J. Math. Soc. Japan
Vol. 30, No. 2, 1978

On transitive groups in which the maximal number of fixed points of involutions is five

By Yutaka HIRAMINE

(Received July 7, 1975)
(Revised Aug. 20, 1977)

§ 1. Introduction.

Let t and μ be integers such that $t \geqq 1$, $\mu \geqq 0$. A finite permutation group (G, Ω) of even order is said to be a (t, μ)-group if G is t-transitive on Ω and μ is the maximal number of the fixed points of involutions in G. All $(2, \mu)$-groups with $\mu \leqq 4$ have been classified; for $\mu=0$ and $\mu=1$ by Bender [2][3], for $\mu=2$ by Hering [12], for $\mu=3$ by King [14] and for $\mu=4$ by Noda [15] and Buekenhout [4]. The (1, 3)-groups have been classified by Buekenhout [5] and (1, 4)-groups have been studied by Rowlinson and Buekenhout [6][20]. In [18][19], Rowlinson has shown that a simple $(1, \mu)$-group with one conjugate class of involutions is one of the known simple groups when $1 \leqq \mu \leqq 7$.

In this paper we shall consider primitive (1, 5)-groups. Let $(\tilde{G}, \Omega)$ be a primitive (1, 5)-group and G be a minimal normal subgroup of $\tilde{G}$.

If G is solvable, G is an elementary abelian p-group for some prime p. In this case we can easily show that $p=5$. Moreover $\tilde{G}$ is a group of automorphisms of an affine space satisfying one of the following:

(1) Dimension of the affine space is 2 or 3.

(2) If T is a Sylow 2-subgroup of $\tilde{G}_\alpha$ $(\alpha \in \Omega)$ then T is cyclic or generalized quaternion and $|C_G(z)|=5$ where z is a unique involution in T.

If G is not solvable, G is a direct product of r isomorphic nonabelian simple groups. In this case, the permutation group (G, Ω) is a $(1, \mu)$-group where $\mu \in \{1, 3, 5\}$ and we can easily show that $r=1$, with the exception of the following case

$$G=G_1 \times G_2 \cong A_5 \times A_5$$

where G_i $(1 \leqq i \leqq 2)$ is isomorphic to the alternating group of degree 5 and G is a permutation group on the set $\{(i, j) \mid 1 \leqq i, j \leqq 5\}$, which is defined by $(i, j)^g = (i^{g_1}, j^{g_2})$ for $g=g_1 \cdot g_2 \in G$ with $g_i \in G_i$ $(1 \leqq i \leqq 2)$. Thus we have $\mathrm{Aut}(G) \geqq \tilde{G} \geqq G$, where G is a simple $(1, \mu)$-group ($\mu \in \{1, 3, 5\}$) or the group isomorphic to $A_5 \times A_5$. Since simple (1, 1)-groups and (1, 3)-groups are known simple groups by Bender [3], Buekenhout [5] and Rowlinson [18], we may consider simple (1, 5)-

groups to classify the primitive (1, 5)-groups.

The purpose of this paper is to prove the following theorem.

THEOREM 1. *Let* (G, Ω) *be a* (1, 5)*-group and* T *be a Sylow* 2*-subgroup of* $O^2(G)$. *Then we have one of the following;*

(1) $|T| \leqq 2^8$.

(2) *T has a cyclic subgroup of index* 4.

(3) *$O^2(G)$ has a unique conjugate class of involutions.*

Here $O^2(G)$ is the subgroup of G generated by all elements of odd order.

In our theorem let G be simple. A simple (1, 5)-group satisfying (2) or (3) is known ([7], [18]). In order to classify simple (1, 5)-groups satisfying (1), we shall prove in § 5 the following lemma.

LEMMA 2. *Let G be a simple* (1, 5)*-group which satisfies* (1) *of Theorem* 1. *Then G has a unique conjugate class of involutions or G has sectional* 2*-rank at most* 4. (*A group G is said to have sectional* 2*-rank k if every section of G has* 2*-rank at most k and some section of G has* 2*-rank equal to k.*)

Simple groups with sectional 2-rank at most 4 were decided recently by D. Gerenstein and K. Harada [**10**]. Thus we shall obtain the following theorem.

THEOREM 3. *Let G be a simple* (1, 5)*-group. Then G is isomorphic to one of the simple groups in the following list.*

(1) $L_2(2^n)$, $n \equiv 0 \pmod 4$, *degree*$=2^n \times 5+5$. *G_α is a (unique) subgroup of $N_G(T)$ of index* 5, *where T is a Sylow* 2*-subgroup of G.*

(2) $U_3(2^n)$, $n \equiv 0 \pmod 2$ *degree*$=2^{3n} \times 5+5$. *G_α is a (unique) subgroup of $N_G(T)$ of index* 5.

(3) $L_2(7)$, *degree*$=21$, $G_\alpha \cong T$.

(4) $L_2(9)$, *degree*$=45$, $G_\alpha \cong T$.

(5) $L_2(19)$, *degree*$=285$, $G_\alpha \cong A_4$.

(6) $L_2(19)$, *degree*$=57$, $G_\alpha \cong A_5$.

(7) $L_2(25)$, *degree*$=65$, $G_\alpha \cong PGL(2, 5)$.

(8) $L_3(4)$, *degree*$=21$, (2*-transitive*).

(9) $L_3(3)$, *degree*$=13$, (2*-transitive*).

(10) A_7, *degree*$=21$, $G_\alpha \cong S_5$.

(11) A_9, *degree*$=9$, (7*-transitive*).

(12) J_1, *degree*$=1045$, $G_\alpha \cong N_G(T)$.

By Theorem 3, [**3**], [**14**] and [**21**], we obtain

THEOREM 4. *Let* (G, Ω) *be a* (2, 5)*-group. Then we have the following:*

(1) *A Sylow* 2*-subgroup of G is cyclic or generalized quaternion, or G is one of the following groups:*

(2) *A subgroup of automorphisms of the affine space of dimension* 3 *over* $GF(5)$ *such that*

$$G=G_\alpha \cdot N \rhd N \cong Z_5 \times Z_5 \times Z_5,\ G_\alpha=SL(3, 5).$$

(3) *A subgroup of automorphisms of the affine space of dimension* 2 *over* $GF(5)$ *such that*

$$G=G_{\alpha}\cdot N\triangleright N\cong Z_5\times Z_5,\ G_{\alpha}=GL(2, 5).$$

(4) *A subgroup of* (3) *of index* 2 *containing* $SL(2, 5)$.

(5) *A subgroup of* (3) *such that* $G=G_{\alpha}\cdot N\triangleright N\cong Z_5\times Z_5$, $G_{\alpha}=N_{GL(2,5)}(Q)$, $Q\in Syl._2(SL(2, 5))|G_{\alpha}|=2^5\cdot 3$.

(6) *A subgroup of* (5) *of index* 2 *containing* $N_{SL(2,5)}(Q)$.

(7) $\mathrm{Aut}(L_2(16))$, $|\Omega|=17$.

(8) *A subgroup of* (7) *of index* 2.

(9) $\mathrm{Aut}(U_3(4))$, $|\Omega|=65$.

(10) *A subgroup of* (9) *of index* 2.

(11) S_7, $|\Omega|=7$.

(12) $L_3(3)$, $|\Omega|=13$.

(13) $L_3(4)$, $|\Omega|=21$.

(14) *A subgroup* G *of* $N_{S_{21}}(L_3(4))$ *such that* $|G: L_3(4)|=3$, $|\Omega|=21$.

(15) A_9, $|\Omega|=9$.

In § 3 and § 4, we shall prove Theorem 1. In the Theorem let us remark that $O^2(G)$ is also transitive on Ω.

If $O^2(G)$ contains no involution, then (1) of Theorem 1 holds. If $O^2(G)$ has an involution, $(O^2(G), \Omega)$ is a $(1, \mu)$-group where $\mu\in\{1, 3, 5\}$. When $\mu=1$ or 3, we can easily show that either (2) or (3) of the theorem holds. Hence we may assume $O^2(G)=G$.

The proof is divided into two cases;

Case 1: $Z(T)$ contians no 5-involution.

Case 2: $Z(T)$ contains a 5-involution.

Here an involution is called a μ-involution if it fixes exactly μ ($\mu=0, 1, 2\cdots$) points.

In the first case, we have

PROPOSITION A. *Let* (G, Ω) *be a* $(1, 5)$*-group with no subgroup of index* 2. *If the center of a Sylow* 2*-subgroup* T *of* G *contains no* 5*-involution, then the order of* T *is at most* 2^8.

In the second case, we have

PROPOSITION B. *Let* (G, Ω) *be a* $(1, 5)$*-group with no subgroup of index* 2. *If the center of a Sylow* 2*-subgroup* T *of* G *contains a* 5*-involution, then one of the following holds.*

(1) $|T|\leqq 2^8$.

(2) T *has a cyclic subgroup of index* 4.

(3) G *has a unique conjugate class of involutions.*

We use the standard notation of [**9**] except the following;

$F(X)$: the set of fixed points of a nonempty subset X of G.

$ccl_G(x)$: the G-conjugate class containing an element $x \in G$.
$|H|_2$: maximal power of 2 dividing the order of a subgroup H of G.
$G|_\Delta$: the restriction of G on a subset Δ of Ω.

§ 2. Preliminary results.

We list now some results that will be required in the proof of the theorems.

(2.1) (Rowlinson [**20**] Lemma 1) Let V be the semi-direct product of a 2-group Y by a four-group $\{1, t_1, t_2, t_3\}$. If $|C_Y(t_i)| \leqq 4$ $(i=1, 2, 3)$, then $|Y| \leqq 2^5$.

(2.2) (Hobby, Satz 7.8 (b), III [**13**]) Let P be a p-group for some prime p. If $Z(\Phi(P))$ is cyclic, then $\Phi(P)$ is also cyclic.

(2.3) (Buekenhout and Rowlinson [**6**] Lemma 2) Let T be a Sylow 2-subgroup of G with $O^2(G)=G$ and v be an element of T of order 2^m. If X is a subgroup of T of index 2^m, then X contains a G-conjugate of the involution $v^{2^{m-1}}$.

(2.4) Let G be a transitive permutation group on Ω and H be a stabilizer of a point in Ω. For any element $x \in H$, we have

$$|F(x)| = |C_G(x)| \cdot |ccl_G(x) \cap H| / |H|.$$

PROOF. Set $M = \{(y, \alpha) \mid ccl_G(x) \ni y,\ F(y) \ni \alpha\}$ and $M_\beta = \{z \in G \mid F(z) \ni \beta,\ z \in ccl_G(x)\}$. By transitivity of G, we have $|M_\beta| = |M_\gamma|$ for all $\beta, \gamma \in \Omega$. Now we count the number of elements of M in two ways and get

$$|G : C_G(x)| \cdot |F(x)| = |\Omega| \cdot |M_\alpha| \qquad (\alpha \in \Omega).$$

We may assume $H = G_\alpha$. Hence we have $|M_\alpha| = |ccl_G(x) \cap H|$. Thus we get (2.4).

As a corollary of (2.4), we have

(2.5) Let Δ be a set and T be a 2-group acting transitively and faithfully on Δ. If x is an element of T with $|F(x)| \neq 0$, we have

$$|C_T(x)| \leqq |F(x)|_2 \cdot |T| / |\Delta|.$$

(2.6) Let Ω be a finite set with $|\Omega|$ odd and G be a transitive permutation group on Ω of even order. Assume $F(x)=F(y)$ for all involutions with $|F(x)| > 1$, $|F(y)| > 1$ in a fixed Sylow 2-subgroup of G. Then all involutions lying in a fixed Sylow 2-subgroup of G have the same set of fixed points, G has a unique conjugate class of involutions and G has a strongly embedded subgroup. (Hence if G is a simple group, G is isomorphic to a simple group of Bender type ([**3**]).)

PROOF. Let u be a 1-involution and x be an involution with $|F(x)| > 1$. By transitivity, we may assume $F(u) \subseteq \Omega - F(x)$. The element u is not conjugate to x in G, hence $O(ux)$ is even. There exists a unique involution $y \in \langle ux \rangle$ with

$[u, y]=[x, y]=1$.

When y is a 1-involution, it follows that $F(u)=F(y)$ and $F(y)\subseteq F(x)$, hence $F(u)\subseteq F(x)$, a contrandiction. When y is not a 1-involution, by assumption we get $F(x)=F(y)$ and $F(u)\subseteq F(y)$, hence $F(u)\subseteq F(x)$, a contradiction. Thus the first statement is proved.

Let x, y be involutions with $F(x)\neq F(y)$. Then $O(xy)$ is odd. For otherwise, there exists a unique involution $z\in\langle xy\rangle$ with $[x, z]=[y, z]=1$. By the first statement of (2.6), we have $F(x)=F(z)$ and $F(y)=F(z)$, hence $F(x)=F(y)$, a contradiction. From this, G has a unique conjugate class of involutions.

Let z be an involution and H be a global stabilizer of $F(z)$. If x is an involution contained in H, x centralizes an involution y contained in the kernel of the action of H on $F(z)$. Since $O(xy)$ $(=2)$ is even, it follows that $F(x)=F(y)$ by the preceding paragraph. Hence H is a strongly embedded subgroup of G.

(2.7) Let P be an elementary abelian 2-group of order 2^n and ϕ be an automorphism of P of order 2. Then we have

$$|C_P(\phi)|\geqq 2^{\frac{1}{2}n}.$$

PROOF. Set $P=\sum_{i=1}^{r} C_P(\phi)\cdot x_i$ (the coset decomposition). Then $x_i^{\phi}x_i$ is an element of $C_P(\phi)$ for each i $(1\leqq i\leqq r)$ and $x_i^{\phi}x_i$ is not equal to $x_j^{\phi}x_j$ for $i\neq j$ $(1\leqq i, j\leqq r)$, hence $r\leqq|C_P(\phi)|$. Since $r=|P: C_P(\phi)|$, we have $|P|\leqq|C_P(\phi)|^2$, which gives (2,7).

(2.8) Let G be a finite group and x be an element of G. Then we have $|ccl_G(x)|\leqq|G'|$.

PROOF. If y is an element of $ccl_G(x)$, there exists $g\in G$ with $y=g^{-1}xg$. Since $x^{-1}x^g=[x, g]\in G'$, we have $x^g\in xG'$. Hence we have $|ccl_G(x)|\leqq|xG'|=|G'|$.

§ 3. Proof of Proposition A.

Since G has a 5-involution, $|\Omega|$ is odd. Hence there exists $\alpha\in\Omega$ with $T\leqq G_\alpha$. Set $M^*=M-\{\alpha\}$ for any subset M of Ω. If G has a 3-involution, then G has an odd permutation and hence $G\neq O^2(G)$. Thus G has no 3-involution and $Z(T)$ acts semi-regularly on Ω^*.

Now we suppose $|T|\geqq 2^9$ and show this leads to a contradiction.

(3.1) If a subgroup R of T is contained in T_β for some $\beta\in\Omega^*$, then $R=1$ or R is not normal in T.

PROOF. By semi-regularity of $Z(T)$ on Ω^*, $Z(T)\cap R=1$, so (3.1) holds.

(3.2) $|\Omega|\equiv 1 \pmod 8$.

PROOF. We assume $|T: T_\beta|\leqq 4$ for some $\beta\in\Omega^*$. Since $|T|>4$, $T_\beta\neq 1$. Hence by (3.1), T_β is not normal in T. In particular $|T: T_\beta|=4$, $|T: N_T(T_\beta)|$

$=2$. Hence $ccl_T(T_\beta)=\{T_\beta, T_\beta^t\}$ for $t\in T-N_T(T_\beta)$ and $T\triangleright T_\beta\cap T_\beta^t$, $|T: T_\beta\cap T_\beta^t|=8$. By (3.1) $|T_\beta\cap T_\beta^t|=1$ and so $|T|=2^3$, a contradiction. Thus $|T: T_\beta|\geqq 8$ for any $\beta\in\Omega^*$, which implies $|\Omega^*|\equiv 0 \pmod 8$.

(3.3) $|Z(T)|=2$ or 2^2.

PROOF. Since T has a 5-involution, semi-regularity of $Z(T)$ on Ω^* gives $|Z(T)|\leqq 2^2$.

(3.4) If a subgroup U of T satisfies $|F(U)|=5$, then U has order at most 4.

PROOF. U acts semi-regularly on $\Omega-F(U)$. If (3.4) is false, $|\Omega-F(U)|\equiv 0 \pmod 8$, which is contrary to (3.2).

(3.5) If the center of a subgroup V of T has a 5-involution, then $|V|\leqq 2^5$.

PROOF. Let x be a 5-involution contained in $Z(V)$. V acts on the set $F(x)^*$. Let U be the kernel of this action, then the factor group V/U is isomorphic to a subgroup of S_4, hence V/U is isomorphic to a subgroup of D_8, the dihedral group of order 8, therefore $|V/U|\leqq 2^3$. On the other hand $|U|\leqq 2^2$ by (3.4). Thus we obtain $|V|\leqq 2^5$.

REMARK. (3.1)-(3.5) hold if T has order at least 2^4.

(3.6) For any $\beta\in\Omega^*$, the 2-rank of T_β is at most 1.

PROOF. Suppose T_β contains a four-group Q for some $\beta\in\Omega^*$.

First we assume $|Z(T)|=2$. By considering the class equation for T, there exists $x\in T-Z(T)$ with $|T: C_T(x)|=2$. Since G has no subgroup of index 2, $C_T(x)$ contains a 5-involution by (2.3). If $|Z(C_T(x))|\geqq 8$, then $Z(C_T(x))$ contains a 5-involution and so by (3.5) we get $|C_T(x)|\leqq 2^5$, contrary to $|T|\geqq 2^9$. Thus $|Z(C_T(x))|=4$ holds.

$C_T(x)$ has no element y with $|C_T(x): C_T(x)\cap C(y)|=2$. Suppose false. Since $|Z(C_T(x)\cap C(y))|\geqq|<Z(T), x, y>|\geqq 8$ and $|C_T(x)\cap C(y)|\geqq 2^7$, it follows that $C_T(x)\cap C(y)$ contains no 5-involution, which clearly means $C_T(x)\cap C(y)$ acts semi-regularly on Ω^*. There exists a normal subgroup S of T such that $|T:S|\leqq 2^3$ and $S\leqq C_T(x)\cap C(y)$ as $|T: C_T(x)\cap c(y)|=4$. Applying (2.1) to Q and S, we see that $|S|\leqq 2^5$, hence $|T|\leqq 2^8$, a contradiction. Thus the number of $C_T(x)$-conjugate classes which consist of four elements is odd. On the other hand, T normalizes $C_T(x)$, so that at least one of these, say $ccl_{C_T(x)}(y)$ is T-invariant. It follows that $ccl_T(y)=ccl_{C_T(x)}(y)$ and so $|T: C_T(y)|=4$. Let $ccl_T(y)=\{y=y_1, y_2, y_3, y_4\}$.

If $T>C_T(y)$, then since $|Z(C_T(y))|\geqq 8$, we get a contradiction as before. Therefore $C_T(y)$ is not normal in T. We may assume $C_T(y_1)=C_T(y_3)\neq C_T(y_2)=C_T(y_4)$. Evidently T normalizes $C_T(y_1)\cap C_T(y_2)=C_T(y_3)\cap C_T(y_4)$. $C_T(y_1)\cap C_T(y_2)$ contains a 5-involution, otherwise applying (2.1) again, we get $|C_T(y_1)\cap C_T(y_2)|\leqq 2^5$. Hence $|T|\leqq 2^8$, a contradiction.

We have $|Z(C_T(y_1))|=4$ as above. Thus $Z(C_T(y_1))=\{y_1, y_3, z, 1\}\cong Z(C_T(y_2))=\{y_2, y_4, z, 1\}$ acts semi-regularly on Ω^*, where $\langle z\rangle=Z(T)$.

Let t be a 5-involution in $C_T(y_1)\cap C_T(y_2)$. The restriction of $Z(C_T(y_1))$ on $F(t^u)^*$ is regular for every $u\in T$ and is isomorphic to the restriction of $Z(C_T(y_2))$ on $F(t^u)^*$. By regularity of $Z(C_T(y_1))$ on $F(t^u)^*$ with $u\in T$, it follows that either $F(t^u)=F(t^v)$ or $F(t^u)^*\cap F(t^v)^*=\phi$ holds for $u, v\in T$. We can easily show that $|\beta^T|\geqq 16$ for $\beta\in F(t)^*$ and so $|\{F(t^u)^*|u\in T\}|\geqq 4$. Considering the permutation representation of y_1 and $Z(C_T(y_2))$, it follows that $y_1=w$ on at least two blocks in $\{F(t^u)^*|u\in T\}$ for some w in $Z(C_T(y_2))$. This implies $y_1w^{-1}\in T$ fixes at least 8 points on Ω^*, hence by assumption, $y_1w^{-1}=1$ and y_1 is contained in $Z(C_T(y_2))$, a contradiction.

Assume next that $|Z(T)|=4$. In this case, the class equation for T shows that T contains an element $x\in T-Z(T)$ with $|T:C_T(x)|\leqq 4$. Since $|Z(C_T(x))|\geqq 8$, $C_T(x)$ contains no 5-involution, hence $|T|\leqq 2^8$ as before, which is a contradiction.

(3.7) $|T:T'|\geqq 8$.

PROOF. If $|T:T'|=4$, T is of maximal class. Hence T is dihedral, semi-dihedral, generalized quaternion or cyclic by Theorem 5.4.5 [**9**]. Since G has no subgroup of index 2, G has a unique conjugate class of involutions, but G has a 1-involution, a contradiction.

(3.8) $|T_\beta|=1$ or 2 for all $\beta\in\Omega^*$.

PROOF. By (3.6) T_β is cyclic or generalized quaternion. Suppose T_β contains an element v of order 8. From (3.2) and the cycle structure of v, we have $|F(v^4)|\geqq 9$, whence v^4 is a μ-involution ($\mu\geqq 9$), contrary to the assumption that (G, Ω) is a (1,5)-group. Thus $T_\beta\cong Q_8$, the quaternion group of order 8 or T_β is cyclic of order at most 4.

In the first case, we have $|F(T_\beta)|=3$ by (3.2). Let $F(T_\beta)=\{\alpha, \beta, \gamma\}$. There exists a subset Δ of $\Omega-F(T_\beta)$ such that $\Delta^{T_\beta}=\Delta$, $|\Delta|=4$. Since T_β acts faithfully on Δ, T_β is isomorphic to a subgroup of S_4, so that $Q_8\cong D_8$, a contradiction.

To complete the proof, we need only show that T_β is not isomorphic to Z_4. Suppose $T_\beta=\langle v\rangle$ with $o(v)=4$ for some $\beta\in\Omega^*$. Since G does not contain an odd permutation, it follows that $|F(T_\beta)|=3$ by (3.2). Then $|Z(T)|=2$, and so T has an element x with $|T:C_T(x)|=2$. Considering the T-orbit which contains β, we get $|C_T(v)|=8=|C_T(v^3)|$ by (2.5), whence $|T:T'|=8$ by (2.8) and (3.7) and so $ccl_T(v)=T'v$, $ccl_T(v^3)=T'v^3$. If $T'v=T'v^3$, then $v\sim v^3$, whence we have $|C_T(v)|\leqq 4$ by (2.4), which is contrary to $|C_T(v)|=8$. Thus $T'v\neq T'v^3$, consequently $\langle v\rangle\cap T'=1$.

Let $N_G(T)=N\cdot T$ where N is a Hall $2'$-subgroup of $N_G(T)$. We argue that N normalizes $C_T(x)$. Since T/T' is isomorphic to $Z_2\times Z_4$, the Frattini subgroup $\Phi(T)$ of T is $T'\langle v^2\rangle$ and $T/\Phi(T)\cong Z_2\times Z_2$. If N does not normalize $C_T(x)$, the whole maximal subgroups of T are $C_T(x)$, $C_T(x^a)$ and $C_T(x^{a^2})$ for some $a\in N$. Since $T\neq\langle v\rangle$, v is contained in one of these. Without loss of generality, we may assume v is contained in $C_T(x)$. Furthermore, $Z(C_T(x))$ acts semi-regularly on Ω^*, for otherwise we get $|C_T(x)|\leqq 2^5$ by (3.5), which implies $|T|\leqq 2^6$, a contradiction. Since $|F(v)^*|=2$ and $v\in C_T(x)$, the semi-regularity of $Z(C_T(x))$ on Ω^* gives $|Z(C_T(x))|\leqq 2$, a contradiction. Hence N normalizes $C_T(x)$.

Thus N acts trivially on $T/\Phi(T)$, so we have $[N, T]=1$ by Theorem 5.1.4 [9]. By Grün's theorem ([9] Theorem 7.4.2), the focal subgroup $T\cap G'=\langle T\cap N(T)', T\cap(T')^g|g\in G\rangle$. Hence we have $T\cap G'=\langle T\cap(T')^g|g\in G\rangle$. Since $\langle v\rangle\cap T'=1$, it follows that $T/T'=\langle T'v, T'w\rangle$ for some $w\in T-T'\langle v\rangle$ with $w^2\in T'$. The groups $T'\langle w\rangle$ and $T'\langle v^2w\rangle$ are normal subgroups of T of index 4. We denote one of these X. By (2.3), we can take $u\in ccl_G(v^2)\cap X$. If $T_\gamma\neq\langle u\rangle$ for some $\gamma\in F(u)^*$, then $T_\gamma=\langle u_0\rangle$ with $u_0\in T$ and $u_0^2=u$. Since $\langle u_0\rangle\cap T'=1$, we have $u\notin T'$. On the other hand u is containd in $\Phi(T)\cap X=T'$, a contradiction. Hence it follows that $T_\gamma=\langle u\rangle$ for all $\gamma\in F(u)^*$. Thus there exist elements u_1, $u_2\in T$ such that $ccl_T(u_1)=T'w$, $ccl_T(u_2)=T'v^2w$ by (2.5). If T' contains a 5-involution x, it follows that $T_\gamma=\langle x\rangle$ for $\gamma\in F(x)^*$. For otherwise, there exists $y\in T$ such taht $T_\gamma=\langle y\rangle$, $y^2=x$ and $\langle y\rangle\cap T'=1$, hence $x\notin T'$, a contradiction. Thus $|C_T(x)|\leqq 8$ by (2.5). Since $|T:T'|=8$, $ccl_T(x)=T'x=T'$ by (2.8), a contradiction. Hence T' acts semi-regularly on Ω^*. From this, we have $T\cap(T')^g\leqq T-\{T'v, T'v^3, T'u_1, T'u_2\}=T'\langle vw\rangle<T$ for all $g\in G$, which implies that the focal subgroup $T\cap G'$ is a proper subgroup of T, contrary to $O^2(G)=G$.

(3.9) If u is a 5-involution in T, then $|C_T(u)|=8$, $ccl_T(u)=T'u$ and u inverts every element of T'.

PROOF. Let β be a fixed point of u with $\beta\neq\alpha$. Now $T_\beta=\langle u\rangle$ by (3.8), hence $|C_T(u)|\leqq 8$ by (2.5). Thus (3.9) holds by (3.7) and (2.8).

(3.10) T/T' is an elementary abelian 2-group of order 8.

PROOF. Suppose false. There exists $\bar{v}\in T/T'$ with $O(\bar{v})=4$. Since $|T:T'\langle v\rangle|=2$, by (2.3), $T'\langle v\rangle$ contains a 5-involution, say u. By (3.9), we have $u\notin T'v\cup T'v^3$, hence $u\in T'v^2$. Again by (3.9), v^2 is contained in $ccl_T(u)$ and so v^2 is a 5-involution. Considering the cycle structure of v, we get $|F(v)^*|\neq 0$, contrary to (3.8).

(3.11) Contradiction.

Each subgroup of T of index 2 contains a 5-involution, whence T has at least three conjugate classes of 5-involutions, say $T'u_i$ $1\leqq i\leqq 3$ by (3.10). If $T'u_iu_j$ contains a 5-involution, say u_4, we have $ccl_T(u_4)=T'u_iu_j$ by (3.9) and so u_iu_j is a 5-involution. Hence $|C_T(u_iu_j)|=8$ by (3.9). On the other hand u_i and u_j invert T' by (3.9) and so u_iu_j centralizes T'. Hence $|T'|\leqq|C_T(u_iu_j)|=8$, which implies $|T|\leqq 2^6$, a contradiction. Thus $T'u_iu_j$ contains no 5-involution for $i, j\in\{1, 2, 3\}$. Hence the subgroup $\{T', T'u_1u_2, T'u_2u_3, T'u_3u_1\}$ of T of index 2 contains no 5-involution, a contradiction. Thus Proposition A is proved.

§ 4. Proof of Proposition B.

To prove Proposition B, we assume the following three Hypotheses:

(1) G has at least two conjugate classes of involutions.

(2) T does not have a cyclic subgroup of index 4.

(3) $|T|=2^n\geqq 2^9$

and show these lead to a contradiction.

Since G has a 5-involution, $|\Omega|$ is odd, hence T is contained in G_α for some $\alpha\in\Omega$. Let z be a 5-involution in $Z(T)$, so T acts on $F(z)^*=F(z)-\{\alpha\}$. Let K be the kernel of this action, then T/K is a subgroup of D_8. K acts semi-regularly on $\Omega-F(z)$. By Hypothesis (3), $|K|\geqq 2^6>8$, hence we have

(4.1) $|\Omega|\equiv 5 \pmod{|K|}$ where $|K|>8$.

By Hypothesis (1) and (2.6), we have

(4.2) There exists a 5-involution x_1 in $T-K$.

(4.3) $|T/K|=2$ or 4.

PROOF. By (4.2) we get $|T/K|\neq 1$. To prove (4.3), it will suffice to show that T/K is not isomorphic to D_8. Suppose $T/K\cong D_8$. Then there exists an element $x\in T$ such that x is 2-cycle on $F(z)^*$.

Assume x is not an involution. Considering the cycle structure of x, $o(x)=|\Omega-F(z)|_2\geqq|K|=2^{n-3}$ because x is an odd permutation on $\Omega-F(z)$. By Hypothesis (2), $o(x)=2^{n-3}$ and so $|\Omega-F(K)|_2=|K|$, whence x stabilizes a K-orbit, say $\Omega_0\subseteqq\Omega-F(z)$. The group $K\langle x\rangle$ is transitive on Ω_0. Since $|K|=|\Omega_0|$, there exists an element kx such that $k\in K$ and $F(kx)\cap\Omega_0\neq\emptyset$, and then kx is a 5-involution. On the other hand $kx\equiv x$ on $F(z)^*$. Thus we may assume x is a 5-involution.

Since $|F(x)^*\cap F(z)|=2$, $F(x)\cap(\Omega-F(z))=\{\beta, \gamma\}$ for some $\beta, \gamma\in\Omega$. Now $C_T(x)$ acts on $\{\beta, \gamma\}$. The kernel K_0 of this action does not contain a four-group by (2.1). Hence x is a unique involution in K_0, which is an odd permutation on $\Omega-F(z)$ so that K_0 contains no element of order 4 and so $K_0=\langle x\rangle$, whence $|C_T(x)|\leqq 4$. This implies that $|T:T'|=4$ and T is dihedral or semi-dihedral ([9] Theorem 5.4.5), which is contrary to Hypothesis (1) by (2.3).

(4.4) For all $\beta\in\Omega-F(z)$, T_β is cyclic of order at most 4.

PROOF. Since $T_\beta\cap K=1$, we have $|T_\beta|\leqq 4$ by (4.3). If T_β is isomorphic to $Z_2\times Z_2$, we get $|K|\leqq 2^5$ by (2.1), contrary to Hypothesis (3).

(4.5) T/K is not isomorphic to Z_2.

PROOF. We assume $T/K\cong Z_2$. By (4.2), we can take a 5-involution $x_1\in T$ with $F(x_1)\neq F(K)$. There exists an extremal element z_0 of T in G with $z_0\in ccl_G(x_1)$. Here an element z_0 is said to be an extremal element of T in G if $|C_T(z_0)|\geqq|C_T(u)|$ holds for any $u\in T\cap ccl_G(x_1)$. Let u be an arbitrary 5-involution in $T-K$. Then we obtain $|C_T(u)|=|\langle u\rangle C_K(u)|\leqq 8$. Hence we may assume

$z_0 \in K$ by Hypothesis (1) and (2.3). There exists an element $g \in G$ such that $x_1^g = z_0$, $(C_T(x_1))^g \leqq C_T(z_0)$. It follows that $(C_K(x_1))^g \leqq T$ and $(C_K(x_1))^g \cap K = 1$ since $F(x_1) \neq F(z_0) = F(K)$. Hence $|C_K(x_1)| = 2$ and $|C_T(x_1)| = 4$, which means T is of maximal class, contrary to Hypothesis (1) by (2.3).

(4.6) T/K is not isomorphic to Z_4.

PROOF. Suppose $T/K \cong Z_4$. Set $T/K = \langle Ky \rangle$. Since y is an odd permutation on $F(K)$ and G has no odd permutation on Ω, y is an odd permutation on $\Omega - F(K)$. If $O(y) \neq 4$, we have $O(y) = |\Omega - F(K)|_2 \geqq |K|$, contrary to Hypothesis (2). Hence $O(y) = 4$ and y^2 is a 5-involution. Set $y^2 = x$. By (2.3), we obtain $ccl_G(x) \cap K \neq \emptyset$. Let $u \in ccl_G(x) \cap K$.

We shall argue that there exists an involution in $K \cap ccl_G(x)$ which is an extremal element of T in G. Suppose false. Then we have $u \notin Z(T)$. Let v be an extremal element with $v \in ccl_G(x) \cap T$. There exists an element $g \in G$ such that $u^g = v$ and $(C_T(u))^g \leqq C_T(v)$. Since $F(v) \neq F(u)$, we have $(C_K(u))^g \cap K = 1$ and $(C_K(u))^g \leqq T$. On the other hand, $C_K(u)$ contains a four group because $u \notin Z(T)$. Hence we have $T/K \cong Z_2 \times Z_2$, a contradiction. Thus we may assume that v is contained in K.

There exists an element $h \in G$ such that $x^h = v$ and $(C_T(x))^h \leqq C_T(v)$. Since $F(x) \neq F(v) = F(K)$, we have $(C_K(x))^h \cap K = 1$ and $(C_K(x))^h \leqq T$. Hence $C_K(x) \cong Z_4$ because $C_K(x) \not\cong Z_2$ by Hypothesis (2). Since x is a square of y, x is contained in $\Phi(T)$. Since $|C_T(x)| = 16$, we get $|T : T'| \leqq 16$ by (2.8). Clearly $x \notin Z(\Phi(T))$. If follows that $Z(\Phi(T)) \leqq C_{<x>K}(x) = \langle x \rangle \times C_K(x) \cong Z_2 \times Z_4$. Hence $Z(\Phi(T))$ is cyclic, whence $\Phi(T)$ is also cyclic by (2.2), which means $x \in K$, a contradiction.

REMARK. By the proof of (4.6), we know that in the case $T/K \cong Z_4$, there exists an element $y \in T - K$ such that $O(y) = 4$, $y^2 \in T - K$ and $|F(y^2)| = 5$.

By (4.3), (4.5) and (4.6), we have

(4.7) $T/K \cong Z_2 \times Z_2$.

(4.8) $|T_\beta| = 1$ or 2 for $\beta \in \Omega - F(z)$. $|C_T(x_0)| = 8$ for any 5-involution $x_0 \in T - K$, whence $|T : T'| = 8$, $ccl_T(x_0) = T'x_0$.

PROOF. T_β is cyclic of order at most 4 by (4.4). Since $T/K \cong Z_2 \times Z_2$ and $T_\beta \cap K = 1$, we get $|T_\beta| \neq 4$. Hence $|T_\beta| = 1$ or 2 and (2.5) gives the latter statement.

(4.9) There exists a conjugate class of 5-involutions $ccl_T(x_2) = T'x_2$ contained in $T - \langle x_1 \rangle K$.

PROOF. Suppose false. Let N be a Hall 2'-subgroup of $N_G(T)$. N stabilizes the following normal series: $T/T' \triangleright K\langle x_1 \rangle / T' \triangleright K/T' \triangleright \bar{1}$. Hence $[T, N] = 1$ by Theorems 5.1.4. and 5.3.2. of [**9**]. Thus we have $T \cap G' = \langle T \cap N(T)', T \cap (T')^g | g \in G \rangle = \langle T \cap (T')^g | g \in G \rangle \leqq K\langle x_1 \rangle$, whence $T \cap G'$ is a proper subgroup of T, contrary to $O^2(G) = G$.

(4.10) There is no 1-involution in $T - K$.

PROOF. Suppose false. Let u be a 1-involution in T. Since $\{T', T'x_1, T'x_2, T'x_1x_2\}$ is a subgroup of T of index 2, u is conjugate to some element in $T'x_1x_2$.

We may assume x_1x_2 is a 1-involution. Hence a four-group $\{1, x_1, x_2, x_1x_2\}$ has trivial intersection with K. By (4.8), $C_T(x_1)=\{1, x_1, x_2, x_1x_2\}\times\langle z\rangle$, and so $C_K(x_1)=\langle z\rangle$. Hence $|C_{K<x_1>}(x_1)|=|x_1C_K(x_1)|=4$ and $K\langle x_1\rangle$ is of maximal class, which is contrary to Hypothesis (2).

(4.11) The group T' is an abelian 2-group of 2-rank 2.

PROOF. Since $ccl_T(x_1)=T'x_1$, an involution x_1 inverts T', hence T' is abelian. Furthermore $|C_{T'}(x_1)|\leqq|C_K(x_1)|\leqq 4$ and so the 2-rank of T' is at most 2. Suppose the 2-rank of T' is 1, that is, $T'=\langle d\rangle$ for some $d\in T'$. Since $x_1x_2\notin T'$, it follows that $(x_1x_2)^2\in T'$ and $(x_1x_2)^2\in\langle d^2\rangle$, for otherwise $O(x_1x_2)=2\cdot O(d)=2^{n-2}$, contrary to Hypothesis (2). Hence for some $d_1\in\langle d\rangle$ we have $(x_1x_2)^2=d^2$. Since $[x_1x_2, T']=1$, $(x_1x_2d^{-1})^2=1$. Hence $x_1x_2d^{-1}$ is a 5-involution contained in $T-K$ by (4.10). Thus $x_1x_2d^{-1}$ also inverts T', hence $|T'|=2$, contrary to Hypothesis (3).

(4.12) $ccl_G(x_1)\cap T'\neq\emptyset$.

PROOF. Suppose false. Let y be an element in $K-T'$. Since $\{T', T'x_1x_2, T'x_1y, T'x_2y\}$ is a subgroup of T of index 2 and $ccl_G(x_1)\cap T'=ccl_G(x_1)\cap T'x_1x_2=\emptyset$, there exists an element $tx_iy\in ccl_G(x_1)\cap T'x_iy$ for some $i\in\{1, 2\}$ and $t\in T'$. If y is an involution, then $[tx_i, y]=1$. Hence $C_K(tx_i)=C_{T'<y>}(tx_i)=\Omega_1(T')\langle y\rangle$, whence $|C_K(tx_i)|=8$ by (4.11), which imples that $|F(tx_i)|\geqq 9$, a contradiction. Thus there is no involution in $K-T'$ and so $ccl_G(x_1)\cap T'y=\emptyset$. Since $\{T', T'y, T'x_1x_2, T'x_1x_2y\}$ is a subgroup of T of index 2 and $ccl_G(x_1)\cap T'=ccl_G(x_1)\cap T'y=ccl_G(x_1)\cap T'x_1x_2=\emptyset$, there exists a 5-involution $sx_1x_2y\in ccl_G(x_1)\cap T'x_1x_2y$ for some $s\in T'$, hence sx_1x_2y inverts T'. Since sx_1 and x_2 invert T', sx_1x_2 centralizes T' and so y inverts T'. On the other hand, tx_iy and tx_i invert T', hence y centralizes T'. Thus $T'=\Omega_1(T')$ and we have $|T|=2^5$ by (4.8) and (4.11), contrary to Hypothesis (3).

(4.13) Contradiction

By (4.8) and (4.12), there exists in K an extremal element z_0 of T in G with $z_0\in ccl_G(x_1)$. Hence there exists an element $g\in G$ such that $(C_T(x_1))^g\leqq C_T(z_0)$ and $x_1^g=z_0$. Since $F(x_1)\neq F(z_0)=F(x_1)^g$, the element g does not stabilize $F(K)$ as a set, hence there exists $\beta\in(\Omega-F(K))\cap F(K)^g$. Clearly we have $C_T(x_1)\geqq C_K(x_1)\geqq\Omega_1(T')\cong Z_2\times Z_2$ by (4.11). Hence $T_\beta\geqq\Omega_1(T')^g\cong Z_2\times Z_2$, contrary to (4.8). Thus Proposition B is proved.

§ 5. Proof of Lemma 2.

Throughout this section we assume the following:

(1) G is a simple (1, 5)-group with $|G|_2\leqq 2^8$.

(2) G has at least two conjugate classes of involutions.

(3) Let T be a Sylow 2-subgroup of G. There exist subgroups T_1, T_2 of T with $T_1 \triangleright T_2$, $T_1/T_2 \cong E_{2^5}$.

and show these lead to a contradiction.

We shall often use the following theorem to prove Lemma 2.

THEOREM (K. Harada [**11**]). *If 2-group S has a subgroup A of order 8 with $C_S(A) \leqq A$, then the sectional 2-rank of S is at most 4.*

(5.1) Let Q be a subgroup of T with $Q \cong Z_2 \times Z_2$. If $|F(q_1)^* \cap F(q_2)^*| = 2$ for some $q_1, q_2 \in Q^\#$, then the sectional 2-rank of T is at most 4.

PROOF. $C_T(Q)$ acts on $\Delta_0 = F(q_1)^* \cap F(q_2)^*$, $\Delta_1 = F(q_1)^* - \Delta_0$ and $\Delta_2 = F(q_2)^* - \Delta_0$. If $|C_T(Q)| \geqq 16$, the kernel of this action is not trivial, a contradiction. Hence we have $|C_T(Q)| \leqq 8$. Let A be a subgroup of T of order 8 containing $C_T(Q)$. Then $C_T(A) \leqq C_T(Q) \leqq A$ because A contains Q. By Harada's theorem, the sectinal 2-rank of T is at most 4, which is contrary to (3).

We note that T has order at least 2^5 by the assumption (3), hence in the case that $Z(T)$ has no 5-involution, (3.1)-(3.5) hold (see Remark in (3.5)).

(5.2) Suppose $Z(T)$ contains no 5-involution. If U is a subgroup of T such that $Z(U)$ has a 5-involution u, U is semi-regular on $F(u)^*$ and $|U| \leqq 2^4$.

PROOF. Let u be a 5-involution in $Z(U)$. By (3.5), $|U| \leqq 2^5$. Hence we have only to show $|U| \neq 2^5$. Assume $|U| = 2^5$. Then there exists $v \in U$ with $v|_{F(u)^*} = (\beta)(\gamma)(\delta\varepsilon)$ where $F(u)^* = \{\beta, \gamma, \delta, \varepsilon\}$. By (5.1), $o(v) \neq 2$, so (3.2) gives $o(v) = 4$. $C_T(v)$ acts on $\{\beta, \gamma\}$ and $\{\delta, \varepsilon\}$. Let K_0 be the kernel of this action. Since $|\Omega| \equiv 1 \pmod 8$, K_0 stabilizes a $\langle v \rangle$-orbit of length 4. Since $[K_0, v] = 1$, K_0 is isomorphic to a subgroup of Z_4. Since G contains no odd permutation, $K_0 \not\cong Z_4$, hence $|C_T(v)| = 8$, which is contrary to (3) by Harada's theorem.

(5.3) Suppose $Z(T)$ contains no 5-involution. Then $T_\beta \cong 1$ or $T_\beta \cong Z_2 \times Z_2$ holds for every $\beta \in \Omega^*$.

PROOF. We take an involution $v \in Z(T_\beta)$. Then $C_T(v)|_{F(v)^*}$ is semi-regular, by (5.2). We have $|C_T(v)| \geqq 16$ by (3) and Harada's theorem. Thus $|C_T(v)|_{F(v)^*}| = 4$, $T_\beta \cong Z_2 \times Z_2$.

(5.4) Let T_0 be a subgroup of T containing T_1. Then T_0 does not contain a cyclic subgroup of index 8.

PROOF. Let x be an element of T_0 with $|T_0 : \langle x \rangle| = 8$. If T_1 is a subgroup of T_0 of index 2^n, an element x^{2^n} is contained in T_1 and $|T_1 : \langle x^{2^n} \rangle| = 8$, which is contrary to $T_1/T_2 \cong E_{2^5}$.

(5.5) Suppose $Z(T)$ contains no 5-involution. Then T_1 acts semi-regularly on Ω^*.

PROOF. If T_1 contains a 5-involution u, $|T_1 : C_{T_1}(u)| = |ccl_{T_1}(u)| \leqq |T_1'| \leqq \frac{1}{2^5}|T_1|$ by (2.8). Hence $|C_{T_1}(u)| \geqq 2^5$, contrary to (5.2).

First we consider the case that $Z(T)$ has no 5-involution. Next we show that the same argument can apply to the case that $Z(T)$ has a 5-involution.

If $Z(T)$ has no 5-involution, we have $|T|=2^7$ or 2^8 by (5.3) and (5.5). Suppose $|T|=2^7$, then $T_1\cong E_{2^5}$ and $T_2=1$. There exists a 5-involution x such that x normalizes T_1. By (5.4) and (5.5), we get $|T_1\langle x\rangle : (T_1\langle x\rangle)'|=8$ and x inverts $(T_1\langle x\rangle)'$. Since $(T_1\langle x\rangle)'\leqq T_1\cong E_{2^5}$, x centralizes $(T_1\langle x\rangle)'$. Thus $|(T_1\langle x\rangle)'|\leqq 4$ and we have $|T|\leqq 2^6$, a contradiction. Next we suppose $|T|=2^8$. By (5.3) and (5.4), $|T: T_1|=2^2$ or 2^3 and $|T_2|=2$ or 1, respectively. If $N_T(T_1)$ contains a 5-involution x, we have $|T: T_1|=2^2$ and $T_2\cong Z_2$ by (2.7) and (5.5). Since $|T_1\langle x\rangle : (T_1\langle x\rangle)'|=8$ and x inverts $(T_1\langle x\rangle)'$ $(\leqq T_1)$, we have $(T_1\langle x\rangle)'\cong Z_4\times Z_4$ by (5.4) and (5.5), contrary to $T_1/T_2\cong E_{2^5}$ and $T_2\cong Z_2$. Hence $N_T(T_1)$ acts semi-regularly on Ω^*. By (5.3), we get $|T: N_T(T_1)|=2^2$, $|T: T_1|=2^3$ and $T_2=1$. There exists a 5-involution x which normalizes $N_T(T_1)$. As above x inverts $(\langle x\rangle N_T(T_1))'$. Hence we have $(\langle x\rangle N_T(T_1))'\cong Z_4\times Z_4$ since $(\langle x\rangle N_T(T_1))'\leqq N_T(T_1)\rhd T_1\cong E_{2^5}$ and $\Omega_1((\langle x\rangle N_T(T_1))')\cong Z_2\times Z_2$. But since $|N_T(T_1): T_1|=2$ and $T_1\cong E_{2^5}$, $N_T(T_1)$ does not contain a subgroup isomorphic to $Z_4\times Z_4$. Thus we get a contradiction.

We now consider the case $Z(T)$ has a 5-involution z. If $T|_{F(z)^*}$ is isomorphic to D_8, in the same way as in the proof of (4.3), T has a cyclic subgroup of index 8, contrary to (5.4). Suppose $T|_{F(z)^*}\cong Z_4$. There exists an element $y\in T-K$ such that $O(y)=4$ and y^2 is a 5-involution in $T-K$ (see Remark in (4.6)). Set $y^2=x$. By (2.3), we have $K\cap ccl_G(x)\neq\emptyset$. Since $|K\langle x\rangle : (K\langle x\rangle)'|=8$ and $C_K(x)\cong Z_4$, $(K\langle x\rangle)'$ is a cyclic subgroup of $K\langle x\rangle$ of index 8. Hence T_1 is not contained in $K\langle x\rangle$. Take y_1 in $T_1-K\langle x\rangle$. Clearly $O(y_1)=4$ and y^2 is a 5-involution. Since $|T_1: C_{T_1}(y_1)|=|ccl_{T_1}(y_1)|\leqq |T_1'|\leqq \frac{1}{2^5}|T_1|$, it follows that $|C_{T_1}(y_1)|\geqq 2^5$. $C_{T_1}(y_1)$ acts on $F(y_1^2)^*$ $(\subseteqq \Omega-F(z))$. Let K_1 be the kernel of this action. Since $|\Omega|\equiv 5 \pmod 8$, we have $|C_{T_1}(y_1)|=2^5$ and $C_{T_1}(y_1)/K_1\cong D_8$ There exists an element $u\in C_{T_1}(y_1)$ such that $u|_{F(y_1^2)^*}=(\beta)(\gamma)(\delta\varepsilon)$ where $F(y_1^2)^*=\{\beta, \gamma, \delta, \varepsilon\}$. Considering the cycle structure of u, we get $O(u)=2$, contrary to (5.1). Hence we have $T/K\cong Z_2\times Z_2$ and $T|_{F(K)^*}$ is semi-regular. From this, (5.1)-(5.5) hold for $T_{\Omega-F(K)}$. Thus we obtain a similar contradiction.

§ 6. Proof of Theorem 3.

By Theorem 1, Lemma 2 and the Fong's theorem [7], we know any simple (1, 5)-group G satisfies one of the following:

(1) G has a unique conjugate class of involutions.

(2) G has sectional 2-rank at most 4 and a Sylow 2-subgroup of G has order at most 2^8.

By Rowlinson's Theorem of [18], these are equivalent to the following:

(i) G is a simple group of Bender type.

(ii) $G\cong L_2(q)$ $(q\equiv 1 \pmod 2)$.

(iii) A Sylow 2-subgroup of G is semi-dihedral.

(iv) G is not of type (i)—(iii) and has sectional 2-rank at most 4, moreover $|G|_2 \leqq 2^8$.

CASE (i). We prove the followihg Lemma.

LEMMA 5. *Let G be a simple group of Bender type and T be a Sylow 2-subgroup of G.*

(1) *If H is a (unique) subgroup of $N_G(T)$ of index μ where μ is odd, then G is a simple $(1, \mu)$-group as a permutation group on the cosets G/H.*

(2) *If G is a simple $(1, \mu)$-group on a set Ω where μ is odd, then (G, Ω) is equivalent to a permutation representation obtained by* (1).

PROOF. (1) Since $N_G(T)$ is isomorphic to one point stabilizer as a (1, 1)-permutation representation of G, $N_G(T)$ is a strongly embedded subgroup of G (cf. [3]). Set $G = \bigcup_i N_G(T)X_i$ and $N_G(T) = \bigcup_{j=1}^{\mu} Hy_j$, the left coset decomposition. We can look on G as permutation group on the cosets $\bigcup_{i,j} Hy_jx_i$. Let z be an arbitrary element contained in $T^\#$. Then we have $(Hy_jx_i)z = Hy_jx_i$ if and only if $z \in H^{y_jx_i}$. Since H is a normal subgroup of $N_G(T)$, we have $z \in H^{y_jx_i}$ if and only if $z \in (N(T))^{y_jx_i} = (N(T))^{x_i}$. Since $N_G(T)$ is a strongly embedded subgroup of G, we have $z \in (N_G(T))^{x_i}$ if and only if $x_i \in N_G(T)$. Thus z fixes exactly μ cosets $\bigcup_{j=1}^{\mu} Hy_jx_i$, whence $(G, G/H)$ is a $(1, \mu)$-group.

(2) Let (G, Ω) be as in (2) and H be a stabilizer of a point $\alpha \in \Omega$. Since G have a μ-involution and μ is odd, it follows that $|\Omega|$ is odd, hence H contains a Sylow 2-subgorup T of G. By the structure of G, H is 2-closed. Let x be an involution in T. By (2.4), we have $\mu = |F(x)| = |C_G(x)| \cdot |ccl_G(x) \cap H| / |H|$. Since H is 2-closed and G has a unique conjugate class of involutions, we have $|ccl_G(x) \cap H| = |ccl_G(x) \cap N_G(T)|$, hence

$$\mu = |F(x)| = (|C_G(x)| \cdot |ccl_G(x) \cap N_G(T)| / |N_G(T)|) \times (|N_G(T)| / |H|) = |N_G(T) : H|.$$

From this, it follows that a simple (1, 5)-group of type (i) is (1) or (2) of Theorem 3.

CASE (ii).

LEMMA 6. *A simple* (1, 5)*-group of type* (ii) *is one of the groups listed in* (3)-(7) *of Theorem* 3.

PROOF. Let p be an odd prime and $q = p^n > 3$. Suppose G is a (1, 5)-group on a set Ω which is isomorphic to $L_2(q)$. If H is a stabilizer of a point in Ω. Since $|\Omega|$ is odd, H contains a Sylow 2-subgorup of G. Hence by the Dickson's Theorem ([13] p. 213), H is isomorphic to one of the following:

(a) Dihedral group of order $2z$ where $z | (q-\varepsilon)/2$, $q \equiv \varepsilon \in \{-1, 1\} \pmod 4$.

(b) A_4, $q \equiv 3$ or $5 \pmod 8$.

(c) S_4, $q^2 - 1 \equiv 0 \pmod{16}$.

(d) A_5, $q\equiv 3$ or 5 (mod 8) or $p=5$ or $q^2-1\equiv 0$ (mod 5).

(e) $PSL(2, p^m)$, $n=mt$ and $1\neq t\equiv 1$ (mod 2).

(f) $PGL(2, p^m)$, $n=2mt$ and $t\equiv 1$ (mod 2).

We note a centralizer of an involution of $L_2(q)$ with q odd has order $(q-\varepsilon)$ and $L_2(q)$ has a unique conjugate class of involutions.

If H is of type (a), by (2.4), we nave

$$5=\frac{(q-\varepsilon)(z+1)}{2z}=\frac{(q-\varepsilon)/2}{z}\cdot(z+1).$$

Hence $z+1=5$ and $\dfrac{(q-\varepsilon)/2}{z}=1$, whence $q=7$ or 3^2. Thus (3) or (4) of Theorem 3 holds.

If H is of type (b), we have

$$5=\frac{(q-\varepsilon)\cdot 3}{|A_4|}=\frac{q-\varepsilon}{4}.$$

Thus (5) of Theorem 3 holds.

If H is of type (c), we have

$$5=\frac{(q-\varepsilon)\cdot 9}{|S_4|}=\frac{(q-\varepsilon)\cdot 3}{8}, \text{ which can not occur.}$$

If H is of type (d), we have

$$5=\frac{(q-\varepsilon)\cdot 15}{|A_5|}=\frac{q-\varepsilon}{4}.$$

Hence (6) of Theorem 3 holds.

If H is of type (e), we have

$$5=\frac{(q-\varepsilon)\cdot|PSL(2, p^m)|/(p^m-\varepsilon)}{|PSL(2, p^m)|}=\frac{p^{mt}-\varepsilon}{p^m-\varepsilon},$$

which can not occur since p^m, $t\geqq 3$ and $\varepsilon\in\{-1, 1\}$.

If H is of type (f), we have

$$\begin{aligned}5&=\frac{(q-1)\cdot(p^m)^2}{|PGL(2, p^m)|}\\&=\frac{\{(p^m)^{t-1}+\cdots+(p^m)+1\}\cdot\{(p^m)^{t-1}-(p^m)^{t-2}+\cdots-(p^m)+1\}\cdot p^{2m}}{p^m}.\end{aligned}$$

Hence we get $t=1$ and $p^m=5$. Thus (7) of Theorem 3 holds.

CASE (iii).

LEMMA 7. *Let G be a group isomorphic to $L_3(q)$ or $U_3(q)$ for q odd. If $q\neq 3, 5$ then G has no $(1, 5)$-permutation representation.*

PROOF. Suppose false. Let (G, Ω) be a $(1, 5)$-group and T be a Sylow 2-

subgroup of G_α with $\alpha\in\Omega$. Since T is semi-dihedral or wreathed, G has a unique conjugate class of involutions ([1]). Hence an involution z contained in $Z(T)$ is a 5-involution. $C_G(z)$ is isomorphic to a quotient of either $GL(2, q)$ or $GU(2, q)$ by a central subgroup Z of order $(q-\varepsilon, 3)$ where $\varepsilon=1$ or -1, respectively ([1]). Hence $G_G(Z)$ has a normal subgroup N of index $q-\varepsilon/(q-\varepsilon, 3)$ isomorphic to $SL(2, q)$.

Let K_0 be the kernel of the action of $C_G(z)$ on $F(z)$. Since $q>5$ and $z\in K_0$, N is contained in K_0 and so $C_G(z)/K_0$ is isomorphic to a subgroup of Z_r with $r=q-\varepsilon/(q-\varepsilon, 3)$. Set $K=K_0\cap T$. By (2.6), we have $T\neq K$ and so T/K is isomorphic to Z_2 or Z_4. Hence $|K|^2>T$ because T is semi-dihedral or wreathed. Thus K is a weakly closed subgroup of T and so $N_G(K)$ is transitive on $F(z)$ by the Witt's Theorem. Since $|F(K)|=5$, there exists a 5-element x in $N_G(K)$ such taht $\langle x\rangle$ is transitive on $F(K)=F(z)$. By the structure of T, x centralizes $\Omega_1(Z(K))$, which contains z. Hence x is contained in $C_G(z)$. Thus $C_G(z)/K_0$ contains a cyclic subgroup of order $2\cdot 5$, contrary to $|F(z)|=|F(K)|=5$.

Simple group with semi-dihedral Sylow 2-subgroups are $L_3(q)$ $(q\equiv -1(\mathrm{mod}\,4))$, M_{11} or $U_3(q)\,(q\equiv 1\,(\mathrm{mod}\,4))$ by Third Main Theorem of [1]. By Lemma 7, we can prove that a simple (1, 5)-group of type (iii) is (9) of Theorem 3.

CASE (iv)

LEMMA 8. *Let G be a* (1, 5)*-group on Ω with $O^2(G)=G$ and z be a central involution such that*

(*) $C_G(z)=L_1\cdot L_2\langle u\rangle$,

$$L_1\cong SL(2,\, q_1),\ L_2\cong SL(2,\, q_2),\ u^2=1\,,$$

$$[L_1,\, L_2]=1,\ Z(L_1)=Z(L_2)=L_1\cap L_2=\langle z\rangle\,,$$

$$u^{-1}L_1\cdot L_2u=L_1\cdot L_2\,.$$

Then one of the following holds:

(a) $q_1\leqq 5$ *or* $q_2\leqq 5$.

(b) z *is not a* 5*-involution.*

PROOF. Suppose false. Let T be a Sylow 2-subgroup of G such that $z\in Z(T)$ and $u\in T$. Since $|\Omega|$ is odd, there exists $\alpha\in\Omega$ with $T\leqq G_\alpha$.

Let K_0 be the kernel of the action of $C_G(z)$ on $F(z)$. Since $|F(z)|=5$, $q_1>5$, $q_2>5$ and z is contained in K_0, it follows that L_1 and L_2 are contained in K_0. Hence we have $|T:K|\leqq 2$ where $K=T\cap K_0$. By (2.6), we have $T\neq K$ and so $T/K\cong Z_2$, $u\notin K$. Since the 2-group T is not of maximal class, we have $|C_T(u)|\geqq 8$, hence $|C_K(u)|\geqq 4$. On the other hand we have $|C_K(u)|\leqq 4$ because K acts semi-regularly on $\Omega-F(K)$, hence $|C_T(u)|=8$. By (2.3), we get $ccl_G(u)\cap K\neq\emptyset$. Clearly there exists an extremal element w of T in G with $w\in K\cap ccl_G(u)$. There exists $g\in G$ such that $u^g=w$ and $(C_T(u))^g\leqq C_T(w)$. Since $F(u)\neq F(w)=F(K)$, we get $(C_K(u))^g\cap K=1$. Thus $|C_K(u)|=|(C_K(u))^g|\leqq 2$, a contradiction.

LEMMA 9. *Let G be a finite group isomorphic to $G_2(q)$, $D_4^2(q)$ or $PS_p(4, q)$ for q odd. If q is not equal to 3 or 5, then G has no (1, 5)-permutation representation.*

PROOF. Suppose false. We note that a centralizer of a central involution in the groups $G_2(q)$, $D_4^2(q)$ and $PS_p(4, q)$ for q (>5) odd is of type (*) of Lemma 8 ([8]). Moreover $G_2(q)$ and $D_4^2(q)$ for q odd have a unique conjugate class of involutions and so Lemma 8 shows that $G_2(q)$ and $D_4^2(q)$ $(q>5)$ have no (1, 5)-permutation representation. Since $PS_p(4, q)$ for q (>5) odd has two conjugate classes of involutions, G is isomorphic to $PS_p(4, q)$ for some q with q (>5) odd and central involutions are 1-involutions. Hence noncentral involutions are 5-involutions and $|\Omega|\equiv 1 \pmod 8$ by (3.2). Let z be a central involution of G. Then the following holds ([22]):

$$C_G(z)=L_1L_2\langle u\rangle \qquad [L_1, L_2]=1 \qquad u^2=1$$
$$L_1^u=L_2 \qquad L_1\cong L_2\cong SL(2, q)$$
$$L_1\cap L_2=Z(L_1)=Z(L_2)=\langle z\rangle \qquad ccl_G(z)\not\ni u\,.$$

From this, $M=\{xx^u \mid x\in L_1\}$ is a subgroup of $C_G(u)$ and isomorphic to $L_2(q)$ with $\langle u\rangle\cap M=1$. Let K_0 be the kernel of the action of $L_1\cdot L_1\langle u\rangle\cap C_G(u)$ on $F(u)$. Since $|F(u)|=5$ and $q>5$, M is contained in K_0, hence $\langle u\rangle\times M\leqq K_0$. Thus we have $|\Omega|\equiv 5 \pmod 8$ because $|\langle u\rangle\times M|_2\geqq 8$, which is contray to $|\Omega|\equiv 1 \pmod 8$.

LEMMA 10. *Let q (>5) be equal to an odd power of 3. $Re(q)$ has no (1, 5)-permutation representation. (Here $Re(q)$ is a group of Ree type.)*

PROOF. Suppose false. Let z be an involution of $Re(q)$. The centralizer of z in $Re(q)$ is equal to $\langle z\rangle\times L$ where L is isomorphic to $L_2(q)$. Since $Re(q)$ has a unique conjugate class of involutions, z is a 5-involution. Let K_0 be the kernel of the action of $\langle z\rangle\times L$ on $F(z)$. Then $L\leqq K_0$ because $|F(z)|=5$ and $q \geqq 3^3$. Hence $\langle z\rangle\times L=K_0$, which is contrary to (2.6).

LEMMA 11. *Let q be a power of an odd prime and G be a finite group isomorphic to $U_4(q)$ $(q\not\equiv 7 \pmod 8)$ or $L_4(q)$ $(q\not\equiv 1 \pmod 8)$. If $q>5$, G has no (1, 5)-permutation representation.*

PROOF. We can easily show that a Sylow 2-subgroup of G has order at least 2^9 when $q\equiv 1, 7 \pmod 8$. Moreover $U_4(q)$ with $q\equiv 3 \pmod 8$ and $L_4(q)$ with $q\equiv 5 \pmod 8$ have a unique conjugate class of involutions. Hence by Theorem 1 and Theorem of [18], G has no (1, 5)-permutation representation with the exception of $U_4(q)$ with $q\equiv 5 \pmod 8$ and $L_4(q)$ with $q\equiv 3 \pmod 8$. From this, if the lemma is false, G is isomorphic to $U_4(q)$ with $q\equiv 5 \pmod 8$ or $L_4(q)$ with $q\equiv 3 \pmod 8$. Let z be a central involution of G and $q\equiv\varepsilon\in\{-1, 1\} \pmod 4$. Then $C_G(z)$ has the following structure ([16], [17]):

(a) $C_G(z)=L_1L_2\langle u, w\rangle\triangleright L_1L_2$

$$L_1\cong L_2\cong SL(2, q),\ [L_1, L_2]=1$$

$$L_1\cap L_2=Z(L_1)=Z(L_2)=\langle z\rangle,\ L_1L_2\cap\langle v, w\rangle=1$$

$$\langle v, w\rangle\cong\text{the dihedral group of order } 2(q+\varepsilon)$$

$$u^2=1,\ w^u=w^{-1},\ L_1^u=L_2\,.$$

(b) G has two conjugate classes of involutions:

$$u\sim z\not\sim uz$$

$$C_G(z)\cap C(u)=C_G(z)\cap C(uz)$$

$$\geqq\langle u\rangle\times\langle x_1x_1^u\,|\,x_1\in L_1\rangle,\ \langle uz\rangle\times\langle x_1x_1^u\,|\,x_1\in L_1\rangle\,.$$

First we consider the case that z is a 5-involution. Let K_0 be the kernel of the action of $C_G(z)$ on $F(z)$. Since $q>5$ and $|F(z)|=5$, we have $L_1L_2\leqq K_0$. Set $q+\varepsilon=2^n\cdot d$ with q odd. Since $q\equiv\varepsilon\in\{-1, 1\}$ (mod 4), n is equal to 1, hence $v=w^d$ is an involution and $|\langle u, w\rangle|_2=|\langle u, v\rangle|$. Let T be a Sylow 2-subgroup of $C_G(z)$ with $T\geqq\langle u, v\rangle$. Set $K=T\cap K_0$. If $u\in K_0$, we have $|T:K|=2$. In this case, v is a 5-involution, hence $|C_K(v)|\leqq 4$. On the other hand, we have $\langle z, u\rangle\leqq C_K(v)$, hence $|C_K(v)|=4$. There exists an extremal element v_0 of T in G with $v_0\in K\cap ccl_G(v)$. There exists $g\in G$ such that $v^g=v_0$ and $(C_T(v))^g\leqq C_T(v_0)$. Since $F(v)\neq F(K)=F(v_0)$, we have $(C_K(v))^g\cap K=1$. Thus $|C_K(v)|=|(C_K(v))^g|=2$, a contradiction. If $u\notin K_0$, we have $F(u)\neq F(z)$. Since $\langle x_1x_1^u\,|\,x_1\in L_1\rangle$ is a subgroup of K_0 isomorphic to $L_2(q)$, the set $F(\langle z\rangle\times\langle x_1x_1^u\,|\,x_1\in L_1\rangle)$ is equal to $F(K)$, which shows $|F(u)|\geqq 2^3+1$, a contradiction.

Now we consider the case that z is a 1-involution. In this case uz is a 5-involution by (b). Since $\langle uz\rangle\times\langle x_1x_1^u\,|\,x_1\in L_1\rangle$ is isomorphic to $Z_2\times L_2(q)$ with $q>5$, we get $|F(\langle uz\rangle\times\langle x_1x_1^u\,|\,x_1\in L_1\rangle)|=|F(\langle uz\rangle)|=5$, hence $|\Omega-F(uz)|\equiv 0$ (mod 8), which is contrary to (3.2).

By Lemma 7–11, Theorem 1 and Harada's Theorem ([**10**]), we can easily show that a simple (1, 5)-group of type (iv) is one of the groups listed in (8) (10) (11) and (12) of Theorem 3 and the others in the Harada's list of Main Theorem of [**10**] have no (1, 5)-permutation representation.

§ 7. Proof of Theorem 4.

Let (G, Ω) be a (2, 5)-group and N be a minimal normal subgroup of G.

First we suppose N is an elementary abelian p-group for some prime p and G is not of type (1) of Theorem 4. Clearly p is equal to 5 and G is a subgroup of automorphisms of an affine space over $GF(5)$ of dimension 2 or 3 because G_α contains a four group whose involutions have 1 or 5 fixed points. In the case of $|N|=5^3$, G has no 1-involution.

(7.1) If N is isomorphic to $Z_5\times Z_5\times Z_5$ and G is not of type (1) of Theorem 4, then (2) of Theorem 4 holds.

PROOF. Let G_α be a stabilizer of a point $\alpha\in\Omega$. We may assume G_α is a subgroup of $GL(3, 5)$. Since G_α is transitive on $\Omega-\{\alpha\}$, $|\Omega|-1=2^2\cdot 31$ divides $|G_\alpha|$ and any element of order 31 has a unique fixed point.

If G_α has an elementary abelian normal subgroup A of odd order, we have $|A|=31$ and A acts semi-regularly on $\Omega-\{\alpha\}$. By assumption, G_α contains a four group B, which normalizes A, hence some involution $x\in B$ centralizes A. Since $|C_N(y)|=5$ for any $y\in B$, we have $|F(x)|=5$ and A acts on $F(x)$. Hence A is not semi-regular on $\Omega-\{\alpha\}$, a contradiction.

If G_α has an elementary abelian normal subgroup A of even order, an element $v\in G_\alpha$ of order 31 centralizes A. By semi-regularity of v on $\Omega-\{\alpha\}$, every involution in A have a unique fixed point α, a contradiction.

Thus a minimal normal subgroup A of G_α is the direct product of isomorphic non abelian simple groups. Since A is a subgroup of $GL(3, 5)$, A is a simple group. The order of A is divisible by 31 because A is $\frac{1}{2}$-transitive on $\Omega-\{\alpha\}$. Hence A is contained in $SL(3, 5)$. Let Q be a Sylow 31-subgroup of A. By Sylow's theorem, we have $|A: N_A(Q)|=2^5$ or $2^5 5^3$ and so a Sylow 2-subgroup of A is isomorphic to that of $SL(3, 5)$. Since $A\leqq SL(3, 5)$, we get $A=SL(3, 5)$. If A is a proper subgroup of G_α, it follows that the element $\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$ is contained in G_α, which is a 25-involuton, a contradiction. Hence $G_\alpha=A=SL(3, 5)$, which shows (7.1).

(7.2) If N is isomorphic to $Z_5\times Z_5$, then we have (3), (4), (5) or (6) of Theorem 4.

PROOF. Let G_α be the stabilizer of a point $\alpha\in\Omega$. We may assume G_α is a subgroup of $GL(2, 5)$. Since G_α is transitive on $\Omega-\{\alpha\}$, $|G_\alpha|$ is divisible by $|\Omega-\{\alpha\}|=2^3\cdot 3$. The order of $G_{\alpha\beta}$ for $\beta\in\Omega-\{\alpha\}$ is even because $G_{\alpha\beta}$ contains a 5-involution, hence $|G_\alpha|$ is divisible by $2^4 3$.

If $|G_\alpha|$ is divisible by 5, it follows that $G_\alpha=GL(2, 5)$ or a subgroup of $GL(2, 5)$ of index 2 containing $SL(2, 5)$. An involution in $GL(2, 5)$ fixes one or five points and $SL(2, 5)$ contains no 5-involution, hence we have (3) or (4) of Theorem 4.

If $|G_\alpha|$ is not divisible by 5, we have $|G_\alpha|=2^4\cdot 3$ or $2^5\cdot 3$. The normalizer of a Sylow 3-subgroup of $GL(2, 5)$ has order $2^3\cdot 3$, hence $O(G_\alpha)=1$ and $O_2(G_\alpha)\neq 1$. Since $O(G_\alpha)=1$, an element of order 3 can not centralize $O_2(G_\alpha)$, hence it can not stabilize the following normal series: $O_2(G_\alpha)\triangleright O_2(G_\alpha)\cap SL(2, 5)\triangleright 1$. Since the factor group $O_2(G_\alpha)/O_2(G_\alpha)\cap SL(2, 5)$ is cyclic and a Sylow 2-subgroup of $SL(2, 5)$ is quaternion of order 8, it follows that $O_2(G_\alpha)\cap SL(2, 5)$ is a Sylow

2-subgroup of $SL(2, 5)$. Set $P=O_2(G_\alpha)\cap SL(2, 5)$. G_α is contained in $N_{GL(2,5)}(P)$, which is a subgroup of $GL(2, 5)$ of index 5. Hence we obtain (5) or (6) of Theorem 4.

Next we assume that N is not solvable. In this case N is a simple $(1, \mu)$-group where $\mu\in\{1, 3, 5\}$ or N is isomorphic to $A_5\times A_5$ and G is a subgroup of $\mathrm{Aut}(N)$ containing N. We note N_α is $\frac{1}{2}$-transitive on $\Omega-\{\alpha\}$ for $\alpha\in\Omega$ because G_α is transitive on $\Omega-\{\alpha\}$ and $G_\alpha\triangleright N_\alpha$. From this N is not isomorphic to $A_5\times A_5$.

(7.3) If N is a simple (1, 1)-group, then (7), (8), (9) or (10) of Theorem 4 holds.

PROOF. If N is a simple (1, 1)-group, N is isomorphic to one of the following groups in its usual representation: $L_2(2^n)$, $S_Z(2^n)$, $U_3(2^n)$ $(n\geqq 2)$. Since N is 2-transitive on Ω, it will suffice to consider that G is a (1, 5)-group or not. Let T be a Sylow 2-subgroup of N_α $(\alpha\in\Omega)$ and x be a 5-involution in G_α. Since N_α is 2-closed ([**3**]), x normalizes T and also $Z(T)$, which is an elementary abelian 2-group. We have $|C_{Z(T)}(x)|\leqq 2^2$ by semi-regularity of T on $\Omega-\{\alpha\}$ and so $|Z(T)|\leqq 2^4$ by (2.7), hence $2\leqq n\leqq 4$. From this we can verify (7.3) by [**21**].

(7.4) If N is a simple (1, 3)-group, G is isomorhic to S_7 in its usual representation, that is, (11) of Theorem 4 holds.

PROOF. Let M be the subgroup which consists of all even permutations in G. Since a 3-involution is a even permutation in this case and G contains a 5-involution, we have $|G:M|=2$ and involutions in M are 3-involutions. Since $G_{\alpha\beta}$ contains a 5-involution for $\alpha\neq\beta\in\Omega$, it follows that $|G_{\alpha\beta}: M_{\alpha\beta}|=|G_{\alpha\beta}\cdot M: M|=2$ and so M is a (2, 3)-group. By King's Theorem ([**14**]), M is isomorphic to (a), (b), (f), (g), (h) or (i) of his list. Hence we can easily verify (7.4).

(7.5) If N is a simple (1, 5)-group, then (12), (13), (14) or (15) of Theorem 4 holds.

PROOF. If N is of type (1) or (2) of Theorem 3, any element in $T^\#$ has the same set of fixed points (see the proof of Lemma 5). Here T is a unique Sylow 2-subgroup of N_α $(\alpha\in\Omega)$. Since T is characteristic in N_α, T is a normal subgroup of G_α, hence T fixes $\Omega-\{\alpha\}$ pointwise, a contradiction.

The automorphism groups of the simple groups (3)-(12) of Theorem 3 are known. Hence we can verify (7.5).

References

[1] J. L. Alperin, R. Brauer and D. Gorenstein, Finite groups with quasi-dihedral and wreathed Sylow 2-subgroups, Trans. Amer. Math. Soc., **151** (1970), 1-261.

[2] H. Bender, Endliche zweifach transitive Permutationsgruppen, deren Involutionen keine Fixpunkte haben, Math. Z., **104** (1968), 175-204.

[3] H. Bender, Transitive Gruppen gerader Ordnung, in denen jede Involution genau

einen Punkt festläst, J. Algebra, **17** (1971), 527-554.

[4] F. Buekenhout, Doubly transitive groups in which the maximum number of fixed points of involutions is four, to appear.

[5] F. Buekenhout, Transitive groups in which involutions fix one or three points, J. Algebra, **23** (1972), 438-451.

[6] F. Buekenhout and P. Rowlinson, On (1, 4)-groups II, to appear.

[7] P. Fong, Sylow 2-subgroups of small order, to appear.

[8] P. Fong and W. J. Wong, A characterization of the finite simple groups PS_p (4, q), G_2 (q), $D_4^2(q)$ I, Nagoya Math. J., **36** (1969), 143-184.

[9] D. Gorenstein, Finite Groups, Harper and Row, New York, 1968.

[10] D. Gorenstein and K. Harada, Finite groups whose 2-subgroup are generated by at most 4 elements, to appear.

[11] K. Harada, On finite groups having self-centralizing 2-subgroups of small order, J. Algebra, **33** (1975), 144-160.

[12] C. Hering, Zweifach transitive Permutationsgruppen, in denen 2 die maximal Anzahl von Fixpunkten von Involutionen ist, Math. Z., **104** (1968), 150-174.

[13] B. Huppert, Endliche Gruppen I, Springer-Verlag, Berlin, 1968.

[14] J. D. King, Doubly transitive groups in which involutions fix one or three points, Math. Z., **111** (1969), 311-321.

[15] R. Noda, Doubly transitive groups in which the maximal number of fixed points of involutions is four, Osaka J. Math., **8** (1971), 77-90.

[16] K. W. Phan, A characterization of four-dimensional unimodular groups, J. Algebra, **15** (1970), 252-279.

[17] K. W. Phan, A characterization of the unitary groups $PSU(4, q^2)$, q odd, J. Algebra, **17** (1971), 132-148.

[18] P. Rowlinson, Simple permutation groups in which an involution fixes a small number of points, J. London Math. Soc., (2) **4** (1972), 655-661.

[19] P. Rowlinson, Simple permutation groups in which an involution fixes a small number of points II, to appear.

[20] P. Rowlinson, On (1, 4)-groups I, to appear.

[21] R. Steinburg, Automorphisms of finite linear groups, Canadian J. of Math., **12** (1960), 606-615.

[22] W. J. Wong, A characterization of the finite projective symplectic groups PS_{p_4} (q), J. Algebra, **14** (1970), 1-35.

Yutaka Hiramine
Department of Mathematics
Osaka Kyoiku University
Minamikawabori, Tennoji-ku
Osaka, Japan