J. Math. Soc. Japan
Vol. 27, No. 2, 1975

On Selberg's trace formula

By Ramesh GANGOLLI and Garth WARNER

(Received Sept. 12, 1972)

§ 1. Introduction.

Let G be a connected, non-compact, semisimple Lie group with finite center, and let Γ be a discrete subgroup of G such that the space G/Γ is compact. Fix a G-invariant measure $d\dot{x}$ on G/Γ, and denote by $L_2(G/\Gamma)$ the Hilbert space of measurable functions on G/Γ that are square-integrable with respect to this measure $d\dot{x}$. We shall view an element of $L_2(G/\Gamma)$ as a function on G, invariant under right translations by elements of Γ. G acts on $L_2(G/\Gamma)$ by the left regular representation U. Thus for $f\in L_2(G/\Gamma)$, $x\in G$, $(U(x)f)(y)=f(x^{-1}y)$, $y\in G$. U is a unitary representation of G, whose study is important in the theory of automorphic functions.

Under the hypothesis that G/Γ is compact, it is well-known (see e. g. Gelfand et al. [3]) that the representation U decomposes into a discrete direct sum of irreducible unitary representations of G, and, moreover, that the multiplicity with which any given irreducible unitary representation of G occurs in this decomposition is finite. Except in special cases, not much is known about which representations occur in U, and what their multiplicities are.

Now let K be a maximal compact subgroup of G. Let $U_0, U_1, \cdots$ be the inequivalent irreducible unitary representations of *class one* with respect to K that occur in U, and let $n_0, n_1, \cdots$ be their multiplicities. We can assume that U_0 is the trivial representation of G, and so $n_0=1$. Our object in the present paper is to get some information about the multiplicities n_i $(i=0, 1, \cdots)$.

Let $G=KAN$ be an Iwasawa decomposition of G, and let $\mathfrak{a}=$Lie algebra of A. If $\mathfrak{F}$ is the space of complex valued linear functions on $\mathfrak{a}$, then for every $\lambda\in\mathfrak{F}$ one has the elementary (zonal) spherical function φ_λ on G, defined by $\varphi_\lambda(x)=\int_K \exp(\lambda-\rho)(H(xk))dk$ $(x\in G)$, where ρ is the half-sum of the positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$ and $H(x)$ is the unique element of $\mathfrak{a}$ such that $x\in K\exp H(x)N$. If W is the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$, then it is known that $\varphi_{\lambda'}=\varphi_{\lambda''}$ if and only if λ' and λ'' are conjugate under W.

Returning to the representation U, let $\varphi_0, \varphi_1, \cdots$ be the positive definite elementary spherical functions that correspond to $U_0, U_1, \cdots$ etc. Then, by what we said above, we can find elements $\lambda_j\in\mathfrak{F}$ so that $\varphi_j=\varphi_{\lambda_j}$, $j=0, 1, \cdots$, each

λ_j being determined up to an action of W. Let now $\langle , \rangle$ be the complex bilinear form on $\mathfrak{F}$ induced by the Cartan-Killing form on $\mathfrak{a}$, which we shall also denote by $\langle , \rangle$. Then we shall prove:

THEOREM I. *There exists an integer d such that*

$$\sum_{j\geqq 0} n_j(1-\langle\lambda_j, \lambda_j\rangle+\langle\rho, \rho\rangle)^{-d}<\infty .$$

It should be mentioned that since all the φ_j are positive definite, it is possible to show that $-\langle\lambda_j, \lambda_j\rangle+\langle\rho, \rho\rangle\geqq 0$ for all j. Hence, if we view λ_j as tagging the representation U_j, the above result says that the spectrum $\{n_j\}$ is "tempered" with respect to the parameter λ_j.

In particular, we see that if r is any positive real number, then the number of indices j for which $-\langle\lambda_j, \lambda_j\rangle+\langle\rho, \rho\rangle\leqq r$ is finite. Since $(\langle\lambda_j, \lambda_j\rangle-\langle\rho, \rho\rangle)$ is just the eigenvalue by which the Casimir element of G acts on U_j, we get:

COROLLARY. *Let ω_j be the eigenvalue by which the Casimir element Ω acts on U_j. Then the numbers ω_j are $\leqq 0$ and have no finite point of accumulation on the line.*

A word about proofs is appropriate here. We shall prove Theorem I by applying results of Trombi and Varadarajan [**12**]. It is also possible to prove Theorem I by appealing to classical results of Minakshisundaram and Pleijel. We prefer not to use this method. H. Garland [**2**] has proved a theorem in content similar to the above corollary. Our methods are different.

Actually, it is desirable to have the precise asymptotic behaviour of the spectral multiplicities $\{n_j\}_{j\geqq 0}$. A valuable tool in studying these is the Selberg trace formula. In the situation we study, viz: the class one case, this may be described as follows (cf. Selberg [**10**], Tamagawa [**11**]). Let $I_1(G)$ be the convolution algebra of K-biinvariant integrable functions. Thus $I_1(G)=\{f \mid f\in L_1(G), f(k_1xk_2)=f(x) \text{ for } x\in G,\ k_1, k_2\in K\}$. For $f\in I_1(G)$, the operator $U(f)=\int_G f(x)U(x)dx$ is an integral operator on $L_2(G/\Gamma)$ with kernel

$$K_f(x, y)=\sum_{\gamma\in\Gamma} f(x\gamma y^{-1}) \tag{1.1}$$

where the series converges absolutely for almost all pairs (x, y). Following Selberg and Tamagawa, one says that $f\in I_1(G)$ is *admissible* if the series on the right of (1.1) converges to a continuous function of the pair (x, y), *and* if the operator $U(f)$ is of the trace class. If f is admissible, then the trace of $U(f)$ can be computed in two different ways. On the one hand it equals $\int_{\mathfrak{D}} K_f(x, x)dx$ where $\mathfrak{D}$ is a fundamental domain for Γ in G. On the other hand, one also has that $\mathrm{Trace}\,(U(f))=\sum_{j\geqq 0} n_j\,\mathrm{Trace}\,(U_j(f))$. Now, it is easy to see that $\mathrm{Trace}\,(U_j(f))=\hat{f}(\lambda_j)$ where $\hat{f}$ is the spherical Fourier transform of f, defined by

$$\hat{f}(\lambda)=\int_G f(x)\varphi_{-\lambda}(x)dx \tag{1.2}$$

for any $\lambda\in\mathfrak{F}$ such that $\varphi_{-\lambda}$ is bounded. Thus Selberg's trace formula asserts that for admissible f, we have

$$\int_{\mathfrak{D}} K_f(x, x)dx=\sum_{j\geqq 0} n_j\hat{f}(\lambda_j)\,, \tag{1.3}$$

with the series on the right converging absolutely.

Experience shows that it is valuable to have a large class of functions which are known to be admissible. For example, consider a function $f\in I_c^\infty(G)$, the space of K-biinvariant, C^∞ functions with compact support. It is known in this case that the left side of (1.1) is a continuous function of (x, y), and that $U(f)$ is Hilbert-Schmidt, so that $\sum_{j\geqq 0} n_j|\hat{f}(\lambda_j)|^2<\infty$. See e.g. Gelfand et al. [**3**]. However, unless one knows something about the absolute convergence of the right side of (1.3), it is not possible to apply Selberg's formula to such an f. As a result of Theorem I, we shall see that the class of admissible functions is really rather wide. In fact, we have:

THEOREM II. *There exists an integer p with the following property: If f is a continuous spherical function such that* i) $\sum_{\gamma\in\Gamma} f(x\gamma y^{-1})$ *converges uniformly on compacta in $G\times G$;* ii) *f is of class C^{2p};* iii) *$f\in L_1(G)$ and $\Omega^p f\in L_1(G)$, then f is admissible.*

We should note that, in particular, every function in $I_c^\infty(G)$ is admissible.

Next let us define the space $\mathcal{I}^1(G)$ (cf. Trombi and Varadarajan [**12**]) as follows. Let us denote by $\Xi(x)$ the elementary spherical function $\varphi_0(x)$. Also, for any $x\in G$, let $x=k\exp X$, $k\in K$, $X\in\mathfrak{p}$, where $\mathfrak{p}$ is the orthogonal complement of the Lie algebra $\mathfrak{k}$ of K in the Lie algebra $\mathfrak{g}$ of G. Defining $|\cdot|$ to be the norm on $\mathfrak{p}$ induced by the Cartan-Killing form, we then write $\sigma(x)=|X|$ where $x=k\exp X$. Now for any left invariant differential operator D on G and any integer $r\geqq 0$, define the semi-norm $\nu_{D,r}$ on $I^\infty(G)$ by: $\nu_{D,r}(f)=\sup_{x\in G}|(Df)(x)|(\Xi(x)^{-2}(1+\sigma(x))^r)$. Then

$$\mathcal{I}^1(G)=\{f\in I^\infty(G)\mid \text{ for each } D, r, \nu_{D,r}(f)<\infty\}\,.$$

The space $\mathcal{I}^1(G)$ clearly contains $I_c^\infty(G)$. It is the K-biinvariant, L_1-analogue of the Schwartz-space $\mathcal{C}(G)$ of Harish-Chandra. It is easily seen that $\mathcal{I}^1(G)\subset I_1(G)$, and, for any D, $Df\in\mathcal{I}^1(G)$ if $f\in\mathcal{I}^1(G)$. Hence Theorem II implies immediately that every $f\in\mathcal{I}^1(G)$ is admissible. We then have:

THEOREM III. *The map $f\to\mathrm{Trace}(U(f))$ is continuous in the topology induced on $\mathcal{I}^1(G)$ by the semi-norms $\nu_{D,r}$.*

§ 2. Notation.

G is a connected semisimple Lie group with finite center, K is a maximal compact subgroup of G.

Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G, K respectively, and let $\langle\cdot,\cdot\rangle$ denote the Cartan-Killing form. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Δ will stand for the roots of $(\mathfrak{g}, \mathfrak{a})$. For each $\lambda\in\Delta$, let $\mathfrak{g}_\lambda$ be the root space corresponding to λ. Fix an order on the real dual of $\mathfrak{a}$, and let Δ_+ be the positive roots in this order, and let $\{\alpha_1, \cdots, \alpha_l\}$ be the simple roots, so that $l=\dim\mathfrak{a}$. We put $\mathfrak{a}^+=\{H\in\mathfrak{a}\,|\,\alpha_i(H)>0\ (i=1, \cdots, l)\}$. Now let $\mathfrak{n}=\sum_{\alpha\in\Delta_+}\mathfrak{g}_\alpha$, and let N be the analytic subgroup of G with Lie algebra $\mathfrak{n}$. Then $\mathfrak{n}$ is nilpotent, exp maps $\mathfrak{n}$ diffeomorphically onto N, and $G=KAN$ is an Iwasawa decomposition of G. For any $x\in G$, write $x=k(x)\exp H(x)n(x)$ with $k(x)\in K$, $H(x)\in\mathfrak{a}$, $n(x)\in N$.

Let $\mathfrak{g}_C$, $\mathfrak{a}_C$ etc. be the complexifications of $\mathfrak{g}$, $\mathfrak{a}$, etc. and let us write $\mathfrak{F}$ for the complex dual of $\mathfrak{a}_C$.

We shall denote by $\mathfrak{F}_R$, (resp. $\mathfrak{F}_I$) the subspace of $\mathfrak{F}$ consisting of linear functionals that are real (resp. purely imaginary) on $\mathfrak{a}$. The form $\langle\cdot,\cdot\rangle$ will be extended to $\mathfrak{a}_C$ as a bilinear form. As is well known, it is nondegenerate on $\mathfrak{a}_C\times\mathfrak{a}_C$. For any $\lambda\in\mathfrak{F}$, we write H_λ for the unique element of $\mathfrak{a}_C$ such that $\langle H_\lambda, H\rangle=\lambda(H)$ for all $H\in\mathfrak{a}_C$. On the other hand $\langle\cdot,\cdot\rangle$ is known to be positive definite on $\mathfrak{a}\times\mathfrak{a}$, hence also on $\mathfrak{F}_R\times\mathfrak{F}_R$. Thus using it we can define the structure of a Hilbert space on $\mathfrak{a}_C\times\mathfrak{a}_C$ and also on $\mathfrak{F}\times\mathfrak{F}$. We shall denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the corresponding inner product and norm. Note that $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ agree on $\mathfrak{a}_C\times\mathfrak{a}_C$.

A function f on G is *spherical* if $f(k_1xk_2)=f(x)$ for all k_1, $k_2\in K$, $x\in G$. We shall denote by $I^\infty(G)$ the space of spherical C^∞ functions on G, by $I_c^\infty(G)$ those of compact support and by $I_1(G)$ the space of absolutely integrable spherical functions.

Let $\mathfrak{G}$ be the universal enveloping algebra of $\mathfrak{g}_C$. We regard elements of $\mathfrak{G}$ as left-invariant differential operators on G. Let $\mathfrak{K}$, $\mathfrak{A}$ denote the universal enveloping algebras of $\mathfrak{k}_C$, $\mathfrak{a}_C$, regarded as subalgebras of $\mathfrak{G}$.

Denote by $\mathfrak{Q}$ the centralizer of $\mathfrak{K}$ in $\mathfrak{G}$. A spherical function φ on G is called *elementary* if $\varphi(1)=1$ and if it is an eigen-function of each differential operator $q\in\mathfrak{Q}$. These functions can be given a neat description, as follows: Let ρ be the half-sum of the positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$. For any $\lambda\in\mathfrak{F}$, let $\varphi_\lambda(x)$ be defined by $\varphi_\lambda(x)=\int_K\exp(\lambda-\rho)(H(xk))dk$. Then, as λ varies over $\mathfrak{F}$, the functions φ_λ are precisely all the elementary spherical functions on G. Moreover, $\varphi_{\lambda'}=\varphi_{\lambda''}$ if and only if λ' and λ'' are conjugate under the action of the Weyl group W of $(\mathfrak{g}, \mathfrak{a})$. See for example Helgason [7, Chap. X].

Let $f\in I_1(G)$. Then its spherical Fourier transform is defined by the function $\hat{f}(\lambda)=\int_G f(x)\varphi_{-\lambda}(x)dx$. Here the domain of $\hat{f}$ is the set of all λ for which the integral converges absolutely. In particular, $\hat{f}$ is defined for all λ for which $\varphi_{-\lambda}$ is bounded. It is known that φ_λ is a positive definite function when $\lambda\in\mathfrak{F}_I$. Hence $\hat{f}$ is defined at least on $\mathfrak{F}_I$ for $f\in I_1(G)$.

If α is any equivalence class of irreducible unitary representations of G, we say that α is of *class one* if the restriction to K of any representation in α contains the trivial representation of K in its reduction. It is known that if α is of class one and if π is a representation in α, then there is a unique subspace of H_π (the Hilbert space on which π acts) which is acted on trivially by $\pi(k)$, $k\in K$. If v is any unit vector in this space and if we put $\varphi_\alpha(x)=(\pi(x)v, v)$, then it can be shown that φ_α depends only on the class α and not on the choice of the representation π, and that φ_α is an elementary spherical function of positive definite type. Moreover the map $\alpha\mapsto\varphi_\alpha$ is a bijection of the set of equivalence classes of class one irreducible unitary representations of G onto the set of elementary positive definite spherical functions. For these facts, see Helgason [**7**, Chap. X].

Now in the context of §1, let $U_0, U_1, \cdots$ be the class one representations that occur in the decomposition of U on $L_2(G/\Gamma)$. We can and do assume that U_0 is the trivial representation of G. Let $n_0(=1), n_1, n_2, \cdots$ be the multiplicities. Moreover, if φ_j is the elementary spherical function attached to U_j by the above remarks, we see that $\varphi_j=\varphi_{\lambda_j}$ for a suitable $\lambda_j\in\mathfrak{F}$. λ_j is determined up to an action of W on $\mathfrak{F}$. We fix a choice of λ_j for each j.

§3. A theorem of Trombi and Varadarajan.

We need to use a mild extension of a theorem proved by Trombi and Varadarajan [**12**]. We shall state their result and the extension to be used.

Let $x\in G$. Then $x=k\exp X$, $k\in K$, $X\in\mathfrak{p}$. Put $\sigma(x)=|X|$, where $|\cdot|$ is the norm induced on $\mathfrak{p}$ by $\langle\cdot,\cdot\rangle$. Also write $\Xi(x)=\int_K \exp -\rho(H(xk))dk$. Let now $p>0$ be given. For any differential operator $D\in\mathfrak{G}$ and any integer $r>0$, we define for any $f\in I^\infty(G)$,

$$\nu^p_{D,r}(f)=\sup_{x\in G}(1+\sigma(x))^r\,\Xi(x)^{-2/p}|Df(x)|. \tag{3.1}$$

The space $\mathcal{I}^p(G)$ is defined to be the space of those $f\in I^\infty(G)$ such that for each $D\in\mathfrak{G}$ and each integer $r\geqq 0$, we have $\nu^p_{D,r}(f)<\infty$. When equipped with the topology given by the seminorms $f\mapsto\nu^p_{D,r}(f)$, $\mathcal{I}^p(G)$ is a Fréchet space. $I_c^\infty(G)$ is a dense subspace of $\mathcal{I}^p(G)$. (When $p=2$, the space $\mathcal{I}^p(G)$ is precisely that part of Harish-Chandra's Schwartz space $\mathcal{C}(G)$ which consists of spherical functions; cf. [**6**, p. 46]).

Our concern here will be with the space $\mathscr{C}^1(G)$. Since it is known that for some integer $r_0 \geqq 0$ the function $\Xi^2(1+\sigma)^{-r_0}$ is in $L_1(G)$, it follows that for any $f \in \mathscr{C}^1(G)$, any $D \in \mathfrak{G}$, and any integer $r \geqq 0$, we have $(1+\sigma)^r(Df) \in L_1(G)$. Thus surely $\hat{f}$ is defined on $\mathfrak{F}_I$. Actually $\hat{f}$ will have a holomorphic extension to a tube domain in $\mathfrak{F}$. Specifically let $\mathfrak{F}^1 = \{\lambda \in \mathfrak{F} \mid |\mathrm{Re}\, s\lambda(H)| \leqq \rho(H)$ for all $H \in \mathfrak{a}^+$, $s \in W\}$, and put $\mathfrak{F}^1_R = \mathfrak{F}^1 \cap \mathfrak{F}_R$ etc. Now, if $S(\mathfrak{F})$ is the symmetric algebra over $\mathfrak{F}$, we regard each $u \in S(\mathfrak{F})$ as giving a differential operator $\partial(u)$ on $\mathfrak{F}$. Now let $Z(\mathfrak{F}^1)$ be the space of functions F on $\mathfrak{F}^1$ satisfying the following conditions: i) F is holomorphic in the interior $\mathrm{Inf}\, \mathfrak{F}^1$ of $\mathfrak{F}^1$; ii) If $u \in S(\mathfrak{F})$ and $l \geqq 0$ is any integer, then $\zeta_{u,l}(F) = \sup_{\lambda \in \mathrm{Int}\, \mathfrak{F}^1} (1+\|\lambda\|^2)^l |(\partial(u)F)(\lambda)| < \infty$. Also we let $\bar{Z}(\mathfrak{F}^1)$ be the subset of $Z(\mathfrak{F}^1)$ consisting of those F which are W-invariant.

The condition ii) implies easily that for any $u \in S(\mathfrak{F})$ and $F \in Z(\mathfrak{F}^1)$, $\partial(u)F$ is continuous on $\mathfrak{F}^1$. Evidently $Z(\mathfrak{F}^1)$ is an algebra under pointwise multiplication. The seminorms $\zeta_{u,l}$ define the structure of a Fréchet algebra on $Z(\mathfrak{F}^1)$. We now have:

THEOREM 3.1 (Trombi and Varadarajan [**12**, p. 283]). *Let $f \in \mathscr{C}^1(G)$. Then the integral $\hat{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx$ converges absolutely for all $\lambda \in \mathfrak{F}^1$. The function $\hat{f}$ lies in $\bar{Z}(\mathfrak{F}^1)$, and the map $f \mapsto \hat{f}$ is a continuous map of $\mathscr{C}^1(G)$ into $\bar{Z}(\mathfrak{F}^1)$.*

Conversely, given an element $a \in \bar{Z}(\mathfrak{F}^1)$, it is possible to ask if it is in the image of the map $f \mapsto \hat{f}$. For this purpose, following Harish-Chandra, one defines for any $a \in \bar{Z}(\mathfrak{F}^1)$ the wave packet $\varphi_a(x)$ by

$$\varphi_a(x) = \frac{1}{w}\int_{\mathfrak{F}_I} a(\lambda)\varphi_\lambda \boldsymbol{c}(\lambda)^{-1}\boldsymbol{c}(-\lambda)^{-1}d\lambda \tag{3.2}$$

where $\boldsymbol{c}(\lambda)$ is the well-known $\boldsymbol{c}$-function of Harish-Chandra, for which an explicit formula is known, w is the order of W, and $d\lambda$ is the Euclidean measure on $\mathfrak{F}_I$ induced by its isomorphism with $\mathfrak{a}$. (One knows, after the fashion of Gindikin and Karpelevič [**19**], that the function $\boldsymbol{c}(\lambda)^{-1}\boldsymbol{c}(-\lambda)^{-1}$ is a tempered continuous function on $\mathfrak{F}_I$. An explicit formula is known for $\boldsymbol{c}$). Having defined the wave packet φ_a as above, the main result of Trombi and Varadarajan may be described as follows; cf. [**12**; pp. 297-298].

THEOREM 3.2. *Suppose $a \in \bar{Z}(\mathfrak{F}^1)$, and define φ_a as above. Let $D \in \mathfrak{G}$, and let r be a nonnegative integer. Then there exists a continuous semi-norm $\zeta_{D,r}$ on $\bar{Z}(\mathfrak{F}^1)$ such that*

$$|D\varphi_a(x)| \leqq \zeta_{D,r}(a)\, \Xi(x)^2(1+\sigma(x))^{-r}. \tag{3.3}$$

In particular, $\nu_{D,r}(\varphi_a) < \infty$ for each $D \in \mathfrak{G}$, $r \geqq 0$, so that $\varphi_a \in \mathscr{C}^1(G)$. Moreover $\hat{\varphi}_a = a$, and the map $f \mapsto \hat{f}$ is a topological isomorphism of $\mathscr{C}^1(G)$ with $\bar{Z}(\mathfrak{F}^1)$.

This result is, substantially, Theorem 3.10.1 in [**12**]. Actually a somewhat more general result, involving arbitrary $\mathcal{I}^p(G)$, $0<p<2$ (and not merely $\mathcal{I}^1(G)$) is proved there. We shall not, however, use it.

For our needs, we wish to focus on the estimate (3.3). An examination of the technique used in [12] shows that the assumption $a\in\bar{Z}(\mathfrak{F}^1)$ is not crucial for the derivation of (3.3) for a fixed D, r. It is in asserting that (3.3) holds for *every* choice of D, r that this assumption is utilized crucially, and this is what leads to the conclusion that $\varphi_a\in\mathcal{I}^1(G)$. This observation allows us to formulate a mild extension of the above, which we now undertake. Let m, l be nonnegative integers, and let us put $Z_{m,l}(\mathfrak{F}^1)$ for the space of functions F on $\mathfrak{F}^1$ such that: i) F is holomorphic on Int $\mathfrak{F}^1$, and ii) If $u\in S(\mathfrak{F})$ is any element such that degree$(u)\leqq m$, then

$$\zeta_{u,l}^1(F)=\sup_{\lambda\in\mathrm{Int}\,\mathfrak{F}^1}(1+\|\lambda\|^2)^l|(\partial(u)F)(\lambda)|<\infty .$$

Also we put $\bar{Z}_{m,l}(\mathfrak{F}^1)$ for the W-invariants in $Z_{m,l}(\mathfrak{F}^1)$. It is clear that if $m'\geqq m$, $l'\geqq l$, then $Z_{m',l'}(\mathfrak{F}^1)\subset Z_{m,l}(\mathfrak{F}^1)$. Put $Z_m(\mathfrak{F}^1)=\bigcap_{l\geqq 0} Z_{m,l}(\mathfrak{F}^1)$. Then $Z(\mathfrak{F}^1)=\bigcap_{m\geqq 0} Z_m(\mathfrak{F}^1)$. Similar statements hold for $\bar{Z}_{m,l}(\mathfrak{F}^1)$.

Since $\boldsymbol{c}(\lambda)^{-1}\boldsymbol{c}(-\lambda)^{-1}$ is tempered, it follows that there exists an integer l such that if $a\in Z_{0,l}$ then the integral defining φ_a converges absolutely (cf. [**12**], Lemma 3.5.3).

Now suppose $D\in\mathfrak{G}$ and the integer $r\geqq 0$ are given. We can then ask: What conditions on a will guarantee that $\nu_{D,r}(\varphi_a)<\infty$. Clearly it is not necessary to demand that a be in *all* the spaces $Z_{m,l}(\mathfrak{F}^1)$. Indeed if we examine the work in [12], we see that if $a\in Z_{m,l}(\mathfrak{F}^1)$, then the larger m is, the more rapidly φ_a will decrease, while the larger the integer l is, the smoother the function φ_a will be. (The situation is analogous to the problem of determining conditions on a function g on $\boldsymbol{R}^n$ in order that its Fourier transform $\hat{g}$ and all derivatives of order $\leqq k$ of $\hat{g}$ should decay faster than $(1+|x|)^{-r}$ on $\boldsymbol{R}^n$. In that case also, we do not *need* to demand that g is in the Schwartz space.)

For any spherical f on G, and integer $k\geqq 0$, let us agree to say that f has continuous derivatives of order up to k if given any $D\in\mathfrak{G}$ such that degree $(D)\leqq k$, Df exists and is continuous. Using the estimates in [**12**, § 3] and bearing in mind the above remarks we get the following result immediately.

PROPOSITION 3.3. *Let $k, r\geqq 0$ be given integers. Then there exist integers $m, l\geqq 0$ (depending on k, r) such that if $a\in Z_{m,l}(\mathfrak{F}^1)$, then the integral defining φ_a exists, and φ_a has continuous derivatives of order up to k. Moreover, if D is any element of G such that* $\deg(D)\leqq k$, *then there exists an element $u\in S(\mathfrak{F})$ such that* $\deg(u)\leqq m$ *and*

$$|(D\varphi_a)(x)|\,\zeta_{u,l}^1(a) \mapstochar\!\circ\!\!\dashv (x)^2(1+\sigma(x))^{-r} . \tag{3.4}$$

The proof of this result is substantially contained in § 3 of [**12**]. There is no point in reproducing it. It is only necessary to retrace the steps in § 3 of [**12**], bearing in mind that at each stage, a need only belong to one of the spaces $Z_{m,l}$ with a sufficiently high m, l.

COROLLARY 3.4. *Let r_0 be the smallest integer such that $\Xi^2(x)(1+\sigma(x))^{-r_0}$ is in $L_1(G)$. Then there exist integers m_0, l_0 such that if $a \in Z_{m_0,l_0}(\mathfrak{F}^1)$, then*

$$|\varphi_a(x)| \leqq C_{a,m_0,l_0}\, \Xi(x)^2(1+\sigma(x))^{-r_0} \tag{3.5}$$

where C_{a,m_0,l_0} is a constant.

This is immediate upon choosing $D=1$, $r=r_0$ in (3.4).

§ 4. Regular growth of certain functions.

We make the following definition, following Selberg [**10**].

DEFINITION 4.1. Let f be a nonnegative continuous function on G. We say that f is of *regular growth*, if there exists a neighbourhood V of the identity in G and a real number $C_{V,f} > 0$ such that for all $x \in G$, we have,

$$f(x) \leqq C_{V,f} \int_{xV} f(z)dz\,. \tag{4.1}$$

The relevance of this notion is brought out by the following well-known proposition; see e. g. Selberg [**10**] or Gelfand et al. [**3**].

PROPOSITION 4.2. *Suppose f, g are continuous functions on G such that:* i) $|f| < g$; ii) $g \in L_1(G)$, *and* iii) *g is of regular growth.*

Let U be the representation of G on $L_2(G/\Gamma)$, and let $U(f)$ be the operator $\int_G f(x)U(x)dx$ on $L_2(G/\Gamma)$. Then under the above conditions, $U(f)$ is a Hilbert-Schmidt operator, with the continuous kernel $K_f(x, y) = \sum_{\lambda\in\Gamma} f(x\gamma y^{-1})$, the series converging uniformly on compacta in $G \times G$.

Our aim at present is to establish that a class of functions on G have regular growth. Let $x \in G$. Then it is well-known that $x = k_1 \exp A(x) k_2$ where $A(x) \in$ closure $(\mathfrak{a}^+)$. Let $\lambda \in \mathfrak{F}_R$ be a real valued linear function on $\mathfrak{a}$. Then obviously $x \mapsto \exp \lambda(A(x))$ is a nonnegative continuous function on G.

LEMMA 4.3. *Let $r \geqq 0$ be an integer, and $\lambda \in \mathfrak{F}_R$. Then the function $(1+\sigma(x))^{-r} \exp \lambda(A(x))$ is of regular growth.*

PROOF. One knows [**6**] that $\sigma(x^{-1}) = \sigma(x)$ and $\sigma(xy) \leqq \sigma(x) + \sigma(y)$, $x, y \in G$. From this we conclude easily that

$$(1+\sigma(x))^{-1} \leqq (1+\sigma(y^{-1}x))(1+\sigma(y))^{-1}\,, \qquad x,\ y \in G\,. \tag{4.2}$$

So,

$$(1+\sigma(x))^{-r} \leqq (1+\sigma(y^{-1}x))^{r}(1+\sigma(y))^{-r} \tag{4.3}$$

and

$$(4.4)\qquad \begin{aligned}&(1+\sigma(x))^{-r}\exp\lambda(A(x))\\ &\leqq(1+\sigma(y^{-1}x))^{r}(1+\sigma(y))^{-r}\exp|\lambda(A(x)-A(y))|\exp\lambda(A(y))\\ &=(1+\sigma(y^{-1}x))^{r}\exp|\lambda(A(x)-A(y))|\cdot(1+\sigma(y))^{-r}\exp\lambda(A(y)).\end{aligned}$$

Now let $\delta>0$, and, for each $x\in G$, put $V_\delta(x)=\{y\in G\,|\,\sigma(y^{-1}x)\leqq\delta\}$. Clearly, if $y\in V_\delta(x)$, then $1+\sigma(y^{-1}x)\leqq 1+\delta$. According to a Lemma of Langlands [**9**, p. 104], given any $\varepsilon>0$, we can find a $\delta>0$ such that if $y\in V_\delta(x)$, then

$$|\lambda(A(x)-A(y))|\leqq\varepsilon\|\lambda\|$$

for any $\lambda\in\mathfrak{F}_R$. Fix an $\varepsilon>0$, and choose δ with this property. Then, for $y\in V_\delta(x)$, we have

$$(4.5)\qquad \begin{aligned}&(1+\sigma(x))^{-r}\exp\lambda(A(x))\\ &\leqq(1+\delta)^{r}\exp\varepsilon\|\lambda\|\cdot(1+\sigma(y))^{-r}\exp\lambda(A(y)).\end{aligned}$$

Integrating this over $V_\delta(x)$ with respect to y, we get

$$(4.6)\qquad \begin{aligned}&\mathrm{Volume}\,(V_\delta(x))\cdot(1+\sigma(x))^{-r}\exp\lambda(A(x))\\ &\leqq(1+\delta)^{r}\exp\varepsilon\|\lambda\|\cdot\int_{V_\delta(x)}(1+\sigma(y))^{-r}\exp\lambda(A(y))dy.\end{aligned}$$

We now observe that $V_\delta(x)=xV_\delta(e)$, where e is the identity of G, so $\mathrm{Volume}\,(V_\delta(x))=\mathrm{Volume}\,(V_\delta(e))$. It follows that

$$(4.7)\qquad \begin{aligned}&(1+\sigma(x))^{-r}\exp\lambda(A(x))\\ &\leqq(1+\delta)^{r}\exp\varepsilon\|\lambda\|(\mathrm{Volume}\,(V_\delta(e)))^{-1}\int_{xV_\delta(e)}(1+\sigma(y))^{-r}\exp\lambda(A(y))dy.\end{aligned}$$

Since $V_\delta(e)$ is a neighborhood of the identity, the lemma follows. Q. E. D.

Combining this with the preceding observations, we are able to prove the following proposition.

PROPOSITION 4.4. *There exist integers m_0, $l_0\geqq 0$ such that if $a\in Z_{m_0,l_0}(\mathfrak{F}^1)$, then φ_a is well-defined and, further, $|\varphi_a|$ has a continuous majorant g such that g is in $L_1(G)$ and has regular growth.*

PROOF. Let $r\geqq 0$ be an integer. We have seen in § 3 that we can find m, $l\geqq 0$ so that if $a\in Z_{m,l}(\mathfrak{F}^1)$, then φ_a is well defined, and further that

$$(4.8)\qquad |\varphi_a(x)|\leqq C_a\,\Xi(x)^2(1+\sigma(x))^{-r}$$

where $C_a>0$ is a constant depending on a.

Now the function $\Xi(x)$ is spherical, whence $\Xi(x)=\Xi(\exp A(x))$, where $A(x)$ was defined above. Since it is known that $\Xi(\exp(A(x))\leqq C\exp-\rho(A(x))\,(1+\sigma(A(x)))^d$ for a suitable $C>0$ and integer $d>0$, and since $\sigma(A(x))=\sigma(x)$, it follows that

$$\mapsto\!\!\!\circ\!\!\!\dashv (x)^2(1+\sigma(x))^{-r} \leqq C \exp -2\rho(A(x)))(1+\sigma(x))^{d-r} . \tag{4.9}$$

We have already seen that the function $\exp -2\rho(A(x))(1+\sigma(x))^{d-r}$ is of regular growth for $r \geqq d$. On the other hand, there exists an integer p such that $(\exp -2\rho(A(x)))(1+\sigma(x))^{-p} \in L_1(G)$; see e.g. [**5**, p. 279]. Hence, if we choose r_0 so that $r_0 \geqq p+d$ and choose m_0, l_0 corresponding to this r_0 so that (4.8) holds, we get the conclusion of the proposition immediately. Q.E.D.

COROLLARY 4.5. *There exist integers $m_0, l_0 \geqq 0$ such that if $a \in Z_{m_0,l_0}(\mathfrak{F}^1)$, and φ_a is the wave-packet corresponding to a, then the operator $U(\varphi_a)$ on $L_2(G/\Gamma)$ is a Hilbert-Schmidt operator.*

This follows easily from Propositions 4.2 and 4.4.

§ 5. An estimate for the spectrum.

We wish to convert the information provided by 4.5 into an estimate on the spectrum.

LEMMA 5.1. *Suppose a is a function on $\mathfrak{F}^1$ such that:* i) *a is holomorphic on* Int $\mathfrak{F}^1$, *C^∞ on $\mathfrak{F}^1$;* ii) *$a \in \bar{Z}_{m,l}(\mathfrak{F}^1)$ with $m, l \geqq 0$ so large that the integral defining φ_a converges absolutely for each $x \in G$, and $\varphi_a \in L_1(G)$;* iii) *For each $u \in S(\mathfrak{F})$, there exists an integer $k \geqq 0$ such that $\sup_{\lambda \in \mathfrak{F}^1} (1+\|\lambda\|^2)^{-k} |(\partial(u)a)(\lambda)| < \infty$.*

Then $\hat{\varphi}_a = a$.

PROOF. Let $b \in \bar{Z}(\mathfrak{F}^1)$. Then it follows from i), iii) that $ab \in \bar{Z}(\mathfrak{F}^1)$. Therefore $\varphi_{ab} \in \mathcal{I}^1(G)$ by the theorem of Trombi and Varadarajan, and $\hat{\varphi}_{ab} = ab$. Also, $\varphi_b \in \mathcal{I}^1(G)$ and $\hat{\varphi}_b = b$. Now we claim that

$$\varphi_{ab} = \varphi_a * \varphi_b . \tag{5.1}$$

This is seen as follows. We have, by definition,

$$\varphi_a(x) = \frac{1}{w} \int_{\mathfrak{F}_I} a(\lambda)\varphi_\lambda(x)\boldsymbol{c}(\lambda)^{-1}\boldsymbol{c}(-\lambda)^{-1} d\lambda . \tag{5.2}$$

Hence, since $\varphi_b \in \mathcal{I}^1(G)$,

$$\begin{aligned} (\varphi_a * \varphi_b)(x) &= \int_G \varphi_a(y)\varphi_b(y^{-1}x)dy \\ &= \frac{1}{w} \int_G \Big(\int_{\mathfrak{F}_I} a(\lambda)\varphi_\lambda(y)\boldsymbol{c}(\lambda)^{-1}\boldsymbol{c}(-\lambda)^{-1} d\lambda \Big) \varphi_b(y^{-1}x)dy . \end{aligned} \tag{5.3}$$

Since $\varphi_b \in L_1(G)$ and $a(\lambda)$ satisfies hypothesis ii), we see that Fubini's theorem applies to this integral. Interchanging the order of integration, one gets

$$(\varphi_a * \varphi_b)(x) = \frac{1}{w} \int_{\mathfrak{F}_I} a(\lambda)\boldsymbol{c}(\lambda)^{-1}\boldsymbol{c}(-\lambda)^{-1} \int_G \varphi_\lambda(y)\varphi_b(y^{-1}x)dyd\lambda . \tag{5.4}$$

Now consider

$$\begin{aligned}(5.5)\qquad \int_G \varphi_\lambda(y)\varphi_b(y^{-1}x)dy &= \int_G \varphi_\lambda(xy^{-1})\varphi_b(y)dy \\ &= \int_G\int_K \varphi_\lambda(xky^{-1})dk\varphi_b(y)dy \\ &= \int_G \varphi_\lambda(x)\varphi_\lambda(y^{-1})\varphi_b(y)dy \\ &= \varphi_\lambda(x)\int_G \varphi_\lambda(y^{-1})\varphi_b(y)dy\end{aligned}$$

where we used the sphericity of φ_b at step 2, and the property $\int_K \varphi_\lambda(xky)dk = \varphi_\lambda(x)\varphi_\lambda(y)$ of φ_λ. Now for $\lambda \in \mathfrak{F}_I$, we know that $\varphi_\lambda(y^{-1}) = \overline{\varphi_\lambda(y)} = \varphi_{-\lambda}(y)$, since in this case φ_λ is positive definite. It follows that the last term on the right in (5.5) is $\varphi_\lambda(x)\hat{\varphi}_b(\lambda)$. But $\hat{\varphi}_b = b$. Hence we get

$$(5.6)\qquad \int_G \varphi_\lambda(y)\varphi_b(y^{-1}x)dy = \varphi_\lambda(x)b(\lambda)$$

and so (5.4) gives

$$(5.7)\qquad (\varphi_a * \varphi_b)(x) = \frac{1}{w}\int_{\mathfrak{F}_I} a(\lambda)b(\lambda)\varphi_\lambda(x)\mathbf{c}(\lambda)^{-1}\mathbf{c}(-\lambda)^{-1}d\lambda = \varphi_{ab}(x).$$

This establishes (5.1).

Because $ab \in \bar{Z}(\mathfrak{F}^1)$, $\varphi_{ab} \in \mathcal{I}^1(G)$ and $\hat{\varphi}_{ab} = ab$. On the other hand, since $\varphi_a \in L_1(G)$, $\varphi_b \in \mathcal{I}^1(G)$, we have that $\hat{\varphi}_a$ is defined and holomorphic on $\mathfrak{F}^1$, $\hat{\varphi}_b = b$. Thus $\widehat{\varphi_a * \varphi_b} = \hat{\varphi}_a\hat{\varphi}_b = \hat{\varphi}_a b$. It follows that $ab = \hat{\varphi}_a b$. All that we need to observe now is that b may be chosen so that it is never zero. For example we may take $b(\lambda) = \exp -\langle\lambda, \lambda\rangle$. We can thus conclude that $\hat{\varphi}_a = a$ on $\mathfrak{F}^1$. Q. E. D.

We remark that the hypotheses of this lemma could be weakened considerably, by using approximate identities in $\mathcal{I}^1(G)$ whose Fourier transforms can be calculated.

LEMMA 5.2. *Let $\mathfrak{F}_R^1$ be the set of real parts of elements of $\mathfrak{F}^1$. Thus $\mathfrak{F}_R^1 = \{\lambda \in \mathfrak{F}_R \mid |s\lambda(H)| \leqq \rho(H)$ for all $H \in \mathfrak{a}^+$, $s \in W\}$. Let $\mathfrak{C}_\rho$ be the closed convex hull of the elements $\{s\rho, s \in W\}$ in $\mathfrak{F}_R$. Then $\mathfrak{F}_R^1 = \mathfrak{C}_\rho$.*

PROOF. In view of the Helgason and Johnson [8] characterization of bounded, elementary, spherical functions, it follows from the work of Trombi and Varadarajan [**12**] that $\mathfrak{F}_R^1$ is contained in $\mathfrak{C}_\rho$. Consider, then, the reverse inclusion. Identify $\mathfrak{a}$ with its dual $\mathfrak{F}_R$ in the usual way. Let ${}^+\mathfrak{a}$ be the dual cone to $\mathfrak{a}^+$, i.e. the set of all $H \in \mathfrak{a}$ such that $\langle H, H^+\rangle > 0$ for all $H^+ \in \mathfrak{a}^+$. Then, as is known (cf. Helgason and Johnson [8]), the closure of the set

$$\bigcup_{s \in W} s\{\mathfrak{a}^+ \cap (-{}^+\mathfrak{a} + H_\rho)\}$$

is the closed, convex hull of the set of points $H_{s\rho}(s \in W)$, i.e. is $\mathfrak{C}_\rho$. This being so, fix an element

$$H \in t\{\mathfrak{a}^+ \cap (-{}^+\mathfrak{a}+H_\rho)\} \qquad (t \in W)$$

and write $H=t(-{}^+H+H_\rho)$ (${}^+H \in {}^+\mathfrak{a}$). Let $s \in W$. Then we must prove that

$$|\langle sH, H^+\rangle| \leqq \langle H_\rho, H^+\rangle$$

for all H^+ in $\mathfrak{a}^+$. Write

$$\begin{aligned}\langle sH-H_\rho, H^+\rangle &= \langle st(-{}^+H+H_\rho)-(-{}^+H+H_\rho)+(-{}^+H+H_\rho)-H_\rho, H^+\rangle \\ &= \langle st(-{}^+H+H_\rho)-(-{}^+H+H_\rho), H^+\rangle - \langle {}^+H, H^+\rangle .\end{aligned}$$

Because $-{}^+H+H_\rho \in \mathfrak{a}^+$, it follows from a general lemma (cf. Harish-Chandra [**5**, p. 280]) that

$$\langle st(-{}^+H+H_\rho)-(-{}^+H+H_\rho), H^+\rangle \leqq 0 .$$

On the other hand, ${}^+H \in {}^+\mathfrak{a}$ and so, by definition, $-\langle {}^+H, H^+\rangle < 0$. Therefore

$$\langle sH, H^+\rangle \leqq \langle H_\rho, H^+\rangle .$$

Finally, let s_0 be that element in W which takes $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. Then the map $H^+ \mapsto -s_0 H^+$, the so-called opposition involution, takes $\mathfrak{a}^+$ bijectively onto itself. We have:

$$\begin{aligned}-\langle sH, H^+\rangle &= \langle s_0 sH, -s_0 H^+\rangle \\ &\leqq \langle H_\rho, -s_0 H^+\rangle \\ &= -\langle H_{s_0\rho}, H^+\rangle \\ &= -\langle H_{-\rho}, H^+\rangle = \langle H_\rho, H^+\rangle .\end{aligned}$$

Therefore

$$|\langle sH, H^+\rangle| \leqq \langle H_\rho, H^+\rangle ,$$

as we wished to prove. Q. E. D.

We now come to the main result of this section. Recall that the representation U contains the subrepresentations U_j, $j \geqq 0$ of class one, occurring with multiplicities n_j. Recall also that φ_{λ_j} is the (positive definite) elementary spherical function associated with U_j.

THEOREM 5.3. *There exists an integer d such that* $\sum_{j \geqq 0} n_j(1-\langle\lambda_j, \lambda_j\rangle+\langle\rho, \rho\rangle)^{-d} < \infty$.

PROOF. Let r be a positive integer and let $a_r(\lambda)$ be the function $(1-\langle\lambda, \lambda\rangle+\langle\rho, \rho\rangle)^{-r}$. We claim that $a_r(\lambda)$ is holomorphic on $\mathfrak{F}^1$. To see this it is enough to verify that for $\lambda \in \mathfrak{F}^1$, $1-\langle\lambda, \lambda\rangle+\langle\rho, \rho\rangle$ is never zero. Let $\lambda=\lambda_R+i\lambda_I$. Then $1-\langle\lambda, \lambda\rangle+\langle\rho, \rho\rangle = 1+\langle\rho, \rho\rangle-\langle\lambda_R, \lambda_R\rangle+\langle\lambda_I, \lambda_I\rangle+2i\langle\lambda_R, \lambda_I\rangle$. Now $\lambda_R \in \mathfrak{F}^1_R$ so $\lambda_R \in \mathfrak{C}_\rho$ by Lemma 5.2. Hence, by the very definition of $\mathfrak{C}_\rho$, $\langle\lambda_R, \lambda_R\rangle \leqq \langle\rho, \rho\rangle$. It follows that the real part of $1-\langle\lambda, \lambda\rangle+\langle\rho, \rho\rangle$ is always $\geqq 1$ for $\lambda \in \mathfrak{F}^1$ which proves our claim.

Clearly $a_r(\lambda)$ is W-invariant. It is easy to prove that if m, l are any given integers $\geqq 0$, then for large enough r, $a_r \in \bar{Z}_{m,l}(\mathfrak{F}^1)$. It follows from Proposition

4.4 that for large enough r, the wave-packet φ_{a_r} is defined and $|\varphi_{a_r}|$ admits an L_1-majorant of regular growth, so that by Proposition 4.2, $U(\varphi_{a_r})$ is a Hilbert-Schmidt operator on $L_2(G/\Gamma)$. Now, if U_j is the subrepresentation of U mentioned above, as we have remarked in § 1, $\mathrm{Trace}\,(U_j(f)*U_j(f))=|\hat{f}(\lambda_j)|^2$ for any spherical integrable f. It follows that

$$\mathrm{Trace}\,(U(\varphi_{a_r})*U(\varphi_{a_r}))=\sum_{j\geqq 0} n_j|\hat{\varphi}_{a_r}(\lambda_j)|^2<\infty$$

since $U(\varphi_{a_r})$ is Hilbert-Schmidt.

Now it is easy to check that if $u\in S(\mathfrak{F})$, then $|\partial(u)a_r(\lambda)|\leqq C_{u,r}(1+\|\lambda\|^2)^{-r}$. Hence the function a_r satisfies all the hypotheses of Lemma 5.1 if r is large enough. It follows that $\hat{\varphi}_{a_r}=a_r$. Hence we see that $\sum_{j\geqq 0} n_j(1-\langle\lambda_j,\lambda_j\rangle+\langle\rho,\rho\rangle)^{-2r}<\infty$, proving the assertion of the theorem. Q. E. D.

If Ω is the Casimir operator of $\mathfrak{g}$, then it is known that $\Omega\varphi_\lambda=(\langle\lambda,\lambda\rangle-\langle\rho,\rho\rangle)\varphi_\lambda$ for any $\lambda\in\mathfrak{F}$. If it also happens that φ_λ is positive definite, then one can show that the eigenvalue $\langle\lambda,\lambda\rangle-\langle\rho,\rho\rangle$ is nonpositive; see, e. g. [1]. In our case all the φ_{λ_j} are positive definite. Hence $-\langle\lambda_j,\lambda_j\rangle+\langle\rho,\rho\rangle\geqq 0$. It follows from the convergence of the series $\sum_{j\geqq 0} n_j(1-\langle\lambda_j,\lambda_j\rangle+\langle\rho,\rho\rangle)^{-d}$ that if $r\geqq 0$, then the number of indices j for which $-\langle\lambda_j,\lambda_j\rangle+\langle\rho,\rho\rangle\leqq r$ is finite. Since the numbers $\langle\lambda_j,\lambda_j\rangle-\langle\rho,\rho\rangle$ are precisely the eigenvalues by which the Casimir operator Ω acts on U_j, we get:

COROLLARY 5.5. *Let ω_j be the scalar by which the Casimir operator Ω acts on U_j. Then $\omega_j\leqq 0$ and the numbers $\{\omega_j\}_{j\geqq 0}$ have no finite point of accumulation.*

§ 6. Admissibility of functions in $\mathcal{I}^1(G)$.

Recall the notion of admissibility used in § 1.

THEOREM 6.1. *There exists an integer p with the following property: If f is a continuous spherical function such that* i) $\sum_{\gamma\in\Gamma} f(x\gamma y^{-1})$ *converges uniformly on compacta in $G\times G$;* ii) *f is of class C^{2p};* iii) *$f\in L_1(G)$ and $\Omega^p f\in L_1(G)$, then f is admissible.*

PROOF. All we need to do is to show that $\sum_{j\geqq 0} n_j\hat{f}(\lambda_j)$ converges absolutely. Now, the hypothesis $\Omega^p f\in L_1(G)$ implies immediately that its Fourier transform $\widehat{\Omega^p f}$ is bounded in absolute value. However $(\widehat{\Omega^p f})(\lambda_j)=(\langle\lambda_j,\lambda_j\rangle-\langle\rho,\rho\rangle)^p\hat{f}(\lambda_j)$. It follows that $|\hat{f}(\lambda_j)|\leqq C|\langle\lambda_j,\lambda_j\rangle-\langle\rho,\rho\rangle|^{-p}$ wherever $\langle\lambda_j,\lambda_j\rangle-\langle\rho,\rho\rangle\neq 0$. Since there are only a finite number of λ_j for which $\langle\lambda_j,\lambda_j\rangle-\langle\rho,\rho\rangle$ can be zero, we see that a suitable $C'>0$, we have $|\hat{f}(\lambda_j)|\leqq C'|1-\langle\lambda_j,\lambda_j\rangle+\langle\rho,\rho\rangle|^{-p}$. The proposition now follows from Theorem 5.3. Q. E. D.

Now, if f is any C^{2p} function of compact support, f clearly fulfills the above hypothesis. Hence we get:

COROLLARY 6.2. *Let p be the integer of Proposition* 6.1. *Suppose that f is of class C^{2p}, f is spherical and that f has compact support. Then f is admissible, so that $U(f)$ is an integral operator of trace class on $L_2(G/\Gamma)$, with continuous kernel $K_f(x, y)=\sum_{\gamma\in\Gamma} f(x\gamma y^{-1})$.*

We understand from Varadarajan that a more general version of this result has been proved by Harish-Chandra (unpublished) in the context of noncompact G/Γ, working on the discrete spectrum of $L_2(G/\Gamma)$. Our methods are, however, different.

We also have:

COROLLARY 6.3. *Every function $f\in\mathscr{C}^1(G)$ is admissible.*

PROOF. If $f\in\mathscr{C}^1(G)$, it is obvious that f satisfies hypotheses ii) and iii) of Theorem 6.1. So all we need to do is to show that $\sum_{\gamma\in\Gamma} f(x\gamma y^{-1})$ converges uniformly on compacta of $G\times G$.

Now, by the definition of the space $\mathscr{C}^1(G)$, for any integer $r\geqq 0$ there exists a real number $C_{f,r}$ such that

$$|f(x)|\leqq C_{f,r}\,\Xi(x)^2(1+\sigma(x))^{-r}\,. \tag{6.1}$$

If we choose r large enough, then the argument used in the proof of Proposition 4.4 shows that $|f|$ has an L_1 majorant of regular growth. From this it follows in a standard manner (cf. [3]) that $\sum_{\gamma\in\Gamma} f(x\gamma y^{-1})$ converges uniformly on compacta in $G\times G$. Q. E. D.

By virtue of this proposition, for each $f\in\mathscr{C}^1(G)$, Trace $(U(f))$ is defined.

THEOREM 6.4. *The map $f\mapsto$ Trace $(U(f))$ is continuous in the topology of $\mathscr{C}^1(G)$.*

PROOF. Suppose f_n is a sequence in $\mathscr{C}^1(G)$ such that $f_n\to 0$ in $\mathscr{C}^1(G)$. We wish to show that Trace $(U(f_n))\to 0$.

For any $f\in\mathscr{C}^1(G)$, and integer $r\geqq 0$, let $\nu_r(f)=\sup_{x\in G}|f(x)|\,\Xi(x)^{-2}(1+\sigma(x))^r$. Clearly $\nu_r(f)$ is finite for each r, and

$$\begin{aligned}|f(x)|&\leqq \nu_r(f)\,\Xi(x)^2(1+\sigma(x))^{-r}\\ &\leqq \nu_r(f)C\exp -2\rho(A(x))(1+\sigma(x))^{d-r}\end{aligned} \tag{6.2}$$

by virtue of (4.9).

However,by virtue of Lemma 4.3, it follows that if we write $F_r(x)$ for the function $e^{-2\rho(A(x))}(1+\sigma(x))^{d-r}$, then we can find a constant $C(r,\rho,\delta)$ such that

$$F_r(x)\leqq C(r,\rho,\delta)\int_{xV_{\delta}(e)} F_r(y)dy\,. \tag{6.3}$$

It follows that if $f\in\mathscr{C}^1(G)$, then

$$|f(x)|\leqq C\cdot C(r,\rho,\delta)\nu_r(f)\int_{V_{\delta}(e)} F_r(xy)dy \tag{6.4}$$

so

(6.5) $$|f(x\gamma x^{-1})| \leqq CC(r, \rho, \delta)\nu_r(f)\int_{V_\delta(e)} F_r(x\gamma x^{-1}y)dy$$

and

(6.6) $$\begin{aligned} |K_f(x, x)| &= |\sum_{\gamma\in\Gamma} f(x\gamma x^{-1})| \\ &\leqq \sum_{\gamma\in\Gamma} |f(x\gamma x^{-1})| \\ &\leqq C\cdot C(r, \rho, \delta)\nu_r(f)\sum_{\gamma\in\Gamma}\int_{V_\delta(e)} F_r(x\gamma x^{-1}y)dy\,. \end{aligned}$$

Since Γ is discrete, and $V_\delta(e)$ is compact, the set $\Gamma\cap x^{-1}V_\delta(e)^{-1}V_\delta(e)x$ contains only a finite number of elements, say N (N independent of x). It follows that for large r,

(6.7) $$|K_f(x, x)| \leqq C\cdot C(r, \rho, \delta)\nu_r(f)N\int_G F_r(y)dy\,.$$

Hence

(6.8) $$\begin{aligned} |\mathrm{Trace}\,(U(f))| &= \left|\int_{\mathfrak{D}} K_f(x, x)dx\right| \\ &\leqq \mathrm{Vol}\,(\mathfrak{D})C\cdot C(r, \rho, \delta)N\nu_r(f)\|F_r\|_1\,. \end{aligned}$$

Since ν_r is a continuous seminorm on $\mathcal{I}^1(G)$, $\nu_r(f_n)\to 0$ as $f_n\to 0$ in $\mathcal{I}^1(G)$, from which our assertion follows. Q.E.D.

Bibliography

[1] R. Gangolli, Asymptotic behaviour of spectra of compact quotients of certain symmetric spaces, Acta Math., 121 (1968), 151-192.

[2] H. Garland, Arithmetic subgroups and boundary behaviour of associated harmonic forms (Manuscript).

[3] I.M. Gelfand, et al., Representation theory and automorphic forms, W.B. Saunders Co., Philadelphia, 1969.

[4] S. Gindikin and F. Karpelevič, Plancherel measure of Riemannian symmetric spaces of non-positive curvature, Dokl. Akad. Nauk. SSSR, 145 (1962), 252-255.

[5] Harish-Chandra, Spherical functions on a semi-simple Lie group I, Amer. J. Math., 80 (1958), 241-310.

[6] Harish-Chandra, Discrete series for semi-simple Lie groups II, Acta Math., 116 (1966), 1-111.

[7] S. Helgason, Differential geometry and symmetric spaces, Academic Press, New York, 1962.

[8] S. Helgason and K. Johnson, The bounded spherical functions on symmetric spaces, Advances in Math., 3 (1969), 586-593.

[9] R.P. Langlands, The dimension of spaces of automorphic forms, Amer. J. Math., 85 (1963), 99-125.

[10] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric

Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc., **20** (1956), 47–87.

[11] T. Tamagawa, On Selberg's trace formula, J. Fac. Sci., Univ. Tokyo, 8 (1960), 363–386.

[12] P. Trombi and V. Varadarajan, Spherical transforms on semisimple Lie groups, Ann. of Math., **74** (1971), 243–303.

Ramesh GANGOLLI
Department of Mathematics
University of Washington
Seattle, Washington 98195
U.S.A.

Garth WARNER
Department of Mathematics
University of Washington
Seattle, Washington 98195
U.S.A.