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High energy resolvent estimates, I, first order operators

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§1. Introduction.

In this paper we consider first order pseudo-differential operators on $\boldsymbol{R}^n$ and study uniform estimates of the resolvents as a spectral parameter goes to infinity. The uniform estimates for higher order partial differential operators and first order systems whose coefficients approach constants at infinity will be studied on the basis of the results of this paper in subsequent papers. The results there will be used to give asymptotic expansions as $t\rightarrow\infty$ of solutions for hyperbolic or Schrödinger-type equations.

In connection with time-decay for solutions of hyperbolic differential equations, analytic continuation of resolvents through the real line have been investigated by many mathematicians (see, for example, [**9**] and [**10**]). They treated, however, only differential operators which are homogeneous differential operators with constant coefficients outside a bounded set. Our aim is to establish uniform continuity and differentiability of the boundary values onto the real line for resolvents of operators neither constant nor homogeneous at infinity.

Now, we explain notations in order to state our main results. We write $x=(x_1, \cdots, x_n)\in\boldsymbol{R}^n$, $\langle x\rangle=(1+|x|^2)^{1/2}$, $\partial_{x_j}=\partial/\partial x_j$, $D_j=-i\,\partial/\partial x_j$, $D_x^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ for a multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ of nonnegative integers. For a smooth function f on an open set of $\boldsymbol{R}^n$, $\nabla_x f=(\partial_{x_1}f, \cdots, \partial_{x_n}f)$. $\mathcal{S}'$ denotes the space of all tempered distributions on $\boldsymbol{R}^n$. For $u\in\mathcal{S}'$, the Fourier transform and the inverse Fourier transform of u are denoted by $\hat{u}=F[u]$ and $\tilde{u}=F^{-1}[u]$, respectively:

$$\hat{u}(\xi)=\int e^{-ix\xi}u(x)dx, \quad \tilde{u}(x)=\int e^{ix\xi}u(\xi)d\!\!\bar{}\,\xi, \quad d\!\!\bar{}\,\xi=(2\pi)^{-n}d\xi.$$

For real numbers σ and s, we put

$$H^{\sigma,s}=\{f\in\mathcal{S}'\,;\ \|f\|_{H^{\sigma,s}}=\|\langle x\rangle^s\langle D_x\rangle^\sigma f\|_{L_2(R^n)}<\infty\}. \tag{1.1}$$

In particular, $H^{0,s}=L_2^s$ and $H^{\sigma,0}=H^\sigma$ are the weighted L_2-space and the Sobolev space, respectively. We write $\|f\|_s=\|f\|_{0,s}$. For Banach spaces X and Y,

$B(X, Y)$ denotes the Banach space of all bounded linear operators from X to Y. We write

$$B(\sigma, s; \tau, r)=B(H^{\sigma,s}, H^{\tau,r}), \quad B(s, r)=B(0, s; 0, r). \tag{1.2}$$

Consider the pseudo-differential operator

$$A(X, D_x)=b(D_x)+c(X, D_x)+d(X, D_x), \tag{1.3}$$

where $b(\xi)$, $c(x, \xi)\in S^1_{1,0}$ and $d(x, \xi)\in S^0_{1,0}$. (For pseudo-differential operators, see [2] and [3] for example.) We impose on the symbols the following conditions (A.I) and (A.II).

(A.I) $b(\xi)$ and $c(x, \xi)$ are real-valued functions such that for some positive constants K_0 and R_0

$$|\nabla_\xi b(\xi)|, \ |\nabla_\xi a(x, \xi)|\geqq K_0, \quad |x|, \ |\xi|\geqq R_0, \tag{1.4}$$

where $a(x, \xi)=b(\xi)+c(x, \xi)$.

(A.II) There exists a constant $\rho>2$ such that for any multi-indices α and β

$$|D_x^\alpha \partial_\xi^\beta c(x, \xi)|\leqq C_{\alpha\beta}\langle x\rangle^{-\rho-|\alpha|}\langle\xi\rangle^{1-|\beta|}, \tag{1.5}$$

$$|D_x^\alpha \partial_\xi^\beta d(x, \xi)|\leqq C_{\alpha\beta}\langle x\rangle^{1-\rho-|\alpha|}\langle\xi\rangle^{-|\beta|}, \tag{1.6}$$

where the $C_{\alpha\beta}$ are constants independent of $x, \xi\in \boldsymbol{R}^n$.

It follows from (A.I) and (A.II) that the Cauchy problem

$$(D_t-A(X, D_x))u(t, x)=0, \qquad u(0, x)=u_0(x)$$

is well-posed: The propagator $E(t)$ defined by $E(t)u_0(x)=u(t, x)$ forms a continuous group of operators on L_2 (see [2]). Let us denote by iA the generator of the group. Then A is a closed operator whose resolvent $R(z)=(z-A)^{-1}$ exists for $|\mathrm{Im}\, z|\gg 1$. Our problem is whether the resolvent $R(z)$ on $\{\pm\mathrm{Im}\, z\gg 1\}$ can be extended, as a $B(s, -s)$-valued bounded continuous function, to the set $\boldsymbol{N}_\pm=\{z\in\boldsymbol{C};\ \pm\mathrm{Im}\, z\geqq 0,\ |z|>N\}$, where N is a large positive number. In solving the problem the no trapped ray condition (A.III) to be stated below plays a crucial role. We denote by $\{q(t, y, \xi), p(t, y, \xi)\}$ the solution of the Hamilton equation

$$\frac{dq}{dt}=-\nabla_\xi a(q, p), \qquad \frac{dp}{dt}=\nabla_x a(q, p), \ \{q(0), p(0)\}=\{y, \xi\}. \tag{1.7}$$

(A.III) For any $R>0$ there exists T such that

$$|q(t, y, \xi)|>R \qquad \text{for all } t>T, \ |y|\leqq R, \text{ and } |\xi|\geqq R_0.$$

We can now state our main results.

THEOREM 1. *Assume* (A.I)$\sim$(A.III). *Let* $0\leqq\theta<1$ *and* k *be a non-negative integer such that* $0<k+\theta<\rho-2$. *Then there exists a constant* $N>0$ *such that the* *resolvent* $R(z)=(z-A)^{-1}$ *on* $\{\pm\mathrm{Im}\, z\gg 1\}$ *can be extended to a k-times continuously differentiable function on the set* $\boldsymbol{N}_{\pm}=\{z;\ \pm\mathrm{Im}\, z\geqq 0,\ |z|>N\}$ *to* $B(s,-s)$, *where* $s>k+\theta+3/2$, *which is holomorphic in the interior of* $\boldsymbol{N}_{\pm}$ *and has the properties: the derivatives* $R^{(j)}(z)$, $j=0,\cdots,k$, *are bounded, and* $R^{(k)}(z)$ *is uniformly Hölder continuous with exponent* θ. *Furthermore, for any* $r>k+\theta+1$ *there exists a constant* C_r *such that for all* $f\in L_2^r$

$$(1.8)\qquad \int_{-\infty}^{\infty}\left[\langle t\rangle^{k+\theta}\left\|\int_{-\infty}^{\infty}\chi(\lambda)R(\lambda\pm i0)fe^{it\lambda}d\lambda\right\|_{-r}\right]^2dt\leqq C_r^2\|f\|_r^2,$$

where $\chi(\lambda)$ *is a* C^{∞}*-function on* $\boldsymbol{R}^1$ *such that* $\chi(\lambda)=1$ *for* $|\lambda|>N+1$ *and* $\mathrm{Supp}\,\chi\subset\{|\lambda|>N\}$.

If $\rho>3$, *for any* $1<\sigma<\rho-2$ *and* $s>\sigma+1/2$ *there exists a constant* C_s *such that*

$$(1.9)\qquad \left\|\int_{-\infty}^{\infty}\chi(\lambda)R(\lambda\pm i0)e^{it\lambda}d\lambda\right\|_{B(s,-s)}\leqq C_s\langle t\rangle^{-\sigma},\quad t\in\boldsymbol{R}^1.$$

THEOREM 2. *Assume* (A.I), (A.II) *and the condition that there exist symbols* $a_1(x,\xi)$ *and* $b_1(\xi)$ *homogeneous of order* 1 *on* $\{|\xi|>R_0\}$ *such that* $a(x,\xi)-a_1(x,\xi)$, $b(\xi)-b_1(\xi)\in S_{1,0}^0$. *Then the estimate* (1.8) *implies* (A.III).

A close connection between the no trapped ray condition and energy decay has been observed by many mathematicians (see [4]~[6], [8]~[**10**]). Vainberg [**10**] showed that the no trapped ray condition implies the high energy resolvent estimate. Theorem 2 asserts that the converse is true for a large class of operators.

The remainder of this paper is organized as follows. Sections 2, 3, and 4 are devoted to the proof of Theorem 1. Theorem 2 is proved in Section 5. In Section 6 we shall deal with operators with constant leading coefficients, and give estimates sharper than (1.8) and (1.9). The estimates are the best possible ones.

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§ 2. Phase functions.

The purpose of this section is to construct phase functions globally in time by solving the Hamilton-Jacobi equation

$$(2.1)\qquad \partial_t\phi-a(x,\nabla_x\phi)=0,\quad \phi(0,x,\xi)=x\xi$$

or

$$(2.1')\qquad \partial_t\phi+a(\nabla_\xi\phi,\xi)=0,\quad \phi(0,x,\xi)=x\xi$$

for $a(x, \xi)$ satisfying (A.I)$\sim$(A.III). We shall carry out the construction only for $t \geqq 0$, since the other case can be treated similarly. In what follows we shall use the following notation $[[x, y]] = x \cdot y / |x||y|$, $x, y \in \boldsymbol{R}^n \setminus \{0\}$.

2.1. Bicharacteristics. Consider the Hamilton equation

$$\frac{dq}{dt} = -\nabla_\xi a(q, p), \quad \frac{dp}{dt} = \nabla_x a(q, p), \quad \{q(0), p(0)\} = \{y, \xi\}. \tag{2.2}$$

It is easily seen that the equation (2.2) has a unique solution $\{q(t, y, \xi), p(t, y, \xi)\}$ on $\boldsymbol{R}^1$.

For $-1 < \sigma < 1$ and $R, \Xi > R_0$, we put

$$\Gamma_\pm(\sigma, R, \Xi) = \{\{y, \xi\};\ \pm[[y, -\nabla_\xi a(y, \xi)]] \geqq \pm\sigma,\ |y| \geqq R,\ |\xi| \geqq \Xi\}. \tag{2.3}$$

LEMMA 2.1. *For any* $\pm t \geqq 0$ *and* $\{y, \xi\} \in \Gamma_\pm(\sigma, R, 2R_0)$ *with* R *sufficiently large,*

$$|q(t, y, \xi) - (y - t\nabla_\xi a(y, \xi))| \leqq CR^{-\rho}|t|, \tag{2.4}$$

$$|p(t, y, \xi) - \xi| \leqq CR^{-\rho}|\xi|, \tag{2.5}$$

$$|q(t, y, \xi)| \geqq c(|t| + |y|), \tag{2.6}$$

where C *and* c *are positive constants depending only on* σ.

PROOF. The lemma can be shown easily by the successive approximation: $\{q_0(t), p_0(t)\} = \{y - t\nabla_\xi a(y, \xi), \xi\}$ and

$$\begin{cases} q_{j+1}(t) = y - \int_0^t \nabla_\xi a(q_j(s), p_j(s))ds \\ p_{j+1}(t) = \xi + \int_0^t \nabla_x a(q_j(s), p_j(s))ds \end{cases} \tag{2.7}$$

for $j \geqq 0$. Q.E.D.

LEMMA 2.2. *Let* $\tau \geqq 0$, $\Xi \geqq 3R_0$, *and* $R \gg 1$. *For any* $\{y, \xi\} \in \Gamma_-(\sigma, R, \Xi)$ *satisfying* $|q(t, y, \xi)| \geqq R$ *and* $[[q(t), -\nabla_\xi a(q(t), p(t))]] \leqq \sigma$ *for all* $t \in [0, \tau]$, *one has* (2.4), (2.5), *and*

$$|q(t, y, \xi)| \geqq c(\tau - t + |q(\tau, y, \xi)|), \quad 0 \leqq t \leqq \tau, \tag{2.8}$$

where c *is a positive constant depending only on* σ.

PROOF. By virtue of Lemma 2.1 we have only to show that $|p(\tau, y, \xi)| \geqq 2R_0$, which is easily proved by reduction to absurdity. Q.E.D.

Fix σ_0, σ_0', and σ_1 such that $-1 < \sigma_0 < \sigma_0' < \sigma_1 < 0$, and choose R' so large that Lemmas 2.1 and 2.2 hold for any $R \geqq R'$ with σ replaced by σ_0 and σ_1, respectively. By virtue of (A.I), (A.II), and Lemma 2.1 we can choose $R_1 \geqq R'$ such that for any $R \geqq R_1$, $\{q(t, y, \xi), p(t, y, \xi)\} \in \Gamma_+(\sigma_1, R, 2R_0)$ if $t \geqq 0$ and $\{y, \xi\}$

$\in\Gamma_+(0, R, 4R_0)$. Fix such an R_1. Then we have

LEMMA 2.3. (i) *There exist* T_0 *and* $\Xi_0 \geqq 2R_0$ *such that* $\{q(t, y, \xi), p(t, y, \xi)\} \in\Gamma_+(\sigma_1, R_1, 2R_0)$ *for any* $t>T_0$, $|y|\leqq R_1$, *and* $|\xi|>\Xi_0$.

(ii) *There exists* Ξ_1 *such that* $|p(t, y, \xi)|\geqq 2R_0$ *for any* $t\geqq 0$, $y\in\boldsymbol{R}^n$, *and* $|\xi|\geqq\Xi_1$.

PROOF. By (A.III) there exists T_0 such that for any $|y|\leqq R_1$ and $|\xi|\geqq R_0$ one can choose $s\in[0, T_0]$ with $|q(s, y, \xi)|\geqq R_1$ and $[[q(s, y, \xi), -\nabla_\xi a(q(s, y, \xi), p(s, y, \xi))]]\geqq 0$. On the other hand, elementary calculations yield the a priori estimate

$$e^{-Mt}\langle\xi\rangle\leqq\langle p(t, y, \xi)\rangle\leqq e^{Mt}\langle\xi\rangle, \quad t\geqq 0, \quad \{y, \xi\}\in\boldsymbol{R}^{2n}, \tag{2.9}$$

where M is a positive constant. Putting $\Xi_0=4R_0e^{MT_0}$, we thus obtain that $\{q(s, y, \xi), p(s, y, \xi)\}\in\Gamma_+(0, R_1, 4R_0)$, which implies (i) by the property of R_1. (ii) follows from (i) and Lemma 2.2. Q.E.D.

For $R\geqq R_1$ we define the incoming time $\tau(y, \xi; R)$ as follows.

DEFINITION 2.4. For $\{y, \xi\}\in\Gamma_-(\sigma_0', R, \Xi_1)$,

$$\tau(y, \xi; R)=\sup\{t\geqq 0;\ |q(s, y, \xi)|\geqq R \text{ and } [[q(s), -\nabla_\xi a(q(s), p(s))]]\leqq\sigma_1 \text{ for all } s\in[0, t]\},$$

and $\tau(y, \xi; R)=0$ otherwise.

2.2. Incoming phase functions.

PROPOSITION 2.5. *Assume* (A.I) *and* (A.II). *Then for any positive integer* k *there exists a real-valued* C^∞-*function* $\phi_k(t, x, \xi)$ *on* $[0, k]\times\boldsymbol{R}^{2n}$ *such that*

$$\phi_k(0, x, \xi)=x\xi, \tag{2.10}$$

$$\partial_t\phi_k(t, x, \xi)-a(x, \nabla_x\phi_k(t, x, \xi))=0 \tag{2.11}$$

$$\text{on } \Omega_k(\xi)=\{(t, q(t, y, \xi));\ 0\leqq t\leqq k,\ \tau(y, \xi, R_2)>k\} \quad \text{with } |\xi|\geqq\Xi_1,$$

where R_2 *is some constant greater than* R_1, *and for* $0\leqq t\leqq k$ *and* $\{y, \xi\}$ *with* $\tau(y, \xi, R_2)>k$

$$\{\nabla_x\phi_k(t, x, \xi), \nabla_\xi\phi_k(t, x, \xi)\}|_{x=q(t,y,\xi)}=\{p(t, y, \xi), y\}. \tag{2.12}$$

Furthermore, ϕ_k *satisfies the following estimates on* $[0, k]\times\boldsymbol{R}^{2n}$:

$$|\nabla_x\phi_k(t, x, \xi)-\xi|\leqq|\xi|/9, \tag{2.13}$$

$$|\nabla_x\nabla_\xi\phi_k(t, x, \xi)-I|\leqq 1/9, \tag{2.14}$$

where I *is the unit matrix of size* n, *and for any multi-indices* α *and* β *with* $|\alpha+\beta|>0$

(2.15) $$|\partial_x^\alpha \partial_\xi^\beta [\phi_k(t, x, \xi) - x\xi - tb(\xi)]| \leqq C_{\alpha\beta} \langle R_2 + k - t\rangle^{-\rho-|\alpha|+1} \langle t\rangle^{|\beta|} \langle\xi\rangle^{1-|\beta|}.$$

Moreover, for any k and l

(2.16) $$\phi_k = \phi_l \quad \text{on } \{(t, x, \xi);\, (t, x) \in \Omega_k(\xi) \cap \Omega_l(\xi),\ |\xi| \geqq \Xi_1\}.$$

REMARK. It is sufficient for Proposition 2.5 to hold that $\rho > 0$.

For the proof we make some preparations. In view of Lemma 2.2 there exists $\delta_1 > 0$ such that $|q(t, y, \xi)| \geqq \delta_1(R + k - t)$ for $t \in [0, \tau(y, \xi; R)]$, $\{y, \xi\} \in \Gamma_-(\sigma_1, R, \Xi_1)$, and $R \geqq R_1$. Choose a C^∞-function χ_0 on $\boldsymbol{R}^n$ such that $\chi_0(x) = 1$ for $|x| \geqq 2$ and $\chi_0(x) = 0$ for $|x| \leqq 1$. Put

(2.17) $$a_{k,R}(t, x, \xi) = b(\xi) + \chi_0(2x/\delta_1(R + k - t))c(x, \xi).$$

We see that for any multi-indices α and β with $|\alpha| > 0$

(2.18) $$|\partial_x^\alpha \partial_\xi^\beta a_{k,R}(t, x, \xi)| \leqq C_{\alpha\beta} \langle R + k - t\rangle^{-\rho-|\alpha|} \langle\xi\rangle^{1-|\beta|}.$$

Denote by $\{q_k(t, y, \xi), p_k(t, y, \xi)\}$ a bicharacteristic for the Hamiltonian $a_{k,R}(t, x, \xi)$. It is easily seen that

(2.19) $$\{q_k(t, y, \xi), p_k(t, y, \xi)\} = \{q(t, y, \xi), p(t, y, \xi)\}$$
$$\text{for } (t, y, \xi) \text{ with } 0 \leqq t \leqq k \text{ and } \tau(y, \xi, R) > k.$$

For the derivatives of $\{q_k, p_k\}$ we have the following lemma.

LEMMA 2.6. *For any multi-indices α and β there exists $C_{\alpha\beta}$ such that*

(2.20.1) $$|\partial_y^\alpha \partial_\xi^\beta [q_k(t, y, \xi) - (y - t\nabla_\xi b(\xi))]| \leqq C_{\alpha\beta} \langle R + k - t\rangle^{-\rho-|\alpha|+1} (1 + t\langle R + k - t\rangle^{-1})^{|\beta|} \langle\xi\rangle^{-|\beta|}$$

(2.20.2) $$|\partial_y^\alpha \partial_\xi^\beta (p_k(t, y, \xi) - \xi)| \leqq C_{\alpha\beta} \langle R + k - t\rangle^{-\rho-|\alpha|} (1 + t\langle R + k - t\rangle^{-1})^{|\beta|} \langle\xi\rangle^{1-|\beta|}$$

for all $0 \leqq t \leqq k$ and R sufficiently large. In particular

(2.21) $$\begin{cases} |\nabla_y q_k(t, y, \xi) - I| \leqq C_0 \langle R + k - t\rangle^{-\rho} \\ |\nabla_y p_k(t, y, \xi)| \leqq C_0 \langle R + k - t\rangle^{-\rho-1} \langle\xi\rangle. \end{cases}$$

PROOF. (2.20) for $\alpha = \beta = 0$ can be shown easily by using (2.18). Let us show (2.21). We have

(2.22.1) $$\nabla_y q_k(t, y, \xi) = I - \int_0^t [\nabla_x \nabla_\xi a_{k,R}(s, q_k(s), p_k(s)) \nabla_y q_k(s) + \nabla_\xi^2 a_{k,R}(s, q_k(s), p_k(s)) \nabla_y p_k(s)] ds$$

(2.22.2) $$\nabla_y p_k(t, y, \xi)=\int_0^t[\nabla_x^2 a_{k,R}(s, q_k(s), p_k(s))\nabla_y q_k(s)$$
$$+\nabla_\xi\nabla_x a_{k,R}(s, q_k(s), p_k(s))\nabla_y p_k(s)]ds\,.$$

With

$$M(t)=\sup_{0\leqq s\leqq t}|\nabla_y q_k(s)-I|\langle R+k-s\rangle^\rho$$

$$N(t)=\sup_{0\leqq s\leqq t}|\nabla_y p_k(s)|\langle R+k-s\rangle^{\rho+1}\langle\xi\rangle^{-1},$$

we have by (2.22)

(2.23) $$\begin{cases} M(t)\leqq CR^{-\rho}M(t)+CN(t)+C \\ N(t)\leqq CR^{-\rho}M(t)+CR^{-\rho}N(t)+C\,. \end{cases}$$

Thus, $M(t)+CN(t)\leqq 2R^{-\rho}C(1+C)(M(t)+CN(t))+3C$. From this we get (2.21) by choosing R so large that $2R^{-\rho}C(1+C)<1/2$.

(2.20) for $|\alpha+\beta|>0$ can be shown by induction on $|\alpha+\beta|$. Q.E.D.

PROOF OF PROPOSITION 2.5. Choose R_2 such that $C_0R_2^{-\rho}<1/10$. By (2.21), the mapping: $y\to q_k(t, y, \xi)$ on $\boldsymbol{R}^n$ is a bijection. Let $y_k(t, x, \xi)$ be the solution of the equation $x=q_k(t, y, \xi)$. Define $\phi_k(t, x, \xi)$ by

(2.24) $$\phi_k(t, x, \xi)=y_k(t, x, \xi)\cdot\xi$$
$$+\int_0^t\{a_{k,R_2}-\xi\cdot\nabla_\xi a_{k,R_2}\}(s, q_k(s, y_k, \xi), p_k(s, y_k, \xi))ds\,.$$

Then we have

(2.25) $$\begin{cases} \partial_t\phi_k(t, x, \xi)-a_{k,R_2}(t, x, \nabla_x\phi_k(t, x, \xi))=0 \qquad \text{on } [0, k]\times\boldsymbol{R}^{2n} \\ \phi_k(0, x, \xi)=x\xi\,, \end{cases}$$

(2.26) $$\begin{cases} \nabla_x\phi_k(t, x, \xi)=p_k(t, y_k(t, x, \xi), \xi) \\ \nabla_\xi\phi_k(t, x, \xi)=y_k(t, x, \xi)\,. \end{cases}$$

Thus the proposition follows from Lemma 2.6. Q.E.D.

2.3. An outgoing phase function.

PROPOSITION 2.7. *Assume* (A.I) *and* (A.II). *Then there exists a real-valued* C^∞-*function* $\phi_+(t, y, \eta)$ *on* $[0, \infty)\times\boldsymbol{R}^{2n}$ *such that*

(2.27) $$\phi_+(0, y, \eta)=y\eta$$

(2.28) $$\partial_t\phi_+(t, y, \eta)+a(\nabla_\eta\phi_+(t, y, \eta), \eta)=0 \qquad \textit{on } \Omega_+(y)$$
$$=\{(t, p(t, y, \xi))\,;\, t\geqq 0,\ \{y, \xi\}\in\Gamma_+(\sigma_0, R_2, 2R_0)\} \quad \textit{with } |y|\geqq R_2\,,$$

where R_2 *is some constant greater than* R_1, *and*

$$\{\nabla_\eta \phi_+(t, y, \eta), \nabla_y \phi_+(t, y, \eta)\}|_{\eta=p(t,y,\xi)} = \{q(t, y, \xi), \xi\} \tag{2.29}$$

for $t \geqq 0$ *and* $\{y, \xi\} \in \Gamma_+(\sigma_0, R_2, 2R_0)$. *Furthermore,* ϕ_+ *satisfies the following estimates on* $[0, \infty) \times \boldsymbol{R}^{2n}$:

$$|\nabla_y \phi_+(t, y, \eta) - \eta| \leqq \langle \eta \rangle / 9, \tag{2.30}$$

$$|\nabla_\eta \nabla_y \phi_+(t, y, \eta) - I| \leqq 1/9, \tag{2.31}$$

$$|\partial_y^\alpha \partial_\eta^\beta [\phi_+(t, y, \eta) - y\eta + tb(\eta)]| \leqq C_{\alpha\beta} R_2^{-\rho-|\alpha|+1} \langle \eta \rangle^{1-|\beta|}, \quad |\alpha+\beta| > 0. \tag{2.32}$$

REMARK. It is sufficient for this proposition to hold that $\rho > 0$.

PROOF. The construction of $\phi_+(t, y, \eta)$ is similar to that of $\phi_k(t, x, \xi)$, and so we give only a sketch.

In view of Lemma 2.1 there exists $\delta_1 > 0$ such that $|q(t, y, \xi)| \geqq \delta_1(|y| + t)$ for $t \geqq 0$ and $\{y, \xi\} \in \Gamma_+(\sigma_0, R, 2R_0)$, where $R \geqq R_1$. With χ_0 being the C^∞-function in § 2.2, put

$$a_{+,R}(t, x, \xi) = b(\xi) + \chi_0(2x/\delta_1(R+t))c(x, \xi). \tag{2.33}$$

Denote by $\{q_+(t, y, \xi), p_+(t, y, \xi)\}$ a bicharacteristic for the Hamiltonian $a_{+,R}(t, x, \xi)$. It is easily seen that

$$\{q_+(t, y, \xi), p_+(t, y, \xi)\} = \{q(t, y, \xi), p(t, y, \xi)\} \tag{2.34}$$

for $t \geqq 0$ and $\{y, \xi\} \in \Gamma_+(\sigma_0, R, 2R_0)$. We can show the following lemma by an argument similar to that in the proof of Lemma 2.6.

LEMMA 2.8. *For any multi-indices* α *and* β *there exists a constant* $C_{\alpha\beta}$ *such that for all* $(t, y, \xi) \in [0, \infty) \times \boldsymbol{R}^{2n}$

$$\begin{cases} |\partial_y^\alpha \partial_\xi^\beta [q_+(t, y, \xi) - (y - t\nabla_\xi b(\xi))]| \leqq C_{\alpha\beta} R^{-\rho} \langle t \rangle \langle \xi \rangle^{-|\beta|} \\ |\partial_y^\alpha \partial_\xi^\beta (p_+(t, y, \xi) - \xi)| \leqq C_{\alpha\beta} R^{-\rho} \langle \xi \rangle^{1-|\beta|}. \end{cases} \tag{2.35}$$

In particular,

$$|\nabla_\xi p_+(t, y, \xi) - I| \leqq C_0 R^{-\rho}. \tag{2.36}$$

REMARK. An estimate similar to (2.35) is given in [**1**, Lemma 3.7].

Now, choose R_2 such that $C_0 R_2^{-\rho} < 1/10$. By (2.36), the mapping: $\xi \to p_+(t, y, \xi)$ on $\boldsymbol{R}^n$ is a bijection. Let $\xi_+(t, y, \eta)$ be the solution of the equation $\eta = p_+(t, y, \xi)$. Define $\phi_+(t, y, \eta)$ by

$$\begin{aligned} \phi_+(t, y, \eta) &= y \cdot \xi_+(t, y, \eta) \\ &\quad - \int_0^t \{a_{+,R_2} - x \cdot \nabla_x a_{+,R_2}\}(s, q_+(s, y, \xi_+), p_+(s, t, \xi_+)) ds. \end{aligned} \tag{2.37}$$

Then ϕ_+ is the desired phase function having the properties in Proposition 2.7 and satisfying

$$
(2.38)\qquad \begin{cases} \partial_t\phi_+(t, y, \eta)+a_{+,R_2}(t, \nabla_\eta\phi_+(t, y, \eta), \eta)=0 & \text{on } [0, \infty)\times \boldsymbol{R}^{2n} \\ \phi_+(0, y, \eta)=y\eta\,, \end{cases}
$$

$$
(2.39)\qquad \begin{cases} \nabla_y\phi_+(t. y, \eta)=\xi_+(t, y, \eta) \\ \nabla_\eta\phi_+(t. y, \eta)=q_+(t, y, \xi_+(t, y, \eta))\,. \end{cases}
$$

Q.E.D.

§3. Approximate propagators.

The purpose of this section is to construct approximate propagators for the hyperbolic operator $L=D_t-A(X, D_x)$. Here $A(X, D_x)$ is the pseudo-differential operator satisfying (A.I)$\sim$(A.III). Namely, this section is devoted to the proof of the following theorem.

THEOREM 3.1. *For any R sufficiently large there exists a function $F(t)$ on $[0, \infty)$ to $B(s, -s)$, $s>1$, having the following properties:*

(i) *$F(0)=$identity.*

(ii) *For any $\varepsilon>0$ and $s\geqq 0$ there exists a constant C_R depending on R such that*

$$
(3.1)\qquad \|F(t)\|_{B(s,-s)}\leqq C_R\langle t\rangle^{-s+\varepsilon}\,.
$$

(iii) *$LF(t)=G_1(t)+G_2(t)$. Here for any $\varepsilon>0$ and $s, s'\in \boldsymbol{R}^1$ there exists a constant C independent of R such that*

$$
(3.2)\qquad \|G_1(t)\|_{B(0,s;1,s')}\leqq C\langle R+t\rangle^{-s+s'+\varepsilon}\,;
$$

and for any $\varepsilon>0$, $s>1$, $s'<\rho$ there exists a constant C'_R depending on R such that

$$
(3.3)\qquad \|G_2(t)\|_{B(0,s;2,s')}\leqq C'_R\langle t\rangle^{-\min(s,\rho-s')+\varepsilon}\,.
$$

3.1. Outgoing approximate propagators. Let χ_0 be a C^∞-function on $\boldsymbol{R}^n$ such that $\chi_0(x)=1$ for $|x|\geqq 2$ and $\chi_0(x)=0$ for $|x|\leqq 1$. Let $-1<\sigma_0<\sigma'_0<\sigma_1<0$, where σ_0, σ'_0, and σ_1 be the constants fixed before Lemma 2.3, and h_+ be a C^∞-function on $\boldsymbol{R}^1$ such that $h_+(\sigma)=1$ for $\sigma\geqq\sigma'_0$ and $h_+(\sigma)=0$ for $\sigma\leqq\sigma_0$. Put

$$
(3.4)\qquad \psi_+(\xi, y)=\chi_0(\xi/2R_0)\chi_0(y/R_2)h_+([[y, -\nabla_\xi a(y, \xi)]])\,,
$$

where R_2 is the constant appearing in Propositions 2.5 and 2.7. Define an outgoing approximate propagator $F_+(t)$, $t\geqq 0$, by

$$
(3.5)\qquad F_+(t)u(x)=\chi_0(4x/\delta_1(R_2+t))\iint e^{i(x\xi-\phi_+(t,y,\xi))}f_+(t, \xi, y)u(y)dyd\xi\,,
$$

$$
f_+(0, \xi, y)=\psi_+(\xi, y)\,,
$$

where $f_+(t, \xi, y)$ is determined below by solving transport equations. By Theorem C in [2], $A(X, D_x)=\tilde{A}(D_x, X')$ with

(3.6) $$\tilde{A}(\xi, y)=\iint e^{-iz\eta}A(y+z, \xi-\eta)dzd\eta\sim\sum_{l=-\infty}^{1}\tilde{a}_l(\xi, y),$$

where $\tilde{a}_1(\xi, y)=a(y, \xi)$ and

(3.7) $$\tilde{a}_l(\xi, y)=\sum_{|\alpha|=1-l}\frac{i^{|\alpha|}}{\alpha!}\partial_y^\alpha\partial_\xi^\alpha a(y, \xi)+\sum_{|\alpha|=-l}\frac{i^{|\alpha|}}{\alpha!}\partial_y^\alpha\partial_\xi^\alpha d(y, \xi)$$

for $l\leqq 0$. We see from (2.28) and the definition of ϕ_+ that

$$\partial_t\phi_+(t, y, \eta)+a(\nabla_\xi\phi_+(t, y, \eta), \eta)=0 \qquad \text{on the set}$$
$$\{(t, y, p(t, y, \xi));\ t\geqq 0,\ \{y, \xi\}\in \mathrm{Supp}\,\phi_+\}.$$

Thus, by Theorem 2.3 in [2], the transport equations to be solved are

(3.8) $$\partial_t f_+^\nu(t, \eta, y)+\sum_{|\alpha|=1}\partial_x^\alpha a(\nabla_\eta\phi_+(t, y, \eta), \eta)\partial_\eta^\alpha f_+^\nu(t, \eta, y)$$
$$-\Big[\frac{1}{2}\sum_{|\alpha|=2}\partial_x^\alpha a(\nabla_\eta\phi_+(t, y, \eta), \eta)\partial_\eta^\alpha\phi_+(t, y, \eta)$$
$$+i\tilde{a}_0(\eta, \nabla_\eta\phi_+(t, y, \eta))\Big]f_+^\nu(t, \eta, y)=g_+^\nu(t, \eta, y),$$

(3.9) $$f_+^0(0, \eta, y)=\phi_+(\eta, y),\quad f_+^\nu(0, \eta, y)=0 \qquad \text{for } \nu<0,$$

where $g_+^0(t, \eta, y)=0$, and for $\nu<0$

(3.10) $$g_+^\nu(t, \eta, y)=-\sum_{l=\nu}^{1}\sum_{|\alpha|=l+1}^{l-\nu}\frac{i^{|\alpha|+1}}{\alpha!}$$
$$\times\partial_\xi^\alpha[\partial_y^\alpha\tilde{a}_l(\eta, \tilde{\nabla}_\eta\phi_+(t, y;\eta, \xi))f_+^{\nu-l+|\alpha|}(t, \xi, y)]|_{\xi=\eta},$$

(3.11) $$\tilde{\nabla}_\eta\phi_+(t, y;\eta, \xi)=\int_0^1\nabla_\eta\phi_+(t, y, \theta(\eta-\xi)+\xi)d\theta.$$

Denoting $\{Q(\tau), P(\tau)\}=\{q(\tau, y, \xi_+(t, y, \eta)), p(\tau, y, \xi_+(t, y, \eta))\}$ with $\xi_+(t, y, \eta)$ being the solution of $\eta=p(t, y, \xi)$, we have

(3.12) $$f_+^0(t, \eta, y)=\phi_+(\xi_+(t, y, \eta), y)$$
$$\times\exp\Big\{\int_0^t\Big[\frac{1}{2}\sum_{|\alpha|=2}\partial_x^\alpha a(Q(\tau), P(\tau))\partial_\eta^\alpha\phi_+(t, y, P(\tau))+i\tilde{a}_0(P(\tau), Q(\tau))\Big]d\tau\Big\},$$

(3.13) $$f_+^{-1}(t, \eta, y)=\int_0^t\exp\Big\{\int_s^t\Big[\frac{1}{2}\sum_{|\alpha|=2}\partial_x^\alpha a(Q(\tau), P(\tau))\partial_\eta^\alpha\phi_+(t, y, P(\tau))$$
$$+i\tilde{a}_0(P(\tau), Q(\tau))\Big]d\tau\Big\}g_+^{-1}(s, P(s), y)ds,$$

$$g_+^{-1}(s, \eta, y)=-\sum_{l=-1}^{1}\sum_{|\alpha|=l+1}\frac{i^{|\alpha|+1}}{\alpha!}\partial_\xi^\alpha[\partial_y^\alpha\tilde{a}_l(\eta, \tilde{\nabla}_\eta\phi_+(t, y;\eta, \xi))f_+^0(t, y, \xi)]|_{\xi=\eta}.$$

Now we can define the symbol $f_+(t, \xi, y)$ of the outgoing approximate propagator (3.5) by

$$f_+(t, \xi, y) = f_+^0(t, \xi, y) + f_+^{-1}(t, \xi, y) . \tag{3.14}$$

PROPOSITION 3.2. *The following estimates hold for all* $t \geqq 0$:

$$\|F_+(t)\|_{B(0,-s)} \leqq C_s \langle t \rangle^{-s}, \qquad s \geqq 0 \tag{3.15}$$

$$\|LF_+(t)\|_{B(0,0;2,s)} \leqq C_s \langle t \rangle^{-\rho+s}, \qquad s \leqq \rho , \tag{3.16}$$

where C_s *is a constant independent of* R.

For the proof we prepare a lemma.

LEMMA 3.3. *Let* ϕ *be a real-valued function on* $\boldsymbol{R}^{2n}$ *such that for some* $\varepsilon_0 < 1$

$$|\nabla_x \phi(x, \xi) - \xi| \leqq \varepsilon_0 \langle \xi \rangle, \quad |\nabla_x \nabla_\xi \phi(x, \xi) - I| \leqq \varepsilon_0 , \tag{3.17}$$

and for some $M < \infty$

$$(1 + \langle \xi \rangle^{|\beta|-1} |\partial_x^\alpha \partial_\xi^\beta \phi(x, \xi)|)^{n+2} \leqq M, \ 1 \leqq |\alpha| \leqq 2, \ |\beta| \leqq n+2 . \tag{3.18}$$

Let $p(x, \xi)$ *be a function such that*

$$\langle \xi \rangle^{|\beta|} |\partial_{x,y}^\alpha \partial_\xi^\beta [p(x, \xi) \bar{p}(y, \xi)]| \leqq M, \ |\alpha| \leqq 1, \ |\beta| \leqq n+1 . \tag{3.19}$$

Let P_ϕ *be the Fourier integral operator with phase function* ϕ *and symbol* p, *and* P_ϕ^* *be the adjoint operator of* P_ϕ. *Then for every* $\varepsilon > 0$ *there exists a constant* C_ε *such that*

$$\|P_\phi\|_{B(0,0)} = \|P_\phi^*\|_{B(0,0)} \leqq C_\varepsilon (1 + M^\varepsilon) \|p(x, \xi)\|_{L_\infty(\boldsymbol{R}^{2n})} . \tag{3.20}$$

PROOF. Since $B(0, 0)$-norm of P_ϕ is obviously equal to that of P_ϕ^*, we have only to estimate the $B(0, 0)$-norm $\|P_\phi^*\|$ of P_ϕ^*. For $u \in \mathcal{S}$, we have

$$F[P_\phi^* u](\xi) = \int e^{-i\phi(y,\xi)} \bar{p}(y, \xi) u(y) dy .$$

Thus

$$\|P_\phi^* u\|^2 = \|F[P_\phi^* u]\|^2 / (2\pi)^n = \int [Qu](z) \bar{u}(z) dz , \tag{3.21}$$

$$[Qu](z) = \iint e^{i(\phi(z,\xi) - \phi(y,\xi))} \bar{p}(y, \xi) p(z, \xi) u(y) dy d\xi . \tag{3.22}$$

Writing $\tilde{\nabla}_x \phi(z, \xi, y) = \int_0^1 \nabla_x \phi(y + \theta(z-y), \xi) d\theta$, we have that $\phi(z, \xi) - \phi(y, \xi) = (z-y) \tilde{\nabla}_x \phi(z, \xi, y)$. (3.17) shows that the mapping: $\xi \to \eta = \tilde{\nabla}_x \phi(z, \xi, y)$ is a bijection and there exists the inverse mapping $\xi(\eta)$ such that $|\partial_\eta \xi(\eta) - I| \leqq 1/(1-\varepsilon_0)$. Change the variable ξ to η. Then (3.22) becomes

$$(3.23)\qquad Qu(z)=\iint e^{i(z-y)\eta}q(z, \eta, y)u(y)dyd\eta ,$$

$$q(z, \eta, y)=p(z, \xi(\eta))\bar{p}(y, \xi(\eta))|\det[\partial_\eta\xi(\eta)]|.$$

Denote by R a pseudo-differential operator with symbol $r(z, \eta, y)=q(M^2z, M^{-2}\eta, M^2y)$. Since $M^n[Qu](M^2z)=Rv(z)$ with $v(y)=M^nu(M^2y)$, we obtain that $\|Q\|=\|R\|$. By definition of $r(z, \eta, y)$, there exists a constant C depending only on ε_0 such that for $l=0$ or 1

$$\langle\eta\rangle^{|\beta|}|\partial_{z,y}^\alpha\partial_\eta^\beta r(z, \eta, y)|\leqq CM^{4l}\|p\|_\infty^{2(1-l)}, \qquad |\alpha|\leqq l,\ |\beta|\leqq n+1 .$$

Thus, $|\langle\eta\rangle^{|\beta|}[r^{(\beta)}(z, \eta, y)-r^{(\beta)}(z', \eta, y')]|\leqq CM^{4\varepsilon}\|p\|_\infty^{2(1-\varepsilon)}(|z-z'|^\varepsilon+|y-y'|^\varepsilon)$. Hence, Theorems 2.1 and 2.2 in [**3**] give

$$\|R\|\leqq C_\varepsilon(1+M^{4\varepsilon})\|p\|_\infty^2 .$$

This implies (3.20). Q.E.D.

PROOF OF PROPOSITION 3.2. (2.32) *shows that for some constant C independent of t*

$$|\partial_y^\alpha\partial_\xi^\beta\phi_+(t, y, \xi)|\leqq C\langle\xi\rangle^{1-|\beta|}, \qquad t\geqq 0,\ 1\leqq|\alpha|\leqq 2,\ |\beta|\leqq n+2 .$$

By (3.12)$\sim$(3.13), $|\partial_y^\alpha\partial_\xi^\beta f_+(t, \xi, y)|\leqq C\langle\xi\rangle^{-|\beta|}$ for any $t\geqq 0$, $|\alpha|\leqq 1$, $|\beta|\leqq n+1$. Since $F_+(t)=\chi_0(4x/\delta_1(R_2+t))P_{\phi_+}^*$ with $p(x, \xi)=f_+(t, \xi, x)$, Lemma 3.3 yields (3.15).

It remains to show (3.16). With $X(t)=\chi_0(4x/\delta_1(R_2+t))\cdot$, we obtain that

$$(3.24)\qquad \begin{aligned} LF_+(t)u(x)&=X(t)\iint e^{i(x\xi-\phi_+(t,y,\xi))}g_+(t, \xi, y)u(y)dyd\xi \\ &\quad +[L, X(t)]\iint e^{i(x\xi-\phi_+(t,y,\xi))}f_+(t, \xi, y)u(y)dyd\xi \\ &\equiv[G_{+,1}(t)+G_{+,2}(t)]u(x) , \end{aligned}$$

where $[L, X(t)]$ is the commutator of L and $X(t)$ and $g_+(t, \xi, y)$ satisfies

$$(3.25)\qquad |\partial_y^\alpha\partial_\xi^\beta g_+(t, \xi, y)|\leqq C_{\alpha\beta}(t+\langle y\rangle)^{-\rho}\langle\xi\rangle^{-2-|\beta|}$$

for any α and β. Since

$$\begin{aligned} &(-ix)^\alpha\iint e^{i(x\xi-\phi_+(t,y,\xi))}g_+(t, \xi, y)u(y)dyd\xi \\ &=\iint e^{ix\xi}\partial_\xi^\alpha[e^{-i\phi_+(t,y,\xi)}g_+(t, \xi, y)]u(y)dyd\xi \end{aligned}$$

for any α, (2.32) and (3.25) together with the interpolation theorem yield

$$\|G_{+,1}(t)\|_{B(0,0;2,s)}\leqq C\langle t\rangle^{-\rho+s}, \qquad s\leqq\rho .$$

The estimate for $G_{+,2}(t)$ can be obtained using that $D^{\alpha}_{x,t}[\chi_0(4x/\delta_1(R_2+t))]=0$ for $|x|\geqq(\delta_1/2)(R_2+t)$ if $|\alpha|>0$, and that for any $|x|\leqq(3\delta_1/4)(R_2+t)$ and $\{y, \xi\}\in\Gamma_+(\sigma_0, R_2, 2R_0)$

$$|x-\nabla_{\xi}\phi_+(t, y, \xi)|\geqq\delta_2(t+\langle y\rangle),$$

where δ_2 is a positive constant. Hence we get (3.16). Q.E.D.

3.2. Incoming approximate propagators. Let χ_1 be a C^{∞}-function on $\boldsymbol{R}^n$ such that $\chi_1(x)=1$ for $|x|\leqq 1$ and $\chi_1(x)=0$ for $|x|\geqq 2$. For $\varepsilon>0$, let Z_{ε} be the set of all lattice points in $\boldsymbol{R}^n$ multiplied by ε. For $R\geqq R_2$ and a non-negative integer k, put

$$I_k=\{(\alpha, \beta)\in Z^2_{\varepsilon};\ |\beta|=|\alpha|,\ k\leqq\tau(\alpha, (\Xi_1/|\alpha|)\beta, R+1)<k+1\}, \tag{3.26}$$

where τ is the incoming time (see Definition 2.4). Let

$$\tilde{\psi}_+(x, \xi)=\sum_{|\alpha|\leqq 2}D^{\alpha}_x\partial^{\alpha}_{\xi}\psi_+(\xi, x)/\alpha!, \tag{3.27}$$

where $\psi_+(\xi, x)$ is the function defined by (3.4). For a nonnegative integer k, set

$$\begin{aligned}\psi_k(x, \xi)=&(1-\tilde{\psi}_+(x, \xi))\chi_0(\xi/\Xi_1)\Phi(x, \xi)^{-1}\\ &\times\sum_{(\alpha,\beta)\in I_k}\chi_1((x-\alpha)/\varepsilon n)\chi_1([|\alpha|/\varepsilon n][\xi/|\xi|-\beta/|\alpha|]),\end{aligned} \tag{3.28}$$

$$\Phi(x, \xi)=\sum_{\alpha\in Z_{\varepsilon}}\sum_{|\beta|=|\alpha|}\chi_1((x-\alpha)/\varepsilon n)\chi_1([|\alpha|/\varepsilon n][\xi/|\xi|-\beta/|\alpha|]). \tag{3.29}$$

Definition 2.4, Lemmas 2.1 and 2.2 imply that there exists a positive constant C such that the points α in (3.28) are included in $\{|x|\leqq C(R+k)\}$ for any $k\geqq 0$ and R sufficiently large. Choosing another R_2 if necessary, we may and shall assume that $|\alpha|\leqq C(R+k)$ for any α in (3.28), $k\geqq 0$, and $R\geqq R_2$. Then we obtain that

$$\sum_{k\geqq 0}\psi_k(x, \xi)=(1-\tilde{\psi}_+(x, \xi))\chi_0(\xi/\Xi_1), \tag{3.30}$$

$$|\partial^{\alpha}_x\partial^{\beta}_{\xi}\psi_k(x, \xi)|\leqq C_{\alpha\beta}\langle R+k\rangle^{|\beta|}\langle\xi\rangle^{-|\beta|},\qquad |\alpha+\beta|\geqq 0. \tag{3.31}$$

Let $\sigma'_0<\sigma'_1<\sigma_1<\sigma''_1<0$. Choosing ε sufficiently small we obtain that for any $\{y, \xi\}\in\operatorname{Supp}\psi_k$

$$\{q(t, y, \xi), p(t, y, \xi)\}\in\Gamma_-(\sigma''_1, R, 2R_0)\qquad\text{for } 0\leqq t\leqq k, \tag{3.32}$$

$$\begin{aligned}&\{q(k, y, \xi), p(k, y, \xi)\}\in\Gamma_+(\sigma'_1, R, 2R_0)\qquad\text{or}\\ &|q(k, y, \xi)|\leqq R+\sup_{x,\xi}|\nabla_{\xi}a(x, \xi)|.\end{aligned} \tag{3.33}$$

Furthermore, for $k\geqq 1$

$$\mathrm{Supp}\,\phi_k \subset \{\{x, \xi\}\,;\ |x| \geqq \delta_1(R+k)\}. \tag{3.34}$$

Now, we define k-th incoming approximate propagator $F_k(t)$ for $0 \leqq t \leqq k$ by

$$F_k(t)u(x) = \iint e^{i(\phi_k(t,x,\xi)-y\xi)} f_k(t, x, \xi) u(y)\,dy\,d\xi, \tag{3.35}$$

$$f_k(t, x, \xi) = \phi_k(y_k(t, x, \xi), \xi) \tag{3.36}$$

$$\times \exp\left\{\int_0^t \left[\frac{1}{2} \sum_{|\alpha|=2} \partial_\xi^\alpha a(Q(\tau), P(\tau)) \partial_x^\alpha \phi_k(\tau, Q(\tau), \xi) + id(Q(\tau), P(\tau))\right] d\tau\right\},$$

where $y_k(t, x, \xi)$ is the solution of $x = q(t, y, \xi)$ and

$$\{Q(\tau), P(\tau)\} = \{q(\tau, y_k(t, x, \xi), \xi), p(\tau, y_k(t, x, \xi), \xi)\}.$$

Furthermore, we put

$$\boldsymbol{F}_k(t) = \sum_{j \geqq k} F_j(t), \qquad 0 \leqq t \leqq k. \tag{3.37}$$

By (2.16),

$$\boldsymbol{F}_k(t)u(x) = \iint e^{i(\phi_k(t,x,\xi)-y\xi)} \sum_{j \geqq k} f_j(t, x, \xi) u(y)\,dy\,d\xi. \tag{3.38}$$

We see that

$$\sum_{j \geqq k} \phi_j(x, \xi) = (1 - \tilde{\phi}_+(x, \xi)) \chi_0(\xi/\Xi_1) \qquad \text{for } |x| \geqq C(R+k), \tag{3.39}$$

where C is a constant independent of k and R. Choose M_0 so large that $M_0 \geqq \sup|\nabla_\xi a(x, \xi)| + 1$ and $\phi_0(x, \xi) = 0$ for $|x| \geqq R + M_0$. Put

$$X_1 = \chi_1(x/(R+M_0))\cdot \quad \text{and} \quad Y_k = \chi_0(4x/\delta_1(R+k))\cdot \tag{3.40}$$

Then we have the following proposition.

PROPOSITION 3.4. (i) *For any $\varepsilon > 0$ and $s, s' \in \boldsymbol{R}^1$ there exists a constant C such that for all $0 \leqq t \leqq k$ and R sufficiently large*

$$\|F_k(t) Y_k\|_{B(s,0)} \leqq C\langle R+k\rangle^{-s+\varepsilon}, \tag{3.41}$$

$$\|L F_k(t) Y_k\|_{B(0,s;1,s')} \leqq C\langle R+k\rangle^{-s+s'+\varepsilon}. \tag{3.42}$$

(ii) *The same estimates as (3.41) and (3.42) hold with $F_k(t)$ replaced by $\boldsymbol{F}_k(t)$.*

(iii) *For any ε and $s, s' \in \boldsymbol{R}^1$ there exists a constant C such that for all $0 \leqq t \leqq k+1$ and R sufficiently large*

$$\|F_{k+1}(t)(Y_{k+1} - Y_k)\|_{B(0,s;2,s')} \leqq C\langle R+k\rangle^{-s+s'+\varepsilon}, \tag{3.43}$$

$$\|(1-\Psi_+)(1-X_1) F_k(k) Y_k\|_{B(0,s;2,s')} \leqq C\langle R+k\rangle^{-s+s'+\varepsilon} \tag{3.44}$$

where $\Psi_+ = F_+(0) = \chi_0(4X/\delta_1 R_2)\phi_+(D_x, X')$.

PROOF. The proposition can be shown in the same way as Proposition 3.2 using (3.30)$\sim$(3.40), Lemma 3.3, and Proposition 2.5.

3.3. Proof of Theorem 3.1. Let $E(t)$ be the fundamental solution of L: $LE(t)=0$ and $E(0)=$identity. Let $\Phi(t)$ be a C^∞-function on $\boldsymbol{R}^1$ such that $\Phi(t)=1$ for $t\leqq 1/4$ and $\Phi(t)=0$ for $t\geqq 1/2$. In view of the assumption (A.III) choose a positive integer T depending only on R such that for all $t\geqq T$ and $\{y, \xi\}$ with $|y|\leqq 2(R+M_0)$ and $|\xi|\geqq R_0$

$$|q(t, y, \xi)|\geqq 3R_2 \text{ and } [[q(t, y, \xi), -\nabla_\xi a(q(t, y, \xi), p(t, y, \xi))]]\geqq\sigma_1. \tag{3.45}$$

We define $F(t)$ for $0\leqq t<1$ by

$$\begin{aligned} F(t)=&F_+(t)+E(t)F_0(0)Y_0+\boldsymbol{F}_1(t)Y_1\\ &+\Phi(t)[(1-\tilde{\varphi}_+(X, D_x))(1-\chi_0(D_x/\Xi_1))+(\tilde{\varphi}_+(X, D_x)-\Psi_+)\\ &+\phi_0(X, D_x)(1-Y_0)+\sum_{j\geqq 1}\phi_j(X, D_x)(1-Y_1)]\,. \end{aligned}\tag{3.46}$$

For $k\leqq t<k+1$ (k : a positive integer),

$$\begin{aligned} F(t)=&F_+(t)+\sum_{j=1}^{k}F_+(t-j)(1-X_1)F_j(j)Y_j\\ &+\sum_{j=0}^{k-T}F_+(t-j-T)E(T)X_1F_j(j)Y_j+\sum_{j=k-T+1}^{k}E(t-j)X_1F_j(j)Y_j\\ &+\boldsymbol{F}_{k+1}(t)Y_{k+1}+\Phi(t-k)[(1-\Psi_+)E(T)X_1F_{k-T}(k-T)Y_{k-T}\\ &+(1-\Psi_+)(1-X_1)F_k(k)Y_k+\boldsymbol{F}_{k+1}(t)(Y_k-Y_{k+1})]\,, \end{aligned}\tag{3.47}$$

where $F_j=0$ for $j<0$. Noting that $X_1F_0(0)=\phi_0(X, D_x)=F_0(0)$, we see that $F(t)$ is a continuous function on $[0, \infty)$ such that $F(0)=$identity and $LF(t)$ has a jump at $t=k$, where k is a positive integer. Theorem 3.1 now follows from Propositions 3.2 and 3.4, (3.46), (3.47), and the following lemma.

LEMMA 3.5. *Let $s\in\boldsymbol{R}^1$ and $T>0$. Then $\|E(t)\|_{B(s,s)}$ and $\|(1-\Psi_+)E(T)X_1\|_{B(0,s;3,s)}$ are bounded on $[0, T]$.*

PROOF. By Theorem 3.2 in [**2**], $E(t)$ can be represented by a Fourier integral operator for $0\leqq t\leqq\delta$, where δ is a sufficiently small positive number. Since $E(t)=[E(\delta)]^jE(t-j\delta)$ for $j\delta\leqq t<(j+1)\delta$, the lemma can be shown in the same way as Proposition 3.4. Q.E.D.

§4. Proof of Theorem 1.

In this section we complete the proof of Theorem 1. To this end we make some preparations.

PROPOSITION 4.1. *Let $R_0(z)=(z-b(D_x))^{-1}$ and $s-1/2=k+\theta$ with k being a* *non-negative integer and* $0<\theta\leqq 1$. *Then there exists a constant* $N>0$ *such that* $R_0(z)$ *on* $\{z;\pm\operatorname{Im} z\gg 1\}$ *can be extended to a k-times continuously differentiable function on the set* $\boldsymbol{N}_\pm=\{z;\pm\operatorname{Im} z\geqq 0,\ |z|>N\}$ *to* $B(s,-s)$ *having the properties:*

(i) *The derivatives* $R_0^{(j)}(z)$, $j=0,\cdots,k$, *are bounded.*

(ii) *When* $0<\theta<1$, *the k-th derivative is uniformly Hölder continuous with exponent* θ; *and when* $\theta=1$, *there exists a constant C such that* $\|\Delta_h^2 R_0^{(k)}(z)\|\leqq Ch$ *for any h and z with* $z,\ z+h\in\boldsymbol{N}_\pm$ *and* $0<h<1$, *where* Δ_h^2 *is the second difference operator.*

Furthermore,

$$\left\|\int_{-\infty}^{\infty}\chi(\lambda)R_0(\lambda\pm i0)e^{it\lambda}d\lambda\right\|_{B(s,-s)}\leqq C_s\langle t\rangle^{-s},\qquad t\in\boldsymbol{R}^1,\ s\geqq 0, \tag{4.1}$$

where $\chi(\lambda)$ *is a* C^∞*-function on* $\boldsymbol{R}^1$ *such that* $\chi(\lambda)=1$ *for* $|\lambda|>N+1$ *and* $\operatorname{Supp}\chi\subset\{|\lambda|>N\}$.

PROOF. The first half can be shown along the line given in [7, § 4]. The estimate (4.1) for $s>1$ is deduced from the estimate

$$\|e^{itb(D_x)}\psi(D_x)\|_{B(s,-s)}\leqq C_s\langle t\rangle^{-s},\qquad t\in\boldsymbol{R}^1,\ s\geqq 0, \tag{4.2}$$

where $\psi(\xi)$ is a C^∞-function on $\boldsymbol{R}^n$ such that $\psi(\xi)=1$ on $\{\xi;|b(\xi)|\geqq N+2\}$ and $\psi(\xi)=0$ on $\{\xi;|b(\xi)|\leqq N+1\}$. We have

$$\int_{-\infty}^{\infty}R_0(\lambda\pm i0)e^{it\lambda}d\lambda=\mp 2\pi iH(\mp t)e^{itb(D_x)},$$

where $H(s)=1$ for $s\geqq 0$ and $H(s)=0$ for $s<0$. This shows (4.1) for $s=0$, which together with the interpolation theorem implies (4.1). Q.E.D.

LEMMA 4.2. *Let* $1/2<s<\rho-3/2$ *and* $A_R=b(D_x)+\chi_0(X/R)d(X,D_x)$ *with R sufficiently large depending on s. Let l be a nonnegative integer. Then the first half of Proposition* 4.1 *holds with* $R_0(z)$ *replaced by* $\langle D\rangle^l(z-A_R)^{-1}\langle D\rangle^{-l}$, *where* $\langle D\rangle^2=1+D_1^2+\cdots+D_n^2$. *Furthermore, there exists a constant C such that for all* $f\in L_2^s$

$$\sup_{T>0}\int_{\Omega_T}\left[\langle t\rangle^{k+\theta}\left\|\int_{-\infty}^{\infty}\chi(\lambda)\langle D\rangle^l(\lambda\pm i0-A_R)^{-1}\langle D\rangle^{-l}fe^{it\lambda}d\lambda\right\|_{-s}\right]^2dt\leqq C^2\|f\|_s^2, \tag{4.3}$$

where $\Omega_T=\{t;T\leqq\langle t\rangle<2T\}$.

If $\rho>3$, (4.1) *holds with* $R_0(\lambda\pm i0)$ *and* $s\geqq 0$ *replaced by* $\langle D\rangle^l(\lambda\pm i0-A_R)^{-1}\langle D\rangle^{-l}$ *and* $1<s<\rho-2$, *respectively.*

PROOF. Choosing R so large that

$$\|\chi_0(X/R)d(X,D_x)R_0(z)\|_{B(s,s)}\leqq 1/2\qquad\text{for all } z\in\boldsymbol{N}_\pm,$$

we have

$$(4.4)\qquad (z-A_R)^{-1}=\sum_{j=0}^{\infty} R_0(z)[\chi_0(X/R)d(X,\ D_x)R_0(z)]^j \quad \text{in} \quad B(s,\ -s).$$

By the Leibniz formula,

$$\|\langle D\rangle^l[\chi_0(X/R)d(X,\ D_x)R_0(z)]^j\langle D\rangle^{-l}\|_{B(s,s)}\leqq Cj^l 2^{-j}.$$

Thus we get the first half of the lemma, which implies (4.3).

Now assume that $\rho>3$. Choosing a C^∞-function ϕ on $\boldsymbol{R}^1$ such that $\phi(\lambda)=1$ on $\operatorname{Supp}\chi$ and $\operatorname{Supp}\phi\subset\{|\lambda|>N\}$, we put

$$(4.5)\qquad V(t)=\int\phi(\lambda)\chi_0(X/R)d(X,\ D_x)R_0(\lambda\pm i0)e^{it\lambda}d\lambda.$$

With $U(t)=\int\chi(\lambda)R_0(\lambda\pm i0)e^{it\lambda}d\lambda$, choose R so large that

$$(4.6)\qquad \|U(t)(*V(t))^j\|_{B(s,-s)}\leqq C2^{-j}\langle t\rangle^{-s}, \qquad j=1,\ 2,\ \cdots,$$

where $*$ denotes the convolution. Then we obtain that

$$(4.7)\qquad \int\chi(\lambda)\langle D\rangle^l(\lambda\pm i0-A_R)^{-1}\langle D\rangle^{-l}e^{it\lambda}d\lambda=\sum_{j=0}^{\infty}U(t)(*V(t))^j.$$

This shows the second half of the lemma. Q. E. D.

Now we can complete the proof of Theorem 1.

PROOF OF THEOREM 1. We shall show the theorem only for $\operatorname{Im}z\leqq 0$. Theorem 3.1 shows that

$$(4.8)\qquad (z-A)\hat{F}(z)=-i+\hat{G}(z) \quad \text{in} \quad B(s,\ -s),\ \ \operatorname{Im}z\leqq 0,$$

for $s>1$, where $\hat{F}(z)=\int_0^\infty F(t)e^{-itz}dt$ and $\hat{G}(z)=\hat{G}_1(z)+\hat{G}_2(z)$. We have from (4.8) that for z with $|z|\gg 1$ and $\operatorname{Im}z\leqq 0$

$$(4.9)\qquad (z-A)[\hat{F}(z)-(z-A_R)^{-1}\hat{G}(z)]=-i+(A-A_R)(z-A_R)^{-1}\hat{G}(z).$$

By (3.2), (A.II), and Lemma 4.2 we can choose R such that for all z with $|z|\gg 1$ and $\operatorname{Im}z\leqq 0$ and s in a compact subset of $(3/2,\ \rho-1/2)$

$$(4.10)\qquad \|(A-A_R)(z-A_R)^{-1}\hat{G}_1(z)\|_{B(s,s)}\leqq 1/4.$$

Fix such an R. Since

$$\begin{aligned}&(A-A_R)(z-A_R)^{-1}\hat{G}_2(z)\\ &=(z-2iN)^{-1}\{(A-A_R)[(2iN-A_R)^{-1}-(z-A_R)^{-1}](2iN-A_R)\langle D\rangle^{-2}\}\{\langle D\rangle^2\hat{G}_2(z)\},\end{aligned}$$

Lemma 4.2 and (3.3) yield

(4.11) $$\|(A-A_R)(z-A_R)^{-1}\hat{G}_2(z)\|_{B(s,s)}\leqq C|z|^{-1}.$$

Thus we can choose N such that

$$\|(A-A_R)(z-A_R)^{-1}\hat{G}(z)\|_{B(s,s)}\leqq 1/2, \quad |z|\geqq N \text{ and } \operatorname{Im} z\leqq 0.$$

Hence, for $z\in \boldsymbol{N}_-$

(4.12) $$(z-A)^{-1}=-i[\hat{F}(z)-(z-A_R)^{-1}\hat{G}(z)]\sum_{j=0}^{\infty}[i(A_R-A)(z-A_R)^{-1}\hat{G}(z)]^j.$$

This implies the first half of the theorem.

The second half of the theorem, (1.9), can be shown in the same way as that of Lemma 4.2. Q.E.D.

REMARK 4.3. The assumption (A.II) is redundant: It is sufficient for Theorem 1 to hold that (1.5) and (1.6) are satisfied for $|\alpha+\beta|\leqq 3n+11$.

§ 5. Proof of Theorem 2.

In this section we give a proof of Theorem 2.

LEMMA 5.1. *Assume the hypotheses of Theorem* 2. *Let* $s>3/2$. *If* (A.III) *does not hold, then there exists a constant* $\delta_0>0$ *such that for any* $T>0$ *there holds*

(5.1) $$\|e^{iAt}\|_{B(s,-s)}\geqq\delta_0$$

on $\{t;\, T-\delta_0\leqq t\leqq T\}$ *or* $\{t;\, -T\leqq t\leqq -T+\delta_0\}$.

PROOF. We see that there exists a C^∞-function $\phi(t, x, \xi)$ on $[-\delta, \delta]\times \boldsymbol{R}^{2n}$, where δ is some positive constant, such that

$$\partial_t\phi(t, x, \xi)-a(x, \nabla_x\phi(t, x, \xi))=0, \qquad \phi(0, x, \xi)=x\xi,$$

and for some $0<\varepsilon_0\leqq 1$ and $C_0>0$

$$|\nabla_x\phi(t, x, \xi)-\xi|\leqq(1-\varepsilon_0)|\xi|+C_0, \qquad |\nabla_x\nabla_\xi\phi(t, x, \xi)-I|\leqq 1-\varepsilon_0.$$

With $\tilde{J}(t, y, \xi)=\det[\partial_y q(t, y, \xi)]$ for $|t|\leqq\delta$, we have that

(5.2) $$\frac{d}{dt}\log\tilde{J}=-[\sum_{|\alpha|=2}\partial_\xi^\alpha a(x, \nabla_x\phi(t, x, \xi))\partial_x^\alpha\phi(t, x, \xi)$$
$$-\sum_{j=1}^{n}\partial_{x_j}\partial_{\xi_j}a(x, \nabla_x\phi(t, x, \xi))]_{x=q(t,y,\xi)}.$$

Using this we rewrite formulas similar to (3.35) and (3.36), and put

(5.3) $$F(t)u(x)=\iint e^{i(\phi(t,x,\xi)-y\xi)}f(t, x, \xi)u(y)dyd\xi,$$

(5.4) $$f(t,\ x,\ \xi)=J(t,\ x,\ \xi)^{-1/2}\exp\left[\int_0^t\left(\frac{1}{2}\sum_{j=1}^n\partial_{x_j}\partial_{\xi_j}a+id\right)(Q(\tau),\ P(\tau))dt\right],$$

$$J(t,\ x,\ \xi)=\tilde{J}(t,\ y(t,\ x,\ \xi),\ \xi),$$

$$\{Q(\tau),\ P(\tau)\}=\{q(\tau,\ y(t,\ x,\ \xi),\ \xi),\ p(\tau,\ y(t,\ x,\ \xi),\ \xi)\},$$

where $y(t,\ x,\ \xi)$ is the solution of $x=q(t,\ y,\ \xi)$. Then we obtain that for some $g\in S^{-1}$

(5.5) $$LF(t)u(x)=\iint e^{i(\phi(t,x,\xi)-y\xi)}g(t,\ x,\ \xi)u(y)dyd\xi.$$

Since (A.III) does not hold, there exists R with the following properties: For any $T>0$ there exists $\{y^0,\ \xi^0\}$ such that $|y^0|\leqq R$, $|\xi^0|=R_0$, and $|q(t,\ y^0,\ \xi^0)|\leqq R$ for $0\leqq t\leqq T$. Choosing another point near $\{y^0,\ \xi^0\}$ if necessary, we may also assume that there exists a neighborhood U of $\{y^0,\ \xi^0\}$ such that for any $\{y,\ \xi\}\in U$

(5.6) $$|q(t,\ y,\ r\xi)|\leqq R \qquad \text{for } 0\leqq t\leqq T \text{ and } r\geqq 1,$$

(5.7) $$\lim_{r\to\infty}|a(y,\ r\xi)|=\lim_{r\to\infty}|b(r\xi)|=\infty.$$

We write $g(t)=\left(\frac{1}{2}\sum_{j=1}^n\partial_{x_j}\partial_{\xi_j}a-\mathrm{Im}\,d\right)(q(t,\ y^0,\ \xi^0),\ p(t,\ y^0,\ \xi^0))$. Assuming that

(5.8) $$\exp\left[\int_0^T g(t)dt\right]\geqq 1,$$

we shall show (5.1) for $t\in[T-\delta_0,\ T]$. (When the left hand side of (5.8) is smaller than 1, (5.1) for $t\in[-T,\ -T+\delta_0]$ can be shown in the same way as below.) Let $\phi(x)$ be a C^∞-function on $\boldsymbol{R}^n$ such that $\phi(x)=1$ for $|x|\leqq 1$, $\phi(x)=0$ for $|x|\geqq 2$, and $\phi(x)\geqq 0$. Put

(5.9) $$h_j(x)=M_j\iint e^{i(x-y)\xi}\phi((\xi-j^2\xi^0)/j)\phi(j^3(y-y^0))dyd\xi,$$

where $j\gg 1$ and the constants M_j are chosen so that $\|h_j\|_s=1$. We see that

(5.10) $$\|h_j\|_r=M_j j^{-5/2}\{|\hat{\phi}(0)|\,\|\phi\|_0+O(j^{-1})\} \quad \text{as} \quad j\to\infty,\ 0\leqq r\leqq s,$$

(5.11) $$\|\langle D_x\rangle^{-1}h_j\|_0=O(j^{-1}) \quad \text{as} \quad j\to\infty.$$

Choosing j sufficiently large, we shall show that there exists $\delta_0>0$ independent of T such that

(5.12) $$\|e^{iAt}h_j\|_{-s}\geqq\delta_0 \qquad \text{for all} \quad t\in[T-\delta_0,\ T] \quad \text{and} \quad j\gg 1.$$

Choosing another δ if necessary, we assume that $K=T/\delta$ is an integer. We write $\{y^k, \xi^k\}=\{q(k\delta, y^0, \xi^0), p(k\delta, y^0, \xi^0)\}$, $k=0, 1, \cdots, K$. Put

$$\Psi_k(x, \xi)=\phi((x-y^k)/\varepsilon_k)\phi((\xi/|\xi|-\xi^k/|\xi^k|)/\varepsilon_k),$$

where $0<\varepsilon=\varepsilon_0<\varepsilon_1<\cdots<\varepsilon_K$ are chosen so that $\Psi_{k+1}(x, \xi)=1$ on $\{\{q(\delta, y, \xi), p(\delta, y, \xi)\}; \{y, \xi\}\in \mathrm{Supp}\Psi_k\}$. (5.5) and (5.11) imply that for any $0\leqq k\leqq K$

$$\|e^{iAk\delta}h_j-F(\delta)^k h_j\|_0=O(j^{-1}) \quad \text{as} \quad j\to\infty. \tag{5.13}$$

We obtain that for any k

$$\sup_{|l|\leqq 1}\|(1-\Psi_{k+1}(X, D_x))F(\delta)\Psi_k(X, D_x)\|_{B(H^l, H^{l+1})}\leqq C\varepsilon^{-M}, \tag{5.14}$$

$$\begin{aligned}&\|\langle D\rangle^{-1}\Psi_{k+1}(X, D_x)F(\delta)\Psi_k(X, D_x)F(\delta)\cdots\Psi_0(X, D_x)h_j\|_0\\ &\leqq C(j\varepsilon^{(k+1)M})^{-1},\end{aligned} \tag{5.15}$$

where C and M are constants independent of ε and j. (5.13)~(5.15) yield

$$\begin{aligned}e^{iA(k+1)\delta}h_j=&\Psi_{k+1}(X, D_x)F(\delta)\Psi_k(X, D_x)F(\delta)\cdots\Psi_0(X, D_x)h_j\\ &+O((j\varepsilon^{MK})^{-1}),\end{aligned} \tag{5.16}$$

for $\varepsilon j\geqq 2$. We have that for some $g_k\in S^{-1}$

$$\begin{aligned}&\Psi_{k+1}(X, D_x)F(\delta)\Psi_k(X, D_x)u(x)\\ &=\iint e^{i(\phi(\delta, x, \xi)-y\xi)}[f_k(x, \xi)+g_k(x, \xi)]u(y)dyd\xi,\\ &f_k(x, \xi)=f(\delta, x, \xi)\Psi_k(y(\delta, x, \xi), \xi).\end{aligned} \tag{5.17}$$

Using (5.3) and (5.4) we get

$$\begin{aligned}&\left\|\iint e^{i(\phi(\delta, x, \xi)-y\xi)}f_k(x, \xi)u(y)dyd\xi\right\|_0\\ &=\exp\left[\int_{k\delta}^{(k+1)\delta}g(t)dt\right]\left[\int\phi^2((\xi/|\xi|-\xi^k/|\xi^k|)/\varepsilon_k)|\hat{u}(\xi)|^2d\xi\right]^{1/2}\\ &\quad+O(\varepsilon_{k+1})\|u\|_0\end{aligned} \tag{5.18}$$

as $\varepsilon_{k+1}\to 0$. We see by definition and (5.10) that $\|h_j\|_0\geqq c$ for some c independent of j. Thus (5.16)~(5.18) yield

$$\|e^{iAT}h_j\|_{-s}\geqq c(R+1)^{-s}\exp\left[\int_0^T g(t)dt\right]+O(\varepsilon_K)+O((j\varepsilon^{MK})^{-1}).$$

Since $\{q(t, y^0, \xi^0), p(t, y^0, \xi^0)\}$ is uniformly continuous on $\boldsymbol{R}^1$, we obtain (5.1) by choosing j sufficiently large after choosing ε_K sufficiently small. Q. E. D.

PROOF OF THEOREM 2. Supposing (A.III) does not hold, we shall derive a contradiction. We treat only the case that (5.1) holds for all $t\in[T-\delta_0, T]$. Choose T so large that $\langle T-\delta_0\rangle^{k+\theta}\delta_0>2C_s$, where $k+\theta$ and C_s are the constants in (1.8). Let h_j be the function given by (5.9). Then (5.12) yields

$$\int_0^T(\langle t\rangle^{k+\theta}\|e^{iAt}h_j\|_{-s})^2dt\geqq 4C_s^2 \tag{5.19}$$

for $j\gg 1$. Choosing ε such that (5.7) holds for any $\{y, \xi\}$ with $|y-y^0|\leqq 2\varepsilon$ and $|\xi-\xi^0|\leqq 2\varepsilon$, put

$$\Phi_j(x, \xi)=\psi((x-y^0)/\varepsilon)\psi((\xi-j^2\xi^0)/2j)\,.$$

We have that $\Phi_j(X, D_x)h_j=h_j+O(j^{-1})$ in L_2^s as $j\to\infty$. For $|z|=N+1$ and $j\gg 1$, put

$$S_{j,z}(x, \xi)=(z-a(x, \xi))^{-1}\Phi_j(x, \xi)\,.$$

By (5.7), $\|S_{j,z}(X, D_x)\|_{B(0,0)}\leqq Cj^{-1}$ for any $|z|=N+1$. Simple calculations show that

$$(z-A)S_{j,z}(X, D_x)=\Phi_j(X, D_x)+T_{j,z}$$

for some $T_{j,z}$ with $\|T_{j,z}\|_{B(0,s)}\leqq Cj^{-1}$ for all $|z|=N+1$. Thus

$$\sup_{|z|=N+1}\|(z-A)^{-1}h_j\|_{-s}=O(j^{-1}) \qquad \text{as} \quad j\to\infty\,. \tag{5.20}$$

By (1.8),

$$e^{iAt}h_j=\frac{1}{2\pi i}\int_\gamma e^{izt}(z-A)^{-1}h_j dz\,,$$

where $\gamma=\{x\,;\,-\infty<x\leqq -N-1\}\cup\{(N+1)e^{i\theta}\,;\,-\pi<\theta\leqq 0\}\cup\{x\,;\,N+1<x<\infty\}$. Hence (5.20) and (1.8) imply that the left hand side of (5.19) is smaller than $2C_s^2$ for sufficiently large j. This is a contradiction. Q. E. D.

§6. The constant leading coefficient case.

In this section an improvement of the estimates in Theorem 1 is given for operators with constant leading coefficients. Consider the pseudo-differential operator

$$A(X, D_x)=b(D_x)+d(X, D_x) \tag{6.1}$$

where $b(\xi)\in S^1$, $d(x, \xi)\in S^0$, and they satisfy the following conditions.

(B.I) $b(\xi)$ is a real-valued function such that $|\nabla_\xi b(\xi)|\geqq K_0$ on $\{|\xi|\geqq R_0\}$ for some positive constants K_0 and R_0.

(B.II) There exists a constant $\rho>1$ such that for any multi-indices α and β

(6.2) $$|D_x^{\alpha}\partial_{\xi}^{\beta}d(x, \xi)|\leqq C_{\alpha\beta}\langle x\rangle^{-\rho-|\alpha|}\langle\xi\rangle^{-|\beta|}, \qquad x, \xi\in \boldsymbol{R}^n .$$

THEOREM 6.1. *Assume* (B.I) *and* (B.II). *Let* $1/2<s<\rho-1/2$ *and* $s-1/2=k+\theta$ (k : *a nonnegative integer*, $0<\theta\leqq 1$). *Then there exists* $N>0$ *such that* $R(z)=(z-A)^{-1}$ *on* $\{z ; \pm\mathrm{Im}\, z\gg 1\}$ *can be extended to a k-times continuously differentiable function on the set* $\boldsymbol{N}_{\pm}=\{z ; \pm\mathrm{Im}\, z\geqq 0, |z|>N\}$ *to* $B(s, -s)$ *which satisfies the following:* (i) *The derivatives* $R^{(j)}(z)$, $j=0, \cdots, k$, *are bounded;* (ii) *when* $0<\theta<1$, $R^{(k)}(z)$ *is uniformly Hölder continuous with exponent* θ ; *and* (iii) *when* $\theta=1$, *there exists a constant* C *such that* $\|\Delta_h^2 R^{(k)}(z)\|\leqq Ch$ *for any* h *and* z *with* $0<h<1$, z *and* $z+2h\in \boldsymbol{N}_{\pm}$. *Furthermore, for all* $f\in L_2^s$

(6.3) $$\sup_{T>0}\int_{\Omega_T}\left[\langle t\rangle^{k+\theta}\left\|\int_{-\infty}^{\infty}\chi(\lambda)R(\lambda\pm i0)fe^{it\lambda}d\lambda\right\|_{-s}\right]^2 dt\leqq C^2\|f\|_s^2 ,$$

where $\Omega_T=\{t ; T\leqq\langle t\rangle<2T\}$ *and* $\chi(\lambda)$ *is a* C^{∞}-*function on* $\boldsymbol{R}^1$ *such that* $\chi(\lambda)=1$ *for* $|\lambda|>N+1$ *and* $\mathrm{Supp}\,\chi<\{|\lambda|>N\}$.

If $\rho>2$, *for any* $1<s<\rho-1$ *there exists a constant* C_s *such that*

(6.4) $$\left\|\int_{-\infty}^{\infty}\chi(\lambda)R(\lambda\pm i0)e^{it\lambda}d\lambda\right\|_{B(s,s)}\leqq C_s\langle t\rangle^{-s}, \qquad t\in \boldsymbol{R}^1 .$$

PROOF. Let $r>k+\theta+3/2$. Then Theorem 1 shows that $R(z)$ can be extended, as a $B(r, -r)$-valued function, to $\boldsymbol{N}_{\pm}$ and has the same regularity properties as in the theorem. Putting $A_R=b(D)+\chi_0(X/R)d(X, D_x)$ as in Lemma 4.2, we have

(6.5) $$R(z)=(z-A_R)^{-1}+(z-A_R)^{-1}(A-A_R)(z-A_R)^{-1} + (z-A_R)^{-1}(A-A_R)R(z)(A-A_R)(z-A_R)^{-1} .$$

Since $A-A_R\in B(-s', s')$ for any $s'>0$, we get the theorem by (6.5) and Lemma 4.2. Q. E. D.

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