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Saturation of fundamental ideals on $\mathcal{P}_\kappa\lambda$

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§0. Preliminaries.

In this paper we will examine how large is the saturation number of familiar ideals on $\mathcal{P}_\kappa\lambda$. It is known that the nonstationary ideal, $\mathrm{NS}_{\kappa\lambda}$ is not λ-saturated and the statement "$\mathrm{NS}_{\kappa\lambda}$ is λ^+-saturated" is a large cardinal hypothesis. On the other hand few results are known on the saturation of other familiar ideals except Johnson's result "the minimal ideal on $\mathcal{P}_\kappa\lambda$, $\mathrm{I}_{\kappa\lambda}$ is not λ^+-saturated", which naturally generalizes that I_κ is not κ^+-saturated. We will improve this in Section 1. Sections 2 and 3 are devoted to extend Matsubara's results on $\mathrm{NS}_{\kappa\lambda}$ to $\mathrm{WNS}_{\kappa\lambda}$, the minimal strongly normal ideal on $\mathcal{P}_\kappa\lambda$.

We work in ZFC and most of our notations are standard. The image of X under f is denoted by $f[X]$, i.e., $f[X]=\{y: y=f(x) \text{ for some } x\in X\}$. Throughout this paper, $\kappa\leqq\lambda$ are uncountable cardinals and κ is regular. For such a pair (κ, λ), $\mathcal{P}_\kappa\lambda=\{x\subset\lambda: |x|<\kappa\}$. More generally, $\mathcal{P}_\gamma A=\{x\subset A: |x|<|\gamma|\}$ for any set A and ordinal γ.

DEFINITION. I is an *ideal* on $\mathcal{P}_\kappa\lambda$ if I is a collection of subsets of $\mathcal{P}_\kappa\lambda$ such that

(i) $\phi\in I$ and $\mathcal{P}_\kappa\lambda\notin I$.

(ii) If $X, Y\subset\mathcal{P}_\kappa\lambda$, $X\in I$ and $Y\subset X$, then $Y\in I$.

(iii) If $X\in I$ and $Y\in I$, then $X\cup Y\in I$.

An ideal I on $\mathcal{P}_\kappa\lambda$ is *κ-complete* if I is closed under union of κ many members.

An ideal I on $\mathcal{P}_\kappa\lambda$ is *fine* if for each $\alpha<\lambda$, $\{x\in\mathcal{P}_\kappa\lambda: \alpha\notin x\}\in I$. For the sake of convenience, throughout this paper, by 'ideal' we mean 'fine κ-complete ideal'.

The diagonal union of $\{X_\alpha: \alpha<\lambda\}$ is defined by:

$$\nabla_{\alpha<\lambda}X_\alpha = \{x \in \mathcal{P}_\kappa\lambda:\ x \in X_\alpha \quad \text{for some } \alpha \in x\}.$$

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An ideal I on $\mathcal{P}_\kappa\lambda$ is *normal* iff $\nabla_{\alpha<\lambda}\{X_\alpha:\ \alpha<\lambda\}\in I$ for any $\{X_\alpha:\ \alpha<\lambda\}\subset I$.

We define $\nabla_< X_a$ for $\{X_a:\ a\in\mathcal{P}_\kappa\lambda\}$ by:

$$\nabla_< X_a$$

$$=\{x\in\mathcal{P}_\kappa\lambda:\ x\in X_a \text{ for some } a\in\mathcal{P}_\kappa\lambda \text{ such that } a\subset x \text{ and } |a|<|x\cap\kappa|\}.$$

An ideal I on $\mathcal{P}_\kappa\lambda$ is *strongly normal* if $\nabla_<\{X_a:\ a\in\mathcal{P}_\kappa\lambda\}\in I$ for any $\{X_a:\ a\in\mathcal{P}_\kappa\lambda\}\subset I$.

A filter $\mathcal{F}$ on $\mathcal{P}_\kappa\lambda$ and an ideal I on $\mathcal{P}_\kappa\lambda$ are *dual* to each other if the following holds:

$$X\in\mathcal{F} \quad \text{iff} \quad \mathcal{P}_\kappa\lambda-X\in I \quad \text{for every} \quad X\subset\mathcal{P}_\kappa\lambda.$$

The dual filter of I will be denoted by I^* and each member of I^* is called *I-measure one*. Let $I^+=\mathcal{P}\mathcal{P}_\kappa\lambda-I=\{X:\ X\notin I\}$ and $X\in I^+$ is called *I-positive.*

For an $x\in\mathcal{P}_\kappa\lambda$, let $\hat{x}=\{y\in\mathcal{P}_\kappa\lambda:\ x\subset y\}$, and $X\subset\mathcal{P}_\kappa\lambda$ is *unbounded* iff $X\cap\hat{x}\neq\phi$ for all $x\in\mathcal{P}_\kappa\lambda$.

Let $\mathrm{I}_{\kappa\lambda}=\{X\subset\mathcal{P}_\kappa\lambda:\ X$ is not unbounded$\}$. $\mathrm{I}_{\kappa\lambda}$ is the minimal ideal on $\mathcal{P}_\kappa\lambda$, and $\hat{x}\in\mathrm{I}_{\kappa\lambda}{}^*$ for any $x\in\mathcal{P}_\kappa\lambda$.

$X\subset\mathcal{P}_\kappa\lambda$ is *closed* iff $\cup D\in X$ for any increasing $\subset$-sequence $D\subset X$ such that $|D|<\kappa$. $C\subset\mathcal{P}_\kappa\lambda$ is said to be a *cub* iff it is closed and unbounded. X is *stationary* iff $X\cap C\neq\phi$ for every cub C.

$\mathrm{NS}_{\kappa\lambda}=\{X\subset\mathcal{P}_\kappa\lambda:\ X$ is not stationary$\}$. So $\mathrm{NS}_{\kappa\lambda}$ is the dual ideal of the cub filter on $\mathcal{P}_\kappa\lambda$ generated by cub sets.

$X\subset\mathcal{P}_\kappa\lambda$ is *strongly closed* iff $\cup D\in X$ for all $D\subset X$ with $|D|<\kappa$ and we call $C\subset\mathcal{P}_\kappa\lambda$ a *strong cub* iff it is strongly closed and unbounded.

$\mathrm{SNS}_{\kappa\lambda}=\{X\subset\mathcal{P}_\kappa\lambda:\ X\cap C=\phi$ for some strong cub set $C\subset\mathcal{P}_\kappa\lambda\}$.

$\mathrm{WNS}_{\kappa\lambda}=\nabla_< I_{\kappa\lambda}$. We say X is *strong stationary* if $X\in\mathrm{WNS}_{\kappa\lambda}{}^+$.

THEOREM (Jech [6], Carr, Levinski and Pelltier [4]). (i) $\mathrm{SNS}_{\kappa\lambda}=\nabla\mathrm{I}_{\kappa\lambda}$.

(ii) $\mathrm{NS}_{\kappa\lambda}$ *is the minimal normal ideal on* $\mathcal{P}_\kappa\lambda$ *and* $\mathrm{NS}_{\kappa\lambda}=\nabla\mathrm{SNS}_{\kappa\lambda}$.

(iii) $\mathrm{WNS}_{\kappa\lambda}$ *is the minimal strongly normal ideal on* $\mathcal{P}_\kappa\lambda$ *and it is proper iff* κ *is Mahlo or* $\kappa=\nu^+$ *with* $\nu^{<\nu}=\nu$.

(iv) $\nabla_<\mathrm{I}_{\kappa\lambda}=\nabla_<\mathrm{SNS}_{\kappa\lambda}=\nabla_<\mathrm{NS}_{\kappa\lambda}=\mathrm{WNS}_{\kappa\lambda}$.

DEFINITION. $X,\ Y\subset\mathcal{P}_\kappa\lambda$ are called *almost disjoint* with respect to I if $X\cap Y\in I$. I is *δ-saturated* for a cardinal δ if there is no pairwise almost disjoint family of size δ of I-positive subsets of $\mathcal{P}_\kappa\lambda$.

§ 1. The saturation of $I_{\kappa\lambda}$.

We prove the next theorem on the saturation number of $\mathrm{I}_{\kappa\lambda}$ which implies

Johnson's result, i.e., Corollary 1.5 below.

THEOREM 1.1. *If* $2^{<\kappa}\leqq\lambda$, *then* $I_{\kappa\lambda}$ *is not* $(\lambda^{<\kappa})^{+}$*-saturated.*

We need some lemmas for the proof. The author is grateful to Y. Matsubara for his suggestions.

LEMMA 1.2. $\mathcal{P}_{\kappa}\lambda$ *is a disjoint union of* $\lambda^{<\kappa}$ *many unbounded subsets. Hence* $I_{\kappa\lambda}$ *is not* $\lambda^{<\kappa}$*-saturated.*

PROOF. Fix an enumeration of $\mathcal{P}_{\kappa}\lambda$, say $\{s_{\alpha}: \alpha<\lambda^{<\kappa}\}$. Inductively we form a disjoint family $\{T_{\alpha}: \alpha<\lambda^{<\kappa}\}$ such that each $T_{\alpha}=\{t_{\alpha}(\xi): \xi<\alpha+1\}$. It is possible since $|\bigcup_{\beta<\alpha}T_{\beta}|<\lambda^{<\kappa}$ and $|\hat{s_{\alpha}}|=\lambda^{<\kappa}$ for any $\alpha<\lambda^{<\kappa}$.

Then put $X_{\xi}=\{t_{\alpha}(\xi): \xi\leqq\alpha<\lambda^{<\kappa}\}$. For each α and $\xi<\lambda^{<\kappa}$, there is a β such that $\xi\leqq\beta$ and $s_{\alpha}\subset s_{\beta}$. Since $t_{\beta}(\xi)\in T_{\beta}$, $s_{\beta}\subset t_{\beta}(\xi)\in X_{\xi}$. So, X_{ξ} is unbounded. By our construction it is clear that $\{X_{\xi}: \xi<\lambda^{<\kappa}\}$ is pairwise disjoint. □

Since $\mathcal{P}_{\kappa}\lambda=\bigcup_{x\in X}\mathcal{P}(x)$ for any unbounded $X\subset\mathcal{P}_{\kappa}\lambda$, we have;

LEMMA 1.3. *If* $2^{<\kappa}\leqq\lambda$, *then* $|X|=\lambda^{<\kappa}$ *for any* $X\in I_{\kappa\lambda}{}^{+}$.

LEMMA 1.4. *For any cardinal* δ, *there is a* $\{g_{\alpha}\in{}^{\delta}\delta: \alpha<\delta^{+}\}$ *such that* sup $\{\sigma<\delta: g_{\alpha}(\sigma)=g_{\beta}(\sigma)\}<\delta$ *whenever* $\alpha<\beta<\delta^{+}$.

PROOF OF THEOREM 1.1. Set $Y_{\xi}=X_{\xi}\cap\hat{s_{\xi}}$ with $\{X_{\xi}: \xi<\lambda^{<\kappa}\}$ and $\{s_{\alpha}: \alpha<\lambda^{<\kappa}\}$ in Lemma 1.2. Since $Y_{\xi}\in I_{\kappa\lambda}{}^{+}$, $|Y_{\xi}|=\lambda^{<\kappa}$ by Lemma 1.3 and our hypothesis.

Let $\{u_{\xi}(\sigma): \sigma<\lambda^{<\kappa}\}$ be an enumeration of Y_{ξ} and $\{g_{\alpha}: \alpha<(\lambda^{<\kappa})^{+}\}$ be a family of functions in Lemma 1.4 where we take $\lambda^{<\kappa}$ for δ.

Set $A_{\alpha}=\{u_{\xi}(g_{\alpha}(\xi)): \xi<\lambda^{<\kappa}\}$. Since $u_{\xi}(g_{\alpha}(\xi))\in A_{\alpha}\cap\hat{s_{\xi}}$ for every $\xi<\lambda^{<\kappa}$, A_{α} is unbounded for any $(\alpha<\lambda^{<\kappa})^{+}$. Since $\{Y_{\xi}: \xi<\lambda^{<\kappa}\}$ is pairwise disjoint, $|A_{\alpha}\cap A_{\beta}|=|\{\xi: g_{\alpha}(\xi)=g_{\beta}(\xi)\}|<\lambda^{<\kappa}$ whenever $\alpha<\beta<(\lambda^{<\kappa})^{+}$. Then, Lemma 1.3 tells us that $A_{\alpha}\cap A_{\beta}\in I_{\kappa\lambda}$.

Now we have shown that $\{A_{\alpha}: \alpha<(\lambda^{<\kappa})^{+}\}$ is an almost disjoint family. Hence $I_{\kappa\lambda}$ is not $(\lambda^{<\kappa})^{+}$-saturated. □

If $2^{<\kappa}>\lambda$, then $\lambda^{<\kappa}\geqq 2^{<\kappa}\geqq\lambda^{+}$. Combining Theorem 1.1 and Lemma 1.2, we get Johnson's Theorem as a corollary.

COROLLARY 1.5 (Johnson [7, Theorem 1.6]). $I_{\kappa\lambda}$ *is not* λ^{+}*-saturated.*

COROLLARY 1.6. *If* $I_{\kappa\lambda}$ *is* λ^{++}*-saturated, then* $\lambda^{<\kappa}=\max(2^{<\kappa}, \lambda)$.

Note that we can blow up 2^{ω} much larger than λ by a c.c.c. forcing and in the extended universe $I_{\kappa\lambda}$ has the same saturation number as in the ground model.

2. Embedding $\mathcal{P}_\kappa\lambda$ into larger sets.

In this section we embed $\mathcal{P}_\kappa\lambda$ into $\mathcal{P}_\kappa\delta$ with $\delta\geqq\lambda$ and get some facts about ideals on $\mathcal{P}_\kappa\lambda$.

Let I be an ideal on $\mathcal{P}_\kappa\lambda$, $\mathcal{G}_x\subset\mathcal{P}(x)$ for each $x\in\mathcal{P}_\kappa\lambda$ and $\{u_\alpha: \alpha<\delta\}$ an enumeration of $\bigcup_{x\in\mathcal{P}_\kappa\lambda}\mathcal{G}_x$ satisfying the following conditions.

(i) $|\mathcal{G}_x|<\kappa$ for any $x\in\mathcal{P}_\kappa\lambda$,

(ii) $\{x\in\mathcal{P}_\kappa\lambda: u_\alpha\in\mathcal{G}_x\}\in I^*$ for every $\alpha<\delta$.

Define $f: \mathcal{P}_\kappa\lambda\to\mathcal{P}_\kappa\delta$ and $J\subset\mathcal{P}\mathcal{P}_\kappa\delta$ by

$$f(x)=\{\alpha<\delta: u_\alpha\in\mathcal{G}_x\} \quad\text{and}\quad J=\{X\subset\mathcal{P}_\kappa\delta: f^{-1}[X]\in I\}=f_*(I).$$

LEMMA 2.1. *J is an ideal on $\mathcal{P}_\kappa\delta$.*

Matsubara [**11**] proved "There is no λ-saturated ideal on $\mathcal{P}_\kappa\lambda$ in case κ is a successor cardinal". Using the above translation from I to J, we slightly improve this in the next theorem.

THEOREM 2.2. *Assume that $\kappa=\nu^+$ and $\nu^{<\mu}=\nu$. Then, there is no $\lambda^{<\mu}$-saturated ideal on $\mathcal{P}_\kappa\lambda$.*

PROOF. Let $\mathcal{G}_x=\mathcal{P}_\mu x$. Then, $\bigcup_{x\in\mathcal{P}_\kappa\lambda}\mathcal{G}_x=\mathcal{P}_\mu\lambda$ and $\delta=\lambda^{<\mu}$. So, (i) and (ii) above are clearly satisfied. Hence we have an ideal J on $\mathcal{P}_\kappa\delta$ which is not δ-saturated [**11**, Theorem 12]. Since $J=f_*(I)$, neither is I δ-saturated. □

Now we turn to strongly normal ideals. We have two cases.

Let $\kappa=\nu^+$ and $\nu^{<\nu}=\nu$ first. By the preceding theorem, we have

COROLLARY 2.3. *If $\kappa=\nu^+$ and $\mathcal{P}_\kappa\lambda$ bears a strongly normal ideal, then no ideal on $\mathcal{P}_\kappa\lambda$ is $\lambda^{<\nu}$-saturated.*

In other words, $\lambda^{<\nu}<\eta$ if there is a η-saturated ideal on $\mathcal{P}_\kappa\lambda$.

Second, let κ be Mahlo. We choose $\{y\subset x: |y|<|x\cap\kappa|\}$ as $\mathcal{G}_x$ for each $x\in\mathcal{P}_\kappa\lambda$. Hence $\bigcup_{x\in\mathcal{P}_\kappa\lambda}\mathcal{G}_x=\mathcal{P}_\kappa\lambda$ and $\delta=\lambda^{<\kappa}$.

LEMMA 2.4. (i) *Let $X_\alpha\subset\mathcal{P}_\kappa\delta$ and $Y_{u_\alpha}=f^{-1}[X_\alpha]\subset\mathcal{P}_\kappa\lambda$ for all $\alpha<\delta$. Put $X=\nabla X_\alpha$ and $Y=\nabla_< Y_{u_\alpha}$. Then $Y=f^{-1}[X]$.*

(ii) *I is strongly normal iff J is normal iff J is strongly normal.*

PROOF. (i) For any $x\in\mathcal{P}_\kappa\lambda$, $x\in Y$ iff $x\in Y_{u_\alpha}$ for some $u_\alpha\in\mathcal{G}_x$ iff $f(x)\in X_\alpha$ for some $\alpha\in f(x)$ iff $f(x)\in\nabla X_\alpha$ iff $x\in f^{-1}[X]$.

(ii) The first equivalence is clear since $X\in J$ iff $f^{-1}[X]\in I$ by our definition. So, we only have to show that J is strongly normal if I is strongly normal.

Let I be strongly normal, $X\in J^+$, $g: X\to\mathcal{P}_\kappa\delta$ such that $g(x)\subset x$ and $|g(x)|$

$<|x\cap\kappa|$ for any $x\in X$, and $Y=f^{-1}[X]$. Define $h: Y\to\mathcal{P}\mathcal{P}_\kappa\lambda$ by: $h(y)=\{u_\alpha:\alpha\in g(f(y))\}$ for any $y\in Y$. Then, $h(y)\subset\mathcal{G}_y$ and $|h(y)|<|f(y)\cap\kappa|$.

If $\{y\in\mathcal{P}_\kappa\lambda$: there is an $\alpha_y\in y\cap\kappa-f(y)\cap\kappa\}\in \mathrm{WNS}_{\kappa\lambda}{}^+$, we have an α such that $A=\{y\in\mathcal{P}_\kappa\lambda: \alpha\notin f(y)\}\in\mathrm{WNS}_{\kappa\lambda}{}^+$ for some $\alpha<\kappa$. But $A\in \mathrm{I}_{\kappa\lambda}$ because $u_\alpha\notin\mathcal{G}_y$ for any $y\in A$.

If $\{y\in\mathcal{P}_\kappa\lambda: f(y)\cap\kappa-y\cap\kappa\neq\phi\}\in\mathrm{WNS}_{\kappa\lambda}{}^+$, there is a $\beta<\kappa$ such that $B=\{y\in\mathcal{P}_\kappa\lambda: u_\beta\in\mathcal{G}_y$ and $\beta\notin y\cap\kappa\}\in\mathrm{WNS}_{\kappa\lambda}{}^+$ since $\mathrm{WNS}_{\kappa\lambda}$ is strongly normal. But $B\in\mathrm{I}_{\kappa\lambda}$. Hence $\{y\in\mathcal{P}_\kappa\lambda: y\cap\kappa=f(y)\cap\kappa\}\in\mathrm{WNS}_{\kappa\lambda}{}^*\subset I^*$.

The reader should also note that the strong normality of $\mathrm{WNS}_{\kappa\lambda}$ and the inaccessibility of κ give us $\{y\in\mathcal{P}_\kappa\lambda: y\cap\kappa$ is inaccessible$\}\in\mathrm{WNS}_{\kappa\lambda}{}^*$.

Now let $Z=\{y\in Y: y\cap\kappa=f(y)\cap\kappa$ is inaccessible$\}\in I^+$ and $c_y=\cup h(y)$ for each $y\in Z$. Then, $c_y\in\mathcal{G}_y$ for any $y\in Z$. Since I is strongly normal, we can find a $c\in\mathcal{P}_\kappa\lambda$ such that $W=\{y\in Z: c_y=c\}\in I^+$. Note that $h(y)\subset\mathcal{P}(c)$ for any $y\in W$ and $|\mathcal{P}(c)|<\kappa$. Since I is κ-complete, we have a $u\in\mathcal{P}_\kappa\lambda$ such that $S=\{y\in W: h(y)=u\}\in I^+$. Hence $T=f[S]\in J^+$ and $g|T$ is constant.

So, J is also strongly normal. □

COROLLARY 2.5. *If* $f_*(I)\supset\mathrm{SNS}_{\kappa\delta}$, *then* $I\supset\mathrm{WNS}_{\kappa\lambda}$ *and* $f_*(I)\supset\mathrm{WNS}_{\kappa\delta}$. *Hence* $f_*(\mathrm{NS}_{\kappa\lambda})\not\supset\mathrm{SNS}_{\kappa\delta}$.

THEOREM 2.6. *Suppose that* $\lambda\leqq\eta$ *is regular,* κ *is Mahlo and there is a strongly normal* η*-saturated ideal on* $\mathcal{P}_\kappa\lambda$. *Then,* $\lambda^{<\kappa}\leqq\eta$.

PROOF. Let I be a strongly normal η-saturated ideal on $\mathcal{P}_\kappa\lambda$ and $\eta>\lambda$. Suppose contrary that $\delta=\lambda^{<\kappa}>\eta$. Then J is a normal η-saturated ideal on $\mathcal{P}_\kappa\delta$. $J|\eta=\{X|\eta: X\in J\}$ is also a normal η-saturated ideal on $\mathcal{P}_\kappa\eta$ where $X|\eta$ is the set $\{x\cap\eta: x\in X\}$. By [1, Corollary 2.4], $\eta^{<\kappa}=\eta$. But $\lambda^{<\kappa}\leqq\eta^{<\kappa}$. □

COROLLARY 2.7. *The saturation number of any strongly normal ideal on* $\mathcal{P}_\kappa\lambda$ *for a Mahlo* κ *does not lie between* λ *and* $\lambda^{<\kappa}$.

We did not use here the strong normality of $f_*(I)$. This idea was already used in [1] to prove;

THEOREM. *If* $\mathcal{P}_\kappa\lambda$ *carries a normal* λ*-saturated ideal and* $cf(\lambda)<\kappa$, *then,* $\lambda^{<\kappa}=\max(2^{<\kappa},\lambda^+)$. *If* $\mathcal{P}_\kappa\lambda$ *bears a normal* λ^+*-saturated ideal, then we have* $\lambda^{<\kappa}\leqq\max(2^{<\kappa},\lambda^+)$.

In this case $\delta=\lambda^+$. We are interested in case $\delta=\lambda^{<\kappa}$ and have some applications to $\mathcal{P}_\kappa\lambda$-combinatorics which we just mention in § 4. We also prove a stronger fact than Lemma 2.4, (ii) later.

§ 3. On $\mathrm{WNS}_{\kappa\lambda}$.

At the end of § 2, we have dealt with strongly normal ideals. We will make more detailed observation on the minimal strongly normal ideal $\mathrm{WNS}_{\kappa\lambda}$.

We already know $\mathrm{WNS}_{\kappa\lambda}$ is not $\lambda^{<\nu}$-saturated if $\kappa=\nu^+$. The question is whether it is $\lambda^{<\kappa}$-saturated or not. Although we know that $\mathrm{NS}_{\kappa\lambda}$ is not $\lambda^{<\kappa}$-saturated if κ is inaccessible (Matsubara, [**12**, Theorem 3]), the method used in § 2 is not available since $f_*(\mathrm{WNS}_{\kappa\lambda})\supset\mathrm{WNS}_{\kappa\delta}\supsetneqq\mathrm{NS}_{\kappa\delta}$ by Lemma 2.4. We trace Matsubara's argument to get analogous fact.

THEOREM 3.1. *If κ is Mahlo, then* $\mathrm{WNS}_{\kappa\lambda}$ *is not $\lambda^{<\kappa}$-saturated.*

We shall prove it by series of lemmas.

LEMMA 3.2. $S_{\kappa\lambda}=\{x\in\mathcal{P}_\kappa\lambda : |x\cap\kappa|=|x|\}\in\mathrm{WNS}_{\kappa\lambda}{}^+$. *Especially if κ is a successor,* $S_{\kappa\lambda}\in\mathrm{WNS}_{\kappa\lambda}{}^*$.

PROOF. Let κ be Mahlo. Note that $X\in\mathrm{WNS}_{\kappa\lambda}{}^*$ iff there is $f:\mathcal{P}_\kappa\lambda\to\mathcal{P}_\kappa\lambda$ such that $C_f=\{x\in\mathcal{P}_\kappa\lambda : f(y)\subset x$ for every $y\subset x$ with $|y|<|x\cap\kappa|\}\subset X$. Thus we only have to show that $S_{\kappa\lambda}\cap C_f\neq\phi$ for every $f:\mathcal{P}_\kappa\lambda\to\mathcal{P}_\kappa\lambda$.

Pick any $f:\mathcal{P}_\kappa\lambda\to\mathcal{P}_\kappa\lambda$ and define a sequence $\{x_\alpha : \alpha<\kappa\}\subset\mathcal{P}_\kappa\lambda$ such that $x_\alpha\subset x_{\alpha+1}\in C_f$ and $|x_\alpha|<|x_{\alpha+1}\cap\kappa|$ for any $\alpha<\kappa$, and $x_\alpha=\bigcup\{x_\beta : \beta<\alpha\}$ if α is a limit ordinal$<\kappa$. Next, define $g:\kappa\to\kappa$ by: $g(\alpha)=|x_\alpha|$ for all $\alpha<\kappa$.

There is an inaccessible $\eta<\kappa$ such that $g[\eta]\subset\eta$ since κ is Mahlo. Now $|x_\eta\cap\kappa|=|\bigcup\{x_\alpha\cap\kappa : \alpha<\eta\}|\geqq|\bigcup\{x_{\alpha+1}\cap\kappa : \alpha<\eta\}|\geqq|\bigcup\{x_\alpha : \alpha<\eta\}|=|x_\eta|$. So, $x_\eta\in S_{\kappa\lambda}$. Let $y\subset x_\eta$ and $|y|<|x_\eta\cap\kappa|$. Since $|x_\eta\cap\kappa|=|x_\eta|=\eta$ is regular, $y\subset x_{\alpha+1}\in C_f$ for some $\alpha<\eta$. Hence $f(y)\subset x_{\alpha+1}\subset x_\eta$. So, $x_\eta\in C_f$ and the proof is completed. □

Recall the definition of *non λ-Shelah ideal* $\mathrm{NSh}_{\kappa\lambda}$ by Carr [**2**]. $X\subset\mathcal{P}_\kappa\lambda$ has the *λ-Shelah property* iff for any $\{f_x : x\in X\}$ with $f_x : x\to x$ for any $x\in X$, there is an $f:\lambda\to\lambda$ such that $\{y\in X\cap\hat{x} : f_y|x=f|x\}\neq\phi$. $\mathrm{NSh}_{\kappa\lambda}=\{X\subset\mathcal{P}_\kappa\lambda : X$ does not have the λ-Shelah property$\}$.

The first proposition of the next corollary in case of a successor λ has already appeared in [**8**].

COROLLARY 3.3. (i) $\{x : |x\cap\kappa|<|x|\}\in\mathrm{NSh}_{\kappa\lambda}{}^*$.

(ii) $\mathrm{WNS}_{\kappa\lambda}\subsetneqq\mathrm{NSh}_{\kappa\lambda}$.

PROOF. (i) Suppose contrary that $X=\{x : |x\cap\kappa|=|x|\}\in\mathrm{NSh}_{\kappa\lambda}{}^+$. Let $f:x\to x\cap\kappa$ be bijective for each $x\in X$. By the λ-Shelah property of X, we have an $f:\lambda\to\kappa$ such that $\{x\in X : f|y=f_x|y\}\in\mathrm{I}_{\kappa\lambda}{}^+$ for any $y\in\mathcal{P}_\kappa\lambda$. Since each f_x is injective, f is also an injection from λ to κ. Contradiction. □

(ii) We only have to show $\mathrm{WNS}_{\kappa\lambda}\subset\mathrm{NSh}_{\kappa\lambda}$ by Lemma 3.2. It is known $\mathrm{NSh}_{\kappa\lambda}$ is strongly normal if $cf(\lambda)\geqq\kappa$. So we deal with the case $cf(\lambda)<\kappa$. Let $X\in\mathrm{NSh}_{\kappa\lambda}{}^{+}$ and $f: X\to\mathcal{P}_{\kappa}\lambda$ such that $f(x)\subset x$ and $|f(x)|<|x\cap\kappa|$ for any $x\in X$. Without loss of generality, we can assume $x\cap\kappa$ is inaccessible for any $x\in X$. Then, $X_1=\{x\in X\cap\hat{\gamma}: f(x)$ has the order type $\gamma\}\in\mathrm{NSh}_{\kappa\lambda}{}^{+}$ for some $\gamma<\kappa$. Let $g_x: \gamma\to f(x)$ be the order preserving map for each $x\in X_1$. There is a $g: \gamma\to\lambda$ such that $X_2=\{x\in X_1: g=g_x\}\in\mathrm{I}_{\kappa\lambda}{}^{+}$. $f(x)=g[\gamma]$ for every $x\in X_2$. Thus $X\in\mathrm{WNS}_{\kappa\lambda}{}^{+}$. □

DEFINITION. I has *the disjointing property* if for any almost disjoint family $\{X_\xi: \xi<\sigma\}$ there exists a disjoint family $\{Y_\xi: \xi<\sigma\}$ such that $(X_\xi-Y_\xi)\cup(Y_\xi-X_\xi)\in I$ for every $\xi<\sigma$.

LEMMA 3.4 (Foreman [**5**, Lemma 10]). *Any countably complete ideal with the disjointing property is precipitous.*

LEMMA 3.5. *Any strongly normal $(\lambda^{<\kappa})^{+}$-saturated ideal on $\mathcal{P}_\kappa\lambda$ with κ Mahlo has the disjointing property hence is precipitous.*

PROOF. Let $\{X_\alpha: \alpha<\lambda^{<\kappa}\}\subset I^{+}$ be an almost disjoint family, $\{s_\alpha: \alpha<\lambda^{<\kappa}\}$ be an enumeration of $\mathcal{P}_\kappa\lambda$. We may assume that $X_\alpha\subset\{y: s_\alpha\subset y,\ |s_\alpha|<|y\cap\kappa|\}$. For any $\alpha<\beta<\lambda^{<\kappa}$, there is a $C_{\alpha,\beta}\in\mathrm{WNS}_{\kappa\lambda}{}^{*}$ such that $C_{\alpha,\beta}\cap X_\alpha\cap X_\beta=\phi$.

Set $C=\{x\in\mathcal{P}_\kappa\lambda: x\in C_{\alpha,\beta}$ whenever $s_\alpha, s_\beta\subset x$ and $|s_\alpha|, |s_\beta|<|x\cap\kappa|\}$ which is in I^{*} since I is strongly normal. We can show that $\{X_\alpha\cap C: \alpha<\lambda^{<\kappa}\}$ is a desired disjoint family. □

LEMMA 3.6. *Suppose that $\delta=\lambda^{<\kappa}$, I is a strongly normal δ-saturated ideal on $\mathcal{P}_\kappa\lambda$, G is a generic filter of $\mathcal{P}\mathcal{P}_\kappa\lambda-I$ and $j: V\to M\cong Ult(V, G)$ is the canonical elementary embedding into the transitive collapse of $Ult(V, G)$. Then $V[G]\models\delta$ is a cardinal, and ${}^{\delta}M\cap V[G]\subset M$.*

PROOF. We only show that ${}^{\delta}M\cap V[G]\subset M$. The other part is clear. Let $\{s_\alpha: \alpha<\delta\}$ and $\langle\tau_\alpha: \alpha<\delta\rangle$ be an enumeration of $\mathcal{P}_\kappa\lambda$ in V and a term for a δ-sequence in $V[G]$ of elements of M respectively. Since I has the disjointing property by Lemma 3.5, we can find, in V, a sequence of functions $\langle f_\alpha: \alpha<\delta\rangle$ such that $\|[f_\alpha]_M=\tau_\alpha\|^{(B)}=\mathbf{1}$, where $[\ ]_M$ and B denote the equivalence class in the ultrapower and corresponding Boolean algebra to I.

Define $g: \mathcal{P}_\kappa\lambda\to V$ by $g(x)=\{f_\alpha(x): s_\alpha\subset x$ and $|s_\alpha|<|x\cap\kappa|\}$. Since I is strongly normal, we have $[g]_M=\{\tau_\alpha{}^{V[G]}: \alpha<\delta\}$. □

LEMMA 3.7. $\{x\in\mathcal{P}_\kappa\lambda: |x|^{<|x\cap\kappa|}=|x|\}\in\mathrm{WNS}_{\kappa\lambda}{}^{+}$.

PROOF. If $\kappa=\nu^{+}$, $\nu^{<\nu}=\nu$ and $\{x: |x|=|x\cap\kappa|=\nu\}\in\mathrm{I}_{\kappa\lambda}{}^{*}\subset\mathrm{WNS}_{\kappa\lambda}{}^{*}$. If κ is Mahlo, we have $\{x: x\cap\kappa$ is strongly inaccessible$\}\in\mathrm{WNS}_{\kappa\lambda}{}^{*}$. Combining these with Lemma 3.2, we get the conclusion. □

LEMMA 3.8. *Let* κ *be Mahlo.* $\{x \in \mathcal{P}_\kappa\lambda : |x|^{<|x\cap\kappa|} = |x|\}$ *splits into* $\lambda^{<\kappa}$ *many disjoint strong stationary subsets.*

PROOF. If we observe the proof of the statement "*Every stationary subsets of* $S_{\kappa\lambda}$ *splits into* λ *many disjoint stationary subsets*" ([**11**], Lemma 14) and replace $NS_{\kappa\lambda}$ by $WNS_{\kappa\lambda}$, we get "*Every strong stationary subsets of* $S_{\kappa\lambda}$ *splits into* λ *many disjoint strong stationary subsets.*" So, $\{x \in \mathcal{P}_\kappa\lambda : |x|^{<|x\cap\kappa|} = |x|\}$ is a disjoint union of λ many strong stationary subsets.

Let $\lambda < \lambda^{<\kappa}$ and suppose contrary that $X = \{x : |x|^{<|x\cap\kappa|} = |x|\}$ is not splitted into $\lambda^{<\kappa}$ many disjoint strong stationary sets. Then, $I = WNS_{\kappa\lambda}|X$ is $\lambda^{<\kappa}$-saturated strongly normal hence precipitous by Lemma 3.5.

Let $j: V \to M \cong \mathrm{Ult}(V, G)$ be as in Lemma 3.6 and $\delta = \lambda^{<\kappa}$ in V. Note that two functions $\{\langle x, x\cap\kappa\rangle : x \in \mathcal{P}_\kappa\lambda\}$ and the identity represent κ and $j[\lambda]$ respectively in M. Hence $M \models |\lambda^{<\kappa}| = |\lambda|$. For each $a \in (\mathcal{P}_\kappa\lambda)^V$, define f_a by $f_a(x) = x \cap a$. Then $M \models$ "$[f_a] \subset j[\lambda]$ and $|[f_a]| \leqq |j(a)| = |a| < \kappa$". So, we have $M \models [f_a] \in \mathcal{P}_\kappa j[\lambda]$. Note that $[f_a] \neq [f_b]$ whenever $a \neq b$. Thus, we have $V[G] \models |(\lambda^{<\kappa})^V| \leqq \lambda$. Hence $\lambda^{<\kappa}$ is collapsed contradicting to that I is $\lambda^{<\kappa}$-saturated. □

Now Theorem 3.1 has been proved and we have a corollary which can be seen on the line of Matsubara's Theorem for stationary sets.

COROLLARY 3.9. *If* κ *is Mahlo, then* $\mathcal{P}_\kappa\lambda$ *splits into* $\lambda^{<\kappa}$ *many strong stationary subsets.*

Recall the next theorem for normal ideals.

THEOREM. *An normal ideal* I *on* $\mathcal{P}_\kappa\lambda$ *is* λ^+*-saturated iff any normal ideal extending* I *is of the form* $I|X$ *for some* $X \in I^+$.

Here is an analogue of it;

THEOREM 3.10. (i) *Let* κ *be Mahlo. A strongly normal ideal* I *on* $\mathcal{P}_\kappa\lambda$ *is* $(\lambda^{<\kappa})^+$*-saturated iff any strongly normal extension of* I *is of the form* $I|A$ *for some* $A \in I^+$.

(ii) *Let* $\kappa = \nu^+$ *and* I *a strongly normal ideal on* $\mathcal{P}_\kappa\lambda$. *Then,* I *is* $(\lambda^{<\nu})^+$*-saturated iff every strongly normal extension of* I *is of the form* $I|A$ *for some* $A \in I^+$.

(iii) *Let* $\lambda^{<\kappa} = \lambda$. *Then every normal ideal extending* $WNS_{\kappa\lambda}$ *is strongly normal. Thus every normal ideal extending* $WNS_{\kappa\lambda}$ *is* λ^+*-saturated if* $WNS_{\kappa\lambda}$ *is* λ^+*-saturated.*

PROOF. (i) Minor adjustment of the proof of the theorem for normal ideal is available.

First assume that I is $(\lambda^{<\kappa})^+$-saturated and $I\subset J$ is strongly normal. Let $\mathcal{W}=\{A_\alpha: \alpha<\eta\}\subset J-I$ be a maximal almost disjoint family for I. Fix an enumeration of $\mathcal{P}_\kappa\lambda$, $\{s_\alpha: \alpha<\lambda^{<\kappa}\}$ and set $A_\alpha=\phi$ if $\eta\leqq\alpha<\lambda^{<\kappa}$.

$A=\{x\in\mathcal{P}_\kappa\lambda: x\notin A_\alpha$ if $s_\alpha\subset x$ and $|s_\alpha|<|x\cap\kappa|\}\in J^*$ since J is strongly normal. $I|A\subset J$ is clear. Suppose $X\in J-I|A$. Then $X\cap A\in J-I$ and there is an A_α such that $X\cap A\cap A_\alpha\in I^+$. But $A\cap A_\alpha\cap\{x: s_\alpha\subset x$ and $|s_\alpha|<|x\cap\kappa|\}$ $=\phi$. Hence $J=I|A$.

For the converse, suppose that I is a strongly normal ideal on $\mathcal{P}_\kappa\lambda$ and $\{A_\alpha: \alpha<\mu\}$ be a maximal almost disjoint family with $\mu>\lambda^{<\kappa}$.

Define J by

$$X\in J \quad \text{iff} \quad |\{\alpha<\mu: X\cap A_\alpha\in I^*\}|\leqq\lambda^{<\kappa}.$$

Then J is a strongly normal ideal$\supset I$ and $J\neq I|A$ for any $A\in I^+$. □

(ii) can be proved similarly by enumerating $\mathcal{P}_\nu\lambda$.

(iii) $\mathrm{WNS}_{\kappa\lambda}=\mathrm{NS}_{\kappa\lambda}|B$ where $B=\{x: \varphi(y)\in x$ if $y\subset x$ and $|y|<|x\cap\kappa|\}$ for any bijection $\varphi: \mathcal{P}_\kappa\lambda\to\lambda$ if $\lambda^{<\kappa}=\lambda$. □

THEOREM 3.11. *If κ is Mahlo and there is a strongly normal ideal I on $\mathcal{P}_\kappa\lambda$ such that $\{x\in\mathcal{P}_\kappa\lambda: X\cap\mathcal{Q}_x\in\mathrm{WNS}_{|x\cap\kappa|x}{}^+\}\in I^*$ for any $X\in I^*$ where $\mathcal{Q}_x=\{y\subset x: |y|<|x\cap\kappa|\}$, then $\mathrm{WNS}_{\kappa\lambda}$ is not $(\lambda^{<\kappa})^+$-saturated.*

PROOF. Suppose contrary that there is an ideal I on $\mathcal{P}_\kappa\lambda$ which satisfies the condition in the statement and $\mathrm{WNS}_{\kappa\lambda}$ is $(\lambda^{<\kappa})^+$-saturated. By Theorem 3.10, $I=\mathrm{WNS}_{\kappa\lambda}|A$ for some $A\in\mathrm{WNS}_{\kappa\lambda}{}^+$. Then we have a $B\in\mathrm{WNS}_{\kappa\lambda}{}^*$ such that $B\cap A\subset\{x: A\cap\mathcal{Q}_x$ is strong stationary in $\mathcal{Q}_x\}\in I^*$.

Since $B\in\mathrm{WNS}_{\kappa\lambda}{}^*$, we can assume that for some function $g: \mathcal{P}_\kappa\lambda\to\mathcal{P}_\kappa\lambda$, $B=\{x: g(y)\subset x$ for all $y\in\mathcal{Q}_x\}$.

By the strongnormality, $C=\{x: |g(y)|<|x\cap\kappa|$ for every $y\in\mathcal{Q}_x\}\in\mathrm{WNS}_{\kappa\lambda}{}^*$. Note that the relation "$y\in\mathcal{Q}_x$" is wellfounded. Pick any x in $C\cap A$ which is minimal in this relation. Then $x\in B$ and $A\cap\mathcal{Q}_x$ is strong stationary in $\mathcal{Q}_x$, since $g|\mathcal{Q}_x: \mathcal{Q}_x\to\mathcal{Q}_x$, $C\cap\mathcal{Q}_x\in\mathrm{WNS}_{|x\cap\kappa|x}{}^*$. Then we have $C\cap A\cap\mathcal{Q}_x\neq\phi$ which is a desired contradiction. □

§4. $\mathcal{P}_\kappa\lambda$ and $\mathcal{P}_\kappa\lambda^{<\kappa}$.

In case $\lambda^{<\kappa}=\lambda$, the structure of $\mathrm{WNS}_{\kappa\lambda}$ and some ideals defined by large cardinal properties such as $\mathrm{NSh}_{\kappa\lambda}$, $\mathrm{NAIn}_{\kappa\lambda}$(non almost λ-ineffable ideal), $\mathrm{NIn}_{\kappa\lambda}$ (non λ-ineffable ideal) are fairly known. $\mathrm{WNS}_{\kappa\lambda}=\mathrm{NS}_{\kappa\lambda}|S$ for some S and the last three ideals are all strongly normal.

On the other hand we know little in case $\lambda^{<\kappa}>\lambda$. So, we further study the behavior of the embedding f defined in §2 and show what large cardinal properties on $\mathcal{P}_\kappa\lambda$ induce the same properties on $\mathcal{P}_\kappa\lambda^{<\kappa}$ in Corollary 4.5.

Let $\delta=\lambda^{<\kappa}$ and κ be strongly inaccessible. Recall the notation in Lemma 2.4. We only use $\{y\subset x: |y|<|x\cap\kappa|\}$ as $\mathcal{G}_x$ for each $x\in\mathcal{P}_\kappa\lambda$ and fix an enumeration $\{s_\alpha: \alpha<\delta\}$ of $\mathcal{P}_\kappa\lambda$. $f: \mathcal{P}_\kappa\lambda\to\mathcal{P}_\kappa\delta$ is defined by: $f(x)=\{\alpha<\delta: s_\alpha\in\mathcal{G}_x\}$. If we regard $\mathcal{G}_x$ as an embedding of $\mathcal{P}_\kappa\lambda$ into $\mathcal{P}_\kappa\mathcal{P}_\kappa\lambda$, we get a natural embedding where $\{x\in\mathcal{P}_\kappa\lambda: a\in\mathcal{G}_x\}\in \mathrm{I}_{\kappa\lambda}{}^*$ holds for any $a\in\mathcal{P}_\kappa\lambda$. However, the use of $\mathcal{G}_x$ causes some notational confusion, hence we adopt the above f to avoid the confusion.

$J=f_*(I)=\{X\subset\mathcal{P}_\kappa\delta: f^{-1}[X]\in I\}$ is an ideal on $\mathcal{P}_\kappa\delta$ for any ideal I on $\mathcal{P}_\kappa\lambda$. We already know J is strongly normal iff I is strongly normal. Note that f is one to one and $f(x)\subset f(y)$ iff $x\subset y$. Hence $f[X]\in \mathrm{I}_{\kappa\delta}$ for any $X\in I_{\kappa\lambda}$.

Note also that our interest is in case $\lambda<\lambda^{<\kappa}=\delta$. In fact, $S=\{x\in\mathcal{P}_\kappa\lambda: f(x)=x\}\in\mathrm{WNS}_{\kappa\lambda}{}^*$ and $\mathrm{WNS}_{\kappa\lambda}=\mathrm{NS}_{\kappa\lambda}|S$ if $\lambda^{<\kappa}=\lambda$.

First we show that f does not depend on the choice of $\{s_\alpha: \alpha<\delta\}$ if $I\supset \mathrm{WNS}_{\kappa\lambda}$;

PROPOSITION 4.1. *Let* $\{s_\alpha: \alpha<\delta\}=\{t_\alpha: \alpha<\delta\}=\mathcal{P}_\kappa\lambda$ *be two bijective enumerations, and* f *and* g *be defined from* s_α*'s and* t_α*'s respectively as above. Then,* $f_*(I)=g_*(I)$ *for* $I\supset\mathrm{WNS}_{\kappa\lambda}$.

PROOF. We prove that $\{x\in\mathcal{P}_\kappa\lambda: f(x)=g(x)\}\in\mathrm{WNS}_{\kappa\lambda}{}^*$. Soppose not. We may assume that there are $\alpha_x (x\in\mathcal{P}_\kappa\lambda)$ such that $X=\{x\in\mathcal{P}_\kappa\lambda: \alpha_x\in f(x)-g(x)\}\in\mathrm{WNS}_{\kappa\lambda}{}^+$. Since $s_{\alpha_x}\in\mathcal{G}_x$ for all $x\in X$ and $\mathrm{WNS}_{\kappa\lambda}$ is strongly normal, there is α such that $Y=\{x\in X: \alpha_x=\alpha\}\in\mathrm{WNS}_{\kappa\lambda}{}^+$. For any $x\in Y$, $t_{\alpha_x}=t_\alpha\notin\mathcal{G}_x$ contradicting to $Y\in\mathrm{I}_{\kappa\lambda}{}^+$. □

DEFINITION. Let $[X]^2=\{(x, y)\in X\times X: x\subsetneqq y\}$ for an $X\subset\mathcal{P}_\kappa\lambda$. An ultrafilter $\mathcal{V}$ on $\mathcal{P}_\kappa\lambda$ has *the partition property* iff for any $F: [\mathcal{P}_\kappa\lambda]^2\to 2$, there exists an $H\in\mathcal{V}$ such that $F|[H]^2$ is constant.

The fact "$f(x)\subset f(y)$ iff $x\subset y$" and Lemma 2.4 (ii) imply;

PROPOSITION 4.2. *If* $cf(\lambda)<\kappa$ *and* $\mathcal{U}$ *is a normal measure on* $\mathcal{P}_\kappa\lambda$ *with the partition property, then* $f_*(\mathcal{U})$ *is a normal measure on* $\mathcal{P}_\kappa\lambda^+$ *with the partition property.*

At the end of this paper, in Corollary 4.5, we observe ideals on $\mathcal{P}_\kappa\lambda$ defined by weak forms of the partition property.

LEMMA 4.3. *Let* $A=f[\mathcal{P}_\kappa\lambda]$ *and let* $c: \delta\to\mathcal{P}_\kappa\lambda$, $d: \mathcal{P}_\kappa\lambda\to\mathcal{P}_\kappa\delta$ *and* $e: \mathcal{P}_\kappa\lambda\to\delta$ *be all bijections. Then,*

(i) $\{x\in\mathcal{P}_\kappa\lambda: c[f(x)]=\mathcal{G}_x\}\in\mathrm{WNS}_{\kappa\lambda}{}^*$.

(ii) $B=\{x\in\mathcal{P}_\kappa\delta: d[\mathcal{G}_{x\cap\lambda}]=\mathcal{G}_x$ *and* $e[\mathcal{G}_{x\cap\lambda}]=x\}\in\mathrm{WNS}_{\kappa\delta}{}^*$.

(iii) $A \in \mathrm{WNS}_{\kappa\delta}^* - \mathrm{NS}_{\kappa\delta}^*$.

(iv) $f_*(\mathrm{I}_{\kappa\lambda}) = \mathrm{I}_{\kappa\delta} | A \supsetneqq \mathrm{I}_{\kappa\delta}$.

(v) $f_*(\nabla_< I) = \nabla f_*(I)$ *for any ideal* I *on* $\mathcal{P}_\kappa\lambda$.

(vi) $f_*(\mathrm{I}_{\kappa\lambda}) \subsetneqq f_*(\mathrm{SNS}_{\kappa\lambda}) \subsetneqq f_*(\mathrm{NS}_{\kappa\lambda}) \subsetneqq f_*(\mathrm{WNS}_{\kappa\lambda}) = \mathrm{SNS}_{\kappa\delta} | A = \mathrm{NS}_{\kappa\delta} | A$.

(vii) $\mathrm{NS}_{\kappa\delta} \subsetneqq \mathrm{SNS}_{\kappa\delta} | A$.

PROOF. (i) Suppose contrary that $\{x \in \mathcal{P}_\kappa\lambda : c[f(x)] \neq \mathcal{Q}_x\} \in \mathrm{WNS}_{\kappa\lambda}^+$. If $\{x : \mathcal{Q}_x - c[f(x)] \neq \phi\} \in \mathrm{WNS}_{\kappa\lambda}^+$, we use strong normality to get a $u \in \mathcal{P}_\kappa\lambda$ such that $X = \{x \in \mathcal{P}_\kappa\lambda : u \in \mathcal{Q}_x - c[f(x)]\} \in \mathrm{WNS}_{\kappa\lambda}^+$. Then we have a $v \in \mathcal{P}_\kappa\lambda$ and an $\alpha < \delta$ such that $u = c(\alpha)$ and $v = s_\alpha$. Now $X \subset \{x : \alpha \in f(x)\} \subset \{x : v \notin \mathcal{Q}_x\} \in \mathrm{I}_{\kappa\lambda}$ which contradicts to $X \in \mathrm{WNS}_{\kappa\lambda}^+$.

If $\{x : c[f(x)] - \mathcal{Q}_x \neq \phi\} \in \mathrm{WNS}_{\kappa\lambda}^+$, there exist a $w \in \mathcal{P}_\kappa\lambda$ and a $\beta < \delta$ such that $w = s_\beta$ and $\{x : c(\beta) \notin \mathcal{Q}_x\} \in \mathrm{WNS}_{\kappa\lambda}^+$. Contradiction.

(ii) First suppose that $X = \{x \in \mathcal{P}_\kappa\delta : d[\mathcal{Q}_{x\cap\lambda}] \neq \mathcal{Q}_x\} \in \mathrm{WNS}_{\kappa\delta}^+$. Then, there is a y such that $X_1 = \{x \in X : d(y) \notin \mathcal{Q}_x\} \in \mathrm{WNS}_{\kappa\delta}^+$, or there is a $z \in \mathcal{P}_\kappa\lambda$ such that $X_2 = \{x \in X : d^{-1}(z) \notin \mathcal{Q}_{x\cap\lambda}\} \in \mathrm{WNS}_{\kappa\lambda}^+$. $\mathcal{Q}_{x\cap\lambda} = \mathcal{Q}_x \cap \mathcal{P}_\kappa\lambda$ for every $x \in \mathcal{P}_\kappa\delta$. Thus both of X_1 and X_2 are in $\mathrm{I}_{\kappa\lambda}$. Contradiction.

Secondly let $Y = \{x \in \mathcal{P}_\kappa\delta : e[\mathcal{Q}_{x\cap\lambda}] \neq x\} \in \mathrm{WNS}_{\kappa\delta}^+$. Again we have a $q \in \mathcal{P}_\kappa\lambda$ with $\{x \in \mathcal{P}_\kappa\delta : e(q) \notin x\} \in \mathrm{WNS}_{\kappa\delta}^+$, or $\{x : e^{-1}(\gamma) \notin \mathcal{Q}_{x\cap\lambda}\} \in \mathrm{WNS}_{\kappa\delta}^+$ for some $\gamma < \delta$, which is a contradiction.

(iii) Suppose contrary that $A \in \mathrm{NS}_{\kappa\delta}^*$. Then there exist a cub $C \subset A$ and a strictly $\subset$-increasing sequence $\{x_n : n \in \omega\} \subset C$ with $\omega_1 \subset y_0 = f^{-1}(x_0)$. Let $x = \bigcup\{x_n : n \in \omega\}$. Since $x \in C \subset A$, $x = f(y)$ for some $y \in \mathcal{P}_\kappa\lambda$.

CLAIM. $y = \bigcup\{y_n : n \in \omega\}$ *where* $y_n = f^{-1}(x_n)$.

PROOF OF THE CLAIM. Since $f(y_n) = x_n \subset x = f(y)$, $y_n \subset y$ for all $n \in \omega$. On the other hand, for $\xi \in y$, there exists $\alpha < \delta$ such that $\{\xi\} = s_\alpha$. Then $\alpha \in f(y) = x$. Hence we find an $n \in \omega$ such that $\alpha \in x_n = f(y_n)$. Then $\xi \in y_n$ and $y \subset \bigcup\{y_n : n \in \omega\}$. □

Since $x_n \subsetneqq x_{n+1}$, $y_n \subsetneqq y_{n+1}$ for all n. Pick any $\gamma_n \in y_{n+1} - y_n$ for each n and let $b = \{\gamma_n : n \in \omega\}$. Then, $b \subset y$ and $|b| = \omega$. By our assumption on y_0, we have $|y \cap \kappa| \geqq \omega_1 > |b|$. So, $b \in \mathcal{Q}_y$. For β with $b = s_\beta$, $\beta \in f(y) = x$ and $\beta \in x_n$ for some $n \in \omega$. We then have $b \in \mathcal{Q}_{y_n}$ contradicting to $\gamma_n \in y_{n+1} - y_n$. Hence $A \notin \mathrm{NS}_{\kappa\lambda}^*$.

$A \in \mathrm{WNS}_{\kappa\delta}^*$ is clear by (ii).

(iv) Suppose that $X \subset \mathcal{P}_\kappa\delta$ and $X \cap A \in I_{\kappa\delta}$. Then there is an $a \in \mathcal{P}_\kappa\delta$ such that $a \not\subset x$ for all $x \in X \cap A$. Let $b = \bigcup\{s_\alpha : \alpha \in a\} \in \mathcal{P}_\kappa\lambda$. Since $a \subset f(b)$ and $f^{-1}[X] = f^{-1}[X \cap A]$, $f^{-1}[X] \cap \hat{b} = \phi$. Hence $I_{\kappa\delta} | A \subset f_*(I_{\kappa\lambda})$. Conversely, if $f^{-1}[X] \in I_{\kappa\lambda}$, then $X \cap A = f[(f^{-1}[X])] \in I_{\kappa\delta}$. The inequality of $I_{\kappa\delta}$ and $I_{\kappa\delta} | A$ is proved by (iii).

(v) is a reformulation of Lemma 2.4 (i).

(vi) We can prove that $f_*(I)\subsetneqq f_*(J)$ whenever $I\subsetneqq J$. Pick any $X\in J-I$. Since f is one to one, $X=f^{-1}[f[X]]$. Hence $f[X]\in f_*(J)-f_*(I)$.

We know $f_*(\mathrm{WNS}_{\kappa\lambda})=f_*(\nabla_< I_{\kappa\lambda})=\nabla f_*(I_{\kappa\lambda})=\nabla(I_{\kappa\delta}|A)$ by (iv) and (v). Since $\nabla(I|Y)=(\nabla I)|Y$ for any ideal I, $f_*(\mathrm{WNS}_{\kappa\lambda})=(\nabla I_{\kappa\delta})|A=\mathrm{SNS}_{\kappa\delta}|A$. Recall that $\nabla_<\nabla_< I=\nabla_< I$ for any ideal I. We also have $f_*(\mathrm{WNS}_{\kappa\lambda})=\mathrm{NS}_{\kappa\delta}|A$.

(vii) is clear by (iii) and (vi).

THEOREM 4.4. $f_*(\mathrm{WNS}_{\kappa\lambda})=\mathrm{WNS}_{\kappa\delta}$.

PROOF. By Lemma 2.4, $J=f_*(\mathrm{WNS}_{\kappa\lambda})$ is strongly normal hence $J\supset\mathrm{WNS}_{\kappa\delta}$. By (iii) and (vi) in the above, $J=\mathrm{NS}_{\kappa\delta}|A$ and $A\in\mathrm{WNS}_{\kappa\delta}{}^*$. $\mathrm{NS}_{\kappa\delta}\subset\mathrm{WNS}_{\kappa\delta}$ implies $J\subset\mathrm{WNS}_{\kappa\delta}$. □

DEFINITION. Let $F:[X]^2\to 2$. Then $H\subset X$ is homogeneous for F iff $F|[H]^2$ is constant. $X\in\mathrm{NP}_{\kappa\lambda}$ iff there is an $F:[X]^2\to 2$ with no unbounded homogeneous set. $\mathrm{NP}_{\kappa\lambda}$ is an ideal on $\mathcal{P}_\kappa\lambda$. We may define similar ideals on $\mathcal{P}_\kappa\lambda$; that is, $X\in\mathrm{NP}^0{}_{\kappa\lambda}$ ($\mathrm{NP}^1{}_{\kappa\lambda}$, $\mathrm{NP}^2{}_{\kappa\lambda}$) if we have an $F:[X]^2\to 2$ with no $\mathrm{SNS}_{\kappa\lambda}$ ($\mathrm{NS}_{\kappa\lambda}$, $\mathrm{WNS}_{\kappa\lambda}$)-positive homogeneous set. We say *Part*(κ,λ) iff $\mathcal{P}_\kappa\lambda\in\mathrm{NP}_{\kappa\lambda}{}^+$. Note that $\mathrm{NP}^i{}_{\kappa\lambda}$ is a normal ideal$\supset\mathrm{NSh}_{\kappa\lambda}$ hence strongly normal if $cf(\lambda)\geqq\kappa$.

$X\in\mathrm{NIn}^i{}_{\kappa\lambda}$ iff there is an $f:X\to\mathcal{P}_\kappa\lambda$ with $f(x)\subset x$ for all $x\in X$ such that $\{x\in X: f(x)=x\cap A\}\in\mathrm{I}_i$ for any $A\subset\lambda$ with $\mathrm{I}_0=\mathrm{SNS}_{\kappa\lambda}$, $\mathrm{I}_1=\mathrm{NS}_{\kappa\lambda}$ and $\mathrm{I}_2=\mathrm{WNS}_{\kappa\lambda}$.

$X\in\mathrm{NSIn}^i{}_{\kappa\lambda}$ iff there is an $f:X\to\mathcal{P}\mathcal{P}_\kappa\lambda$ with $f(x)\subset\mathcal{G}_x$ for all $x\in X$ such that $\{x\in X: f(x)=B\cap\mathcal{G}_x\}\in\mathrm{I}_i$ for any $B\subset\mathcal{P}_\kappa\lambda$.

COROLLARY 4.5. *Let* $\delta=\lambda^{<\kappa}$.

(i) *If* $X\in\mathrm{NP}_{\kappa\lambda}{}^+$, *then* $f[X]\in\mathrm{NP}_{\kappa\delta}{}^+$. *Thus Part*$(\kappa,\lambda)$ *implies Part*(κ,δ).

(ii) $\mathrm{NP}^0{}_{\kappa\lambda}=\mathrm{NP}^1{}_{\kappa\lambda}$.

(iii) *If* $cf(\lambda)\geqq\kappa$, *then* $\mathrm{NP}^0{}_{\kappa\lambda}=\mathrm{NP}^1{}_{\kappa\lambda}=\mathrm{NP}^2{}_{\kappa\lambda}$.

(iv) *If* $X\in\mathrm{NP}^2{}_{\kappa\lambda}$, *then* $f[X]\in\mathrm{NP}^2{}_{\kappa\delta}$.

(v) $\mathrm{NIn}^0{}_{\kappa\lambda}=\mathrm{NIn}^1{}_{\kappa\lambda}$.

(vi) *If* $cf(\lambda)\geqq\kappa$, *then* $\mathrm{NIn}^i{}_{\kappa\lambda}=\mathrm{NSIn}^j{}_{\kappa\lambda}$ *for* $0\leqq i, j\leqq 2$.

(vii) *If* $X\in\mathrm{NSIn}^2{}_{\kappa\lambda}{}^+$, *then* $f[X]\in\mathrm{NSIn}^2{}_{\kappa\delta}{}^+$ *and* $\delta\leqq\lambda^+$.

PROOF. (i) Suppose that $X\in\mathrm{NP}_{\kappa\lambda}{}^+$, $Y=f[X]$ and $F:[Y]^2\to 2$. For each $(u,v)\in[Y]^2$ we can find a unique $(x,y)\in[X]^2$ with $u=f(x)$ and $v=f(y)$. If $G:[X]^2\to 2$ is defined by $G(x,y)=F(f(x),f(y))$, $G|[H]^2$ is constant for some $H\in\mathrm{I}_{\kappa\lambda}{}^+$. Then, $f[H]$ is clearly an unbounded homogeneous set for F.

(ii) Note that $\mathrm{NS}_{\kappa\lambda}=\mathrm{SNS}_{\kappa\lambda}|D$ for some $D\subset\mathcal{P}_\kappa\lambda$ (Matet [**10**]). We only have to show that $X\in\mathrm{NP}^0{}_{\kappa\lambda}$ for all $X\in\mathrm{NP}^1{}_{\kappa\lambda}$. Let $F:[X]^2\to 2$ witness that $X\in\mathrm{NP}^1{}_{\kappa\lambda}$. Since $D\in\mathrm{NS}_{\kappa\lambda}{}^*\subset\mathrm{NP}^0{}_{\kappa\lambda}{}^*$, $X\cap D\in\mathrm{NP}^0{}_{\kappa\lambda}{}^+$ if $X\in\mathrm{NP}^0{}_{\kappa\lambda}{}^+$. So, there is an $H\subset X\cap D$ which is a $\mathrm{SNS}_{\kappa\lambda}$-positive homogeneous set for F. In fact $H\in\mathrm{NS}_{\kappa\lambda}{}^+$ since $H\cap D\in\mathrm{SNS}_{\kappa\lambda}{}^+$.

(iii) $WNS_{\kappa\lambda}=NS_{\kappa\lambda}|S$ for some $S\subset\mathcal{P}_\kappa\lambda$ if $cf(\lambda)\geqq\kappa$. The proof is similar as above.

(iv) Note that $f[H]\in NS_{\kappa\delta}^+$ if $H\in WNS_{\kappa\lambda}^+$ and $\delta^{<\kappa}=\delta$. The similar proof as (i) works together with (iii).

(v) is similar to (ii).

(vi) The fact that $NIn^i_{\kappa\lambda}$'s are the same is proved as in (iii). It is also clear that $NIn^i_{\kappa\lambda}\subset NSIn^i_{\kappa\lambda}$ and $NSIn^i_{\kappa\lambda}\subset NSIn^j_{\kappa\lambda}$ if $i<j$. Hence, we only have to show that $NIn^2_{\kappa\lambda}=NSIn^2_{\kappa\lambda}$. Let $X\in NIn^2_{\kappa\lambda}{}^+$, $f(x)\subset\mathcal{Q}_x$ for all $x\in X$, and $c:\mathcal{P}_\kappa\lambda\to\lambda$ bijective. $S=\{x: c[\mathcal{Q}_x]=x\}\in WNS_{\kappa\lambda}{}^*\subset NIn^2_{\kappa\lambda}{}^*$. We have an $A\subset\lambda$ such that $Y=\{x\in X\cap S: c[f(x)]=x\cap A\}\in NS_{\kappa\lambda}^+$. Put $B=c^{-1}[A]$. Then $f(x)=\mathcal{Q}_x\cap B$ for all $x\in Y$. We also know $Y\in WNS_{\kappa\lambda}^+$ since $WNS_{\kappa\lambda}=NS_{\kappa\lambda}|S$.

(vii) Suppose that $X\in NSIn^2_{\kappa\lambda}{}^+$ and $Y=f[X]$. Since $\delta^{<\kappa}=\delta$, it suffices to show that $Y\in NIn_{\kappa\delta}^+$. Let $g: Y\to\mathcal{P}_\kappa\delta$ such that $g(x)\subset x$ for all $x\in Y$. If we define $h: X\to\mathcal{P}\mathcal{P}_\kappa\lambda$ as $h(z)=\{s_\alpha:\alpha\in g(f(z))\}$, then $h(z)\subset\mathcal{Q}_z$ for any $z\in X$. Hence we have a $B\subset\mathcal{P}_\kappa\lambda$ with $W=\{z\in X: h(z)=B\cap\mathcal{Q}_z\}\in WNS_{\kappa\lambda}^+$. Now it is clear that $f[W]\in\mathcal{P}(Y)\cap NS_{\kappa\delta}^+$ and $g(x)=x\cap f[B]$ for all $x\in f[W]$. It is known that $\lambda^{<\kappa}=\lambda$ if κ is λ-ineffable and $cf(\lambda)\geqq\kappa$. If $cf(\lambda)<\kappa$, then $\delta\geqq\lambda^+$ and κ is at least λ^+-ineffable by the previous paragraph. So, $(\lambda^+)^{<\kappa}=\lambda^+$. □

The proposition (viii) has some interest under similarity to the certain extendibility of large cardinal property below;

If κ is λ-(super) compact, then it is $\lambda^{<\kappa}$-(super) compact. Moreover $\lambda^{<\kappa}=\lambda$ if $cf(\lambda)\geqq\kappa$ and $\lambda^{<\kappa}=\lambda^+$ if $cf(\lambda)<\kappa$.

It may be natural to ask;

QUESTION. *Are they also true if the compactness is replaced by "κ is λ-ineffable"?*

The answer is "Yes" if $cf(\lambda)\geqq\kappa$. The question in case $cf(\lambda)<\kappa$ seems to remain open. It is a motivation of the study about embedding $\mathcal{P}_\kappa\lambda$ into $\mathcal{P}_\kappa\lambda^{<\kappa}$ indeed. But we could only show that the stronger property "$NSIn_{\kappa\lambda}$ is proper" inherits to $\mathcal{P}_\kappa\lambda^{<\kappa}$ and the weaker property $Part(\kappa,\lambda)$ leads to $Part(\kappa,\lambda^{<\kappa})$.

REMARK. S. Kamo proved an interesting fact using this embedding:

THEOREM (Kamo [**9**]). *Suppose that κ is almost κ^+-ineffable and no $\alpha<\kappa$ is almost α^+-ineffable. Then, κ is not κ^+-ineffable.*

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