-N}=e^{1-(1+N)}\le C\exp(-k'd\cos(\kappa\theta)|z_0|^{-\kappa}). \tag{4.11}$$

So it follows from (4.10) and (4.11) that there is a constant $A'=A'(\theta,h)$ such that

$$|\hat{V}_{0,h}(d;z_0,\lambda)|\le A'\exp(-k'd\cos(\kappa\theta)|z_0|^{-\kappa}+c(|z_0|+d)|\lambda|) \tag{4.12}$$

and (4.9) holds for $l=0$. Suppose

$$|\hat{V}_{l-1,h}(d;z_0,\lambda)|\le A'(1+|\lambda|)^{l-1}\exp(-k'd\cos(\kappa\theta)|z_0|^{-\kappa}+c'(|z_0|+d)|\lambda|). \tag{4.13}$$

We have

$$(\vartheta+\lambda+1)^l z_0^h \hat{V}_d(z_0,\lambda) = (\vartheta+\lambda+1)(\vartheta+\lambda+1)^{l-1} z_0^h \hat{V}_d(z_0,\lambda)$$
$$= (\vartheta+\lambda+1)\left\{z_0^{-\kappa}\int_0^d \exp(-\zeta z_0^{-\kappa})(\kappa\zeta\partial_\zeta+\vartheta+\lambda+1)^{l-1}\partial_\zeta^{-h/\kappa}V(z_0,\lambda,\zeta)\,d\zeta\right\}$$
$$+ (\vartheta+\lambda+1)\hat{V}_{l-1,h}(d;z_0,\lambda)$$

and

$$(\vartheta+\lambda+1)\left\{z_0^{-\kappa}\int_0^d \exp(-\zeta z_0^{-\kappa})(\kappa\zeta\partial_\zeta+\vartheta+\lambda+1)^{l-1}\partial_\zeta^{-h/\kappa}V(z_0,\lambda,\zeta)\,d\zeta\right\}$$
$$= z_0^{-\kappa}\int_0^d \exp(-\zeta z_0^{-\kappa})(\vartheta+\lambda+1-\kappa)(\kappa\zeta\partial_\zeta+\vartheta+\lambda+1)^{l-1}\partial_\zeta^{-h/\kappa}V(z_0,\lambda,\zeta)\,d\zeta$$
$$+ z_0^{-\kappa}\int_0^d \exp(-\zeta z_0^{-\kappa})\kappa z_0^{-\kappa}\zeta(\kappa\zeta\partial_\zeta+\vartheta+\lambda+1)^{l-1}\partial_\zeta^{-h/\kappa}V(z_0,\lambda,\zeta)\,d\zeta$$

by integration by parts

$$= z_0^{-\kappa}\int_0^d \exp(-\zeta z_0^{-\kappa})(\vartheta+\lambda+1-\kappa)(\kappa\zeta\partial_\zeta+\vartheta+\lambda+1)^{l-1}\partial_\zeta^{-h/\kappa}V(z_0,\lambda,\zeta)\,d\zeta$$
$$+ z_0^{-\kappa}\int_0^d \exp(-\zeta z_0^{-\kappa})\kappa\partial_\zeta\zeta(\kappa\zeta\partial_\zeta+\vartheta+\lambda+1)^{l-1}\partial_\zeta^{-h/\kappa}V(z_0,\lambda,\zeta))\,d\zeta$$
$$- \kappa z_0^{-\kappa}\, d\exp(-dz_0^{-\kappa})\{(\kappa\zeta\partial_\zeta+\vartheta+\lambda+1)^{l-1}\partial_\zeta^{-h/\kappa}V(z_0,\lambda,\zeta)\}|_{\zeta=d}$$
$$= z_0^{-\kappa}\int_0^d \exp(-\zeta z_0^{-\kappa})(\kappa\zeta\partial_\zeta+\vartheta+\lambda+1)^{l}\partial_\zeta^{-h/\kappa}V(z_0,\lambda,\zeta)\,d\zeta$$
$$- \kappa z_0^{-\kappa}\, d\exp(-dz_0^{-\kappa})\{(\kappa\zeta\partial_\zeta+\vartheta+\lambda+1)^{l-1}\partial_\zeta^{-h/\kappa}V(z_0,\lambda,\zeta)\}|_{\zeta=d}.$$

Hence

$$\hat{V}_{l,h}(d;z_0,\lambda) = (\vartheta+\lambda+1)\hat{V}_{l-1,h}(d;z_0,\lambda) - \kappa z_0^{-\kappa}\, d\exp(-dz_0^{-\kappa})\{(\kappa\zeta\partial_\zeta+\vartheta+\lambda+1)^{l-1}\partial_\zeta^{-h/\kappa}V(z_0,\lambda,\zeta)\}|_{\zeta=d}. \tag{4.14}$$

It follows from (4.4) in Lemma 4.1 that

$$|\kappa z_0^{-\kappa}\, d\exp(-dz_0^{-\kappa})\{(\kappa\zeta\partial_\zeta+\vartheta+\lambda+1)^{l-1}\partial_\zeta^{-h/\kappa}V(z_0,\lambda,\zeta)\}|_{\zeta=d}$$
$$\le A'(1+|\lambda|)^{l-1}\exp(-k'd\cos(\kappa\theta)|z_0|^{-\kappa}+c'(|z_0|+d)|\lambda|)$$

and from the inductive hypothesis (4.13) and Cauchy's integral formula that

$$|(\vartheta+\lambda+1)\hat{V}_{l-1,h}(d;z_0,\lambda)| \le A'(1+|\lambda|)^{l}\exp(-k''d\cos(\kappa\theta)|z_0|^{-\kappa}+c''(|z_0|+d)|\lambda|)$$

for $0<k''\le k'$ and $c''\ge c'$. Thus there are positive constants $0<k_0\le k''$ and $c_0\ge c''$ such that (4.9) holds for finite many l $(0\le l\le L)$.

PROOF OF THEOREM 2.3. Recall

$$K_\delta(z,w',\lambda) = z_0^{-e_*-\gamma^*}\int_0^\delta \exp(-\zeta z_0^{-\gamma^*})\,U(z,w',\lambda,\zeta)\,d\zeta. \tag{4.15}$$

We note by Proposition 2.2 that

$$|U(z,w',\lambda,\zeta)| \le C\exp(c(|z_0|+|\zeta|)|\lambda|), \tag{4.16}$$

from which the statements (1) and (2) in Theorem 2.3 follow if $c \le c_0$. We proceed to show (3). Let us calculate ${}^tP(z,\partial)(z_0^\lambda K_\delta(z,w',\lambda))$. We have

$$ {}^tP(z,\partial)(z_0^\lambda K_\delta(z,w',\lambda)) = z_0^\lambda\,{}^tP(\lambda;z,\partial)K_\delta(z,w',\lambda),$$

where

$$ {}^tP(\lambda;z,\partial) = \sum_{|\alpha|\le m}(-1)^{|\alpha|}(\vartheta+\lambda+1)^{\alpha_0}z_0^{e_\alpha}(\partial^{\alpha'}b_\alpha(z)\cdot). \tag{4.17}$$

We can write

$$\begin{aligned}(\vartheta&+\lambda+1)^{\alpha_0}z_0^{e_\alpha}\partial^{\alpha'}(b_\alpha(z)K_\delta(z,w',\lambda))\\ &= z_0^{-\gamma^*}\int_0^\delta \exp(-\zeta z_0^{-\gamma^*})(\gamma^*\zeta\partial_\zeta+\vartheta+\lambda+1)^{\alpha_0}\partial_\zeta^{-(e_\alpha-e_*)/\gamma^*}\partial^{\alpha'}(b_\alpha(z)U(z,w',\lambda,\zeta))\,d\zeta\\ &\quad + K^R_{\delta,\alpha}(z,w',\lambda).\end{aligned}$$

It follows from Lemma 4.2, by putting $\hat{e}=e_*$, $\kappa=\gamma^*$, $d=\delta$, $l=\alpha_0$, $h=e_\alpha$ and $V(z_0,\lambda,\zeta)=\partial^{\alpha'}(b_\alpha(z)U(z,w',\lambda,\zeta))$, that if $|\arg z_0|<\theta<\pi/2\gamma^*$,

$$|K^R_{\delta,\alpha}(z,w',\lambda)| \le A(1+|\lambda|)^m\exp(-k_0\delta(\cos\gamma^*\theta)|z_0|^{-\gamma^*}+c_0(|z_0|+\delta)|\lambda|)$$

for a constant $A=A(\theta)$. Therefore we have

$$\begin{aligned} {}^tP(z,\partial)(z_0^\lambda K_\delta(z,w',\lambda)) &= z_0^\lambda\,{}^tP(\lambda;z,\partial)K_\delta(z,w',\lambda)\\ &= z_0^{-\gamma^*+\lambda}\int_0^\delta \exp(-\zeta z_0^{-\gamma^*})\,{}^t\mathscr{P}(z,\lambda,\zeta,\partial_z,\partial_\zeta)U(z,w',\lambda,\zeta)\,d\zeta\\ &\quad + z_0^\lambda\sum_\alpha K^R_{\delta,\alpha}(z,w',\lambda).\end{aligned} \tag{4.18}$$

Since ${}^t\mathscr{P}(z,\lambda,\zeta,\partial_z,\partial_\zeta)U(z,w',\lambda,\zeta)=(2\pi i)^{-d}\prod_{j=1}^d(z_j-w_j)^{-1}$ and $z_0^{-\gamma^*}\int_0^\delta\exp(-\zeta z_0^{-\gamma^*})\,d\zeta = 1-\exp(-\delta z_0^{-\gamma^*})$, we have

$$z_0^\lambda\,{}^tP(\lambda;z,\partial)K_\delta(z,w',\lambda) = \frac{z_0^\lambda}{(2\pi i)^d\prod_{j=1}^d(z_j-w_j)} + z_0^\lambda K_\delta^R(z,w',\lambda), \tag{4.19}$$

where $K_\delta^R = (\sum_\alpha K^R_{\delta,\alpha}(z,w',\lambda)-\exp(-\delta z_0^{-\gamma^*}))$. It follows from the estimates of $K^R_{\delta,\alpha}$ given above that if $|\arg z_0|<\theta<\pi/2\gamma^*$, there is a constant $A=A(\theta)$ such that

$$(4.20) \qquad |K_\delta^R(z, w', \lambda)| \le A(1+|\lambda|)^m \exp(-k_0\delta(\cos\gamma^*\theta)|z_0|^{-\gamma^*} + c_0(|z_0|+\delta)|\lambda|).$$

Thus we complete the proof of Theorem 2.3.

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