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Theory of invariants in the geometry of paths.

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Introduction.

In an n-dimensional space X_n referred to a coordinate system x^i $(i=1,2,\cdots,n)$, we consider a system of ordinary differential equations of the m-th order

$$(0.1) \qquad x^{(m)i}+H^i(t, x, x^{(1)}, \cdots, x^{(m-1)})=0$$

where $x^{(r)i}=d^r x^i/dt^r$ $(r=1, 2, \cdots, m)$. Its solutions $x^i=x^i(t)$ which exist under suitable conditions for the functions H^i determine a system of curves called *paths of the m-th order.* We assume in the following that the functions H^i admit continuous derivatives with respect to the $mn+1$ arguments t, x^i, $x^{(1)i}, \cdots$, $x^{(m-1)i}$ up to the order needed.

Hitherto many has been contributed to the theory of invariants of the paths under various transformation groups, by which various geometries of paths were established. First of all, we notice the result of A. Kawaguchi and H. Hombu [8][1]. This is concerned with the theory under the transformation group of coordinates

$$\text{(i)} \qquad \xi^\alpha=\xi^\alpha(x^i), \quad \tau=t,$$

and also with the theory under the transformation group of coordinates and parameter

$$\text{(ii)} \qquad \xi^\alpha=\xi^\alpha(x^i), \quad \tau=\tau(t).$$

The first case was also treated by D. D. Kosambi [7]. We call this *the ordinary geometry of paths.* Later the second case was treated by T. Ohkubo [9] and S. Hokari [4] in the case of the third and m-th order respectively. These studies arrived at remarkable results. We call this type of geometry *intrinsic.*

1) Numbers in brackets refer to the bibliography at the end of the paper.

On the other hand, the geometry of paths of the second order under the so-called rheonomic transformation group

(iii) $$\xi^{\alpha}=\xi^{\alpha}(t, x^{i}), \qquad \tau=t$$

was studied by W. Slebodzinski [10], K. Zorawski [18], A. Wundheiler [16], [17] and also E. Cartan [2]. Especially we must notice the study of E. Cartan. The case for the higher order has been treated by H. Hombu [5] and the present author [11]. We call this *the rheonomic geometry.*

In chapter I, we shall attempt to develop the theory of invariants of the paths (0.1) under the so-called *generalized rheonomic transformation group*

(0.2) $$\xi^{\alpha}=\xi^{\alpha}(t, x^{i}), \quad \tau=\tau(t).$$

We call it *the generalized rheonomic geometry.* We assume that the functions $\xi^{\alpha}(t, x^{i})$, $\tau(t)$ are continuously differentiable up to the order needed, and also that the functional determinat $|\partial\xi^{\alpha}/\partial x^{i}|$ and the derivative $d\tau/dt$ are both not equal to zero. The present author has already studied the case of the third order [12]. The geometry of K-spreads of the second order under an analogous group has been studied by E. Bortolotti [1]. In § 1, we consider generalized rheonomic transformations from two different standpoints. Thus, introducing two kinds of fundamental quantities, i.e., the vector of the first kind and the weighted vector of the second kind, we discuss geometrical relations between them. These relations are fundamental for later purpose of our theory. In § 2, we make clear the transformation laws of the line-element. It is noteworthy that the ordinary differentials $dx^{(r)i}$ ($r=0, 1, \ldots\ldots, m-1$) must be replaced by the following pfaffians to sustain analogous transformation laws:

$$\begin{cases} \mathfrak{d}x^{(r)i}=dx^{(r)i}-x^{(r+1)i}dt & (r=0, 1, \cdots, m-2), \\ \mathfrak{d}x^{(m-1)i}=dx^{(m-1)i}+H^{i}dt. \end{cases}$$

In § 3, we define in the manifold $X_{n+1}^{(m-1)}$—the manifold of the elements $(t, x^{i}, x^{(1)i}, \cdots, x^{(m-1)i})$—the covariant derivative along a path of a weighted vector field of the second kind by use of the parameters of connection Γ^{i}_{j} and Γ which are determined by the functions H^{i} and their derivatives. The transformation laws of these parameters are necessary for the equivalence problems which we discuss in chapter IV. In § 4, we construct a set of pfaffians with vector property of the second kind of weight $p+1$ each of which may be seen as a covariant differential of pfaffian with vector character of the second kind of weight p. Using these results, we define the covariant differential $\delta x^{(r)i}$ $(r=0, 1, \cdots, m-1)$ of our line-element. This result is useful to define covariant derivatives of a vector field (§ 5, § 7). The transformation laws of the coefficients $\Lambda^{(r)i}_{(s)j}$, $\Omega^{(r)i}_{(s)j}$ appeared in them are also necessary for the equivalence problem. In § 5, we define the covariant differential, in the manifold $X^{(m-1)}_{n+1}$, of a weighted vector field of the second kind, using the parameters of connection Γ^i_{jk} and Γ^i_j, Γ already determined in § 3, The transformation law of Γ^i_{jk} plays an important role in the equivalence problem. The covariant derivatives are given there. In § 6, we obtain curvature and torsion tensors by construction of all the commutators of the operators ∇, $\nabla_{(r)j}$ $(r=0, 1, \ldots\ldots, m-1)$ which define the covariant derivatives. Hitherto we observe the weighted vector field of the second kind. In § 7, we discuss the covariant differential of a vector field of the first kind. The parameters of connection ${}^*\Gamma^I_{JK}$ are expressible in terms of Γ^i_{jk}, Γ^i_j and Γ already determined in § 3 and § 5. The covariant derivatives are obtainable in the same manner as in § 5. In § 8, we define the curvature and torsion tensors of our space by the aid of the results in § 7. These are mixed tensors of the first and second kind.

In chapter II, we clarify the relations between two methods of discussing the generalized rheonomic geometry of paths mentioned in chapter I. In § 9, we give some geometrical meanings to the relations between the two sets of parameters Γ^i_{jk}, Γ^i_j, Γ and ${}^*\Gamma^I_{JK}$, and thus explain the relations between two kinds of covariant differentials. In § 10, we obtain the relations between two kinds of covariant derivatives, and in § 11, we show that the components of the mixed curvatures and torsions are expressible in terms of the curvatures and torsions of the second kind.

In chapter III, we discuss the relations between the generalized rheonomic geometry and ordinary, intrinsic and rheonomic geometries, and show that our methods are, after some suitable modifications, also available to the other geometries.

Chapter IV is devoted to the equivalence problem. We show that it is reducible to the problem for a necessary and sufficient condition that the simultaneous partial differential equations of the first order composed of

(i) the ordinary case: the transformation laws of the parameters of connection $\Lambda^{(r)i}_{(s)j}$, Γ^i_{jk};

(ii) the intrinsic case: those of $\Lambda^{(r)i}_{(s)j}$, Γ^i_{jk}, Γ;

(iii) the rheonomic case: those of $\Lambda^{(r)i}_{(s)j}$, Γ^i_{jk}, Γ^i_j;

(iv) the generalized rheonomic case: those of $\Lambda^{(r)i}_{(s)j}$, Γ^i_{jk}, Γ^i_j, Γ

and some other partial differential equations of the first order may have a solution (in (i) or (ii) under some accessory conditions). A necessary and sufficient condition for the equivalence is expressed algebraically in terms of

(i), (ii) the ordinary and intrinsic cases: curvature and torsion tensors and a set of invariants $K^i_{(r)}$ $(r=1, 2, \cdots, m)$,

(iii), (iv) the rheonomic and generalized rheonomic cases: the curvature and torsion tensors,

and their successive covariant derivatives.

Chapter I. The generalized rheonomic geometry.

1. Two kinds of geometric quantities. Let us take $n+1$ independent variables $y^0\equiv t, y^i\equiv x^i$ as the coordinates in the $(n+1)$-dimensional manifold X_{n+1} and denote them by y^I $(I=0, 1, \cdots, n)$ for convenience's sake. This X_{n+1} is the product of the given X_n and the manifold of parameter t. Then, the transformation group (0.2) may be seen to be a coordinate transformation group

$$\eta^A=\eta^A(y^I) \tag{1.1}$$

in X_{n+1}. This is not most general in X_{n+1}, but satisfies the following special relations:

$$\begin{aligned} \frac{\partial\eta^A}{\partial y^I} &= \frac{\partial\xi^\alpha}{\partial x^i}(A=\alpha,\ I=i), \quad = \frac{\partial\xi^\alpha}{\partial t}(A=\alpha,\ I=0), \\ &= 0 \quad (A=0,\ I=i), \quad = \frac{d\tau}{dt}(A=0,\ I=0). \end{aligned} \tag{1.2}$$

The functional determinant $|\partial\eta^A/\partial y^I|$ is not equal to zero on account of our assumptions for (0.2).

Let v and w be two kinds of those geometric objects which have uniquely determined components v^I and w_J in every coordinate system y^I and are subject to the transformation law

$$\bar{v}^A = \frac{\partial \eta^A}{\partial y^I} v^I, \quad \bar{w}_B = \frac{\partial y^J}{\partial \eta^B} w_J \tag{1.3}$$

under (1.1). We call such a geometric object v or w a *contravariant* or *covariant vector of the first kind.* On the other hand, a geometric object v^i or w_j is called a *contravariant* or *covariant vector of the second kind of weight p* if its components obey the transformation law

$$\bar{v}^\alpha = \sigma^p \frac{\partial \xi^\alpha}{\partial x^i} v^i, \quad \bar{w}_\beta = \sigma^p \frac{\partial x^j}{\partial \xi^\beta} w_j. \tag{1.4}$$

A geometric quantity f which is subject to the transformation law

$$\bar{f} = \sigma^p f \tag{1.4$'$}$$

is called a *scalar of weight p.*

Let the components of a vector v^I or w_J of the first kind be (v^o, v^i) or (w_o, w_j). Then we have from (1.2) and (1.3):

$$\begin{cases} \bar{v}^o = \dfrac{1}{\sigma} v^o, \\[2ex] \bar{v}^\alpha = \dfrac{\partial \xi^\alpha}{\partial t} v^o + \dfrac{\partial \xi^\alpha}{\partial x^i} v^i, \end{cases} \qquad \begin{cases} \bar{w}_o = \sigma w_o + \dfrac{\partial x^j}{\partial \tau} w_j, \\[2ex] \bar{w}_\beta = \dfrac{\partial x^j}{\partial \xi^\beta} w_j. \end{cases} \tag{1.5}$$

Hence we can see the following properties: (i) v^0 *is a scalar of weight* -1: (ii) v^i *is a contravariant vector of the second kind of weight* 0 *if and only if* $v^0 \equiv 0$; (i)$'$ w_0 *is a scalar of weight* $+1$ *when and only when* $w_j \equiv 0$; (ii)$'$ w_j *is a covariant vector of the second kind of weight* 0.

Under the transformation (0.2), $x^{(1)i} = dx^i/dt$ is subject to

$$\xi^{[1]\alpha} = \sigma\left(\frac{\partial \xi^\alpha}{\partial x^i} x^{(1)i} + \frac{\partial \xi^\alpha}{\partial t}\right); \tag{1.6}$$

hence if we put for the sake of convenience $y_0^{(1)0} \equiv 1$, $y_0^{(1)i} \equiv x^{(1)i}$, the quantity $y_0^{(1)I}$ with the components $(y_0^{(1)0}, y_0^{(1)i})$ is a contravariant vector of the first kind with respect to the index I. Furthermore, if we put $y_j^{(1)I}\equiv 0$, then $y_J^{(1)I}$ with components $(y_0^{(1)I}, y_j^{(1)I})$ is also an affinor of the first kind with respect to the indices I and J.

If v^I is a contravariant vector with components (v^0, v^i), then the vector

$$V^I\equiv v^I-y_0^{(1)I}\,v^0$$

has components $(0, v^i-x^{(1)i}\,v^0)$. Hence we have (iii) $V^i=v^i-x^{(1)i}v^0$ *is a contravariant vector of the second kind of weight* 0. Similarly, for a covariant vector $w_J=(w_0, w_j)$, we have (iii)′ $W_J=w_J-y_J^{(1)I}w_I$ *is a covariant vector of the first kind with components* $(-x^{(1)j}w_j, w_j)$.

2. Transformations of the line-element. Let $x^i=x^i(t)$ be a curve in X_n. We call a set of quantities $x^i(t)$, $x^{(r)i}$ $(r=1, 2, \cdots, m)$ the line-element of the m-th order. Under the transformation (0.2), they are subject to the transformation

$$(2.1)\quad\begin{cases}\text{(i)}\quad \xi^\alpha=\xi^\alpha(t, x^i),\\[4pt] \text{(ii)}\quad \xi^{[1]\alpha}=\sigma\left(\dfrac{\partial\xi^\alpha}{\partial x^i}x^{(1)i}+\dfrac{\partial\xi^\alpha}{\partial t}\right),\\[8pt] \text{(iii)}\quad \xi^{[2]\alpha}=\sigma^2\left(\dfrac{\partial\xi^\alpha}{\partial x^i}x^{(2)i}+\dfrac{\partial^2\xi^\alpha}{\partial x^i\partial x^j}x^{(1)i}x^{(1)j}+2\dfrac{\partial^2\xi^\alpha}{\partial t\partial x^i}x^{(1)i}+\dfrac{\partial^2\xi^\alpha}{\partial t^2}\right)\\ \qquad\qquad +\sigma^{[1]}\left(\dfrac{\partial\xi^\alpha}{\partial x^i}x^{(1)i}+\dfrac{\partial\xi^\alpha}{\partial t}\right),\\[8pt] \text{(iv)}\quad \xi^{[r]\alpha}=\sigma^r\dfrac{\partial\xi^\alpha}{\partial x^i}x^{(r)i}+\left\{r\sigma^r D_t\dfrac{\partial\xi^\alpha}{\partial x^i}+\dfrac{r(r-1)}{2}\sigma^{r-2}\sigma^{[1]}\dfrac{\partial\xi^\alpha}{\partial x^i}\right\}x^{(r-1)i}\\ \qquad\qquad +\cdots\cdots\quad (r=3, 4, \cdots),\end{cases}$$

where we put $\sigma^{[1]}=d\sigma/d\tau$, and define the operator D_t for a differentiable function $f=f(t, x, x^{(1)}, \cdots, x^{(m-1)})$ as follows:

$$(2.2)\quad \begin{aligned}D_t f&=\frac{\partial f}{\partial t}+\sum_{r=0}^{m-2}\frac{\partial f}{\partial x^{(r)j}}x^{(r+1)j}-\frac{\partial f}{\partial x^{(m-1)j}}H^j\\ &\equiv f_{,t}+\sum_{r=0}^{m-2}f_{,(r)j}\,x^{(r+1)j}-f_{,(m-1)j}H^j.\end{aligned}$$

This is nothing but the derivative of f with respect to t along the path considered. In general, when a set of quantities $(x^i, x^{(1)i}, \cdots, x^{(m)i})$, defined in every coordinate system and parameter t and independent of any curve, is transformed in the same manner as in (2.1), we call it also a *line-element of the m-th order*. It is convenient for us to consider that (0.1) gives the correspondence between a set of values of parameter t and the line-element of the $(m-1)$-th order $(x^i, x^{(1)i}, \cdots, x^{(m-1)i})$ and a set of values of the line-element of the m-th order $(x^i, x^{(1)i}, \cdots, x^{(m-1)i}, x^{(m)i} \equiv -H^i)$. We denote by $X^{(m-1)}_{n+1}$ the manifold of geometric objects $(t, x^i, x^{(1)i}, \cdots, x^{(m-1)i})$.

It is evident from (2.1) that $H^i(t, x, x^{(1)}, \cdots, x^{(m-1)})$ are subject to the transformation

$$\overline{H}^\alpha = \sigma^m \frac{\partial \xi^\alpha}{\partial x^i} H^i - \left\{ m\sigma^m D_t \frac{\partial \xi^\alpha}{\partial x^i} + \frac{m(m-1)}{2} \sigma^{m-2} \sigma^{[1]} \frac{\partial \xi^\alpha}{\partial x^i} \right\} x^{(m-1)i} + \cdots . \tag{2.3}$$

On the other hand a set of differentials $\{dx^{(r)i}\}$ of the line-element of the $(m-1)$-th order is transformed according to

$$d\xi^{[r]\alpha} = \sum_{s=0}^{r} \frac{\partial \xi^{[r]\alpha}}{\partial x^{(s)i}} dx^{(s)i} + \frac{\partial \xi^{[r]\alpha}}{\partial t} dt \qquad (r=0, 1, \cdots, m-1).$$

From these we have again the transformation laws of the line-element of the $(m-1)$-th order and of H^i as follows:

$$\begin{cases} \xi^{[r+1]\alpha} = \sigma \sum_{s=0}^{r} \dfrac{\partial \xi^{[r]\alpha}}{\partial x^{(s)i}} x^{(s+1)i} + \sigma \dfrac{\partial \xi^{[r]\alpha}}{\partial t} & (r=0, 1, \cdots, m-2), \\ -\overline{H}^\alpha = -\sigma \dfrac{\partial \xi^{[m-1]\alpha}}{\partial x^{(m-1)i}} H^i + \sigma \sum_{s=0}^{m-2} \dfrac{\partial \xi^{[m-1]\alpha}}{\partial x^{(s)i}} x^{(s+1)i} + \sigma \dfrac{\partial \xi^{[m-1]\alpha}}{\partial t} . \end{cases} \tag{2.1'}$$

Hence if we put

$$\begin{cases} \mathfrak{d} x^{(r)i} = dx^{(r)i} - x^{(r+1)i} dt & (r=0, 1, \cdots, m-2), \\ \mathfrak{d} x^{(m-1)i} = dx^{(m-1)i} + H^i dt , \end{cases} \tag{2.4}$$

then these pfaffians are subject to the transformation

$$\overline{\mathfrak{d}}\xi^{[r]\alpha} = \sum_{s=0}^{r} \frac{\partial \xi^{[r]\alpha}}{\partial x^{(s)i}} \mathfrak{d} x^{(s)i} \qquad (r=0, 1, \cdots, m-1). \tag{2.5}$$

This shows that $\mathfrak{d}x^{(r)i}$ $(r=0, 1, \cdots, m-1)$ obey the law analogous to the differentials $dx^{(r)i}$ in the ordinary geometry of paths.

From (2.1)′ and (2.1) we have immediately

$$(2.6)\quad \begin{cases} \dfrac{\partial \xi^{[r+1]\alpha}}{\partial x^i} = \sigma D_t \dfrac{\partial \xi^{[r]\alpha}}{\partial x^i} & (r=0, 1, \cdots, m-2), \\ \dfrac{\partial \xi^{[r+1]\alpha}}{\partial x^{(s)i}} = \sigma \left\{ D_t \dfrac{\partial \xi^{[r]\alpha}}{\partial x^{(s)i}} + \dfrac{\partial \xi^{[r]\alpha}}{\partial x^{(s-1)i}} \right\} & \begin{pmatrix} r=1, 2, \cdots, m-2 \\ s=1, 2, \cdots, r \end{pmatrix}, \end{cases}$$

$$(2.7)\quad \begin{cases} \dfrac{\partial \xi^{[r]\alpha}}{\partial x^{(r)i}} = \sigma^r \dfrac{\partial \xi^\alpha}{\partial x^i} & (r=0, 1, \cdots, m-1), \\ \dfrac{\partial \xi^{[r]\alpha}}{\partial x^{(r-1)i}} = r\sigma^r D_t \dfrac{\partial \xi^\alpha}{\partial x^i} + \dfrac{r(r-1)}{2} \sigma^{r-2} \sigma^{[1]} \dfrac{\partial \xi^\alpha}{\partial x^i} & (r=1, 2, \cdots, m-1). \end{cases}$$

These make clear the transformation of the line-element.

3. Covariant derivative along a path of a vector field of the second kind. Let f be a differentiable scalar field on $X_{n+1}^{(m-1)}$. Then the derivative of f along a path is given by $D_t f$. For the transformation of t, we can easily prove that $\overline{D}_\tau f = \sigma D_t f$ which shows that D_t is a scalar operator of weight $+1$. (See e. g. T. Suguri [12]).

Even if v^i is a vector field of weight p, the derivative $D_t v^i$ is no more a vector. We will determine the functions Γ^i_j and Γ using H^i and their derivatives such that

$$(3.1)\quad \delta_t v^i = D_t v^i + \Gamma^i_j v^j + p\Gamma v^i$$

is a vector of weight $p+1$[2]. This is nothing but a covariant derivative of v^i along a path. Thus Γ^i_j and Γ must be transformed as

$$(3.2)\quad \frac{\partial \xi^\beta}{\partial x^j} \overline{\Gamma}^\alpha_\beta = \sigma \frac{\partial \xi^\alpha}{\partial x^i} \Gamma^i_j - \sigma D_t \frac{\partial \xi^\alpha}{\partial x^j},$$

$$(3.3)\quad \sigma \overline{\Gamma} = \sigma^2 \Gamma - \sigma^{[1]}.$$

Differentiating (2.3) successively in $x^{(m-1)j}$ and using (2.7), we have

2) This is due to the fact that D_t is a scalar operator of weight $+1$.

$$\sigma \frac{\partial \xi^\beta}{\partial x^j} \overline{H}^\alpha{}_{,[m-1]\beta} = \sigma^2 \frac{\partial \xi^\alpha}{\partial x^i} H^i{}_{,(m-1)j} - m\sigma^2 D_t \frac{\partial \xi^\alpha}{\partial x^j} - \frac{m(m-1)}{2} \sigma^{[1]} \frac{\partial \xi^\alpha}{\partial x^j}, \tag{3.4}$$

$$\sigma^{m-2} \frac{\partial \xi^\beta}{\partial x^j} \frac{\partial \xi^\gamma}{\partial x^k} \overline{H}^\alpha{}_{,[m-1]\beta[m-1]\gamma} = \frac{\partial \xi^\alpha}{\partial x^i} H^i{}_{,(m-1)j(m-1)k}\,{}^{3)}, \tag{3.5}$$

$$\sigma^{m-2} \frac{\partial \xi^\alpha}{\partial x^i} \overline{H}^\varepsilon{}_{,[m-1]\varepsilon[m-1]\alpha} = H^l{}_{,(m-1)l(m-1)i}, \tag{3.6}$$

$$\sigma^{2m-3} \frac{\partial \xi^\alpha}{\partial x^i} \frac{\partial \xi^\beta}{\partial x^j} \overline{G}_{\alpha\beta} = G_{ij}, \tag{3.7}$$

where G_{ij} is a tensor of the second kind of weight $-(2m-3)$ and is defined by

$$\overline{G}_{ij} = H^l{}_{,(m-1)l(m-1)i(m-1)j}. \tag{3.8}$$

If we assume that the determinant $|G_{ij}|$ is not equal to zero, then we can define by $G^{ij}\,G_{jk} = \delta^i_k$ the tensor G^{ij} of weight $2m-3$ which transforms as

$$\overline{G}^{\alpha\beta} = \sigma^{2m-3} \frac{\partial \xi^\alpha}{\partial x^i} \frac{\partial \xi^\beta}{\partial x^j} G^{ij}. \tag{3.9}$$

Differentiating (3.6) in $x^{(m-2)j}$ and using (2.7) and (3.9), we have

$$\begin{aligned} &\sigma \frac{\partial \xi^\beta}{\partial x^j} \overline{G}^{\alpha\gamma} \overline{H}^\varepsilon{}_{,[m-1]\varepsilon[m-1]\gamma[m-2]\beta} \\ &= \sigma^2 \left\{ \frac{\partial \xi^\alpha}{\partial x^i} G^{ik} H^l{}_{,(m-1)l(m-1)k(m-2)j} - (m-1) D_t \frac{\partial \xi^\alpha}{\partial x^j} \right\} \\ &\quad - \frac{(m-1)(m-2)}{2} \sigma^{[1]} \frac{\partial \xi^\alpha}{\partial x^j}. \end{aligned} \tag{3.10}$$

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3) When H^i are polynomials of the first degree with respect to the highest derivatives $x^{(m-1)j}$, i. e.

$$H^i = A^i_j(t, x, x^{(1)}, \ldots, x^{(r)})\, x^{(m-1)j} + B^i(t, x, x^{(1)}, \ldots, x^{(s)}) \quad (r, s \leqq m-2),$$

we must use $A^i_{j,(r)k}$ in place of $H^i{}_{,(m-1)j(m-1)k}$.

Hence, eliminating $\sigma^{[1]}$ or $D_t(\partial\xi^\alpha/\partial x^i)$ from (3.4) and (3.10) we find that the functions defined by

$$\Gamma^i_j = G^{ik} H^l{}_{,(m-1)\,l\,(m-1)\,k\,(m-2)\,j} - \frac{m-2}{m} H^i{}_{,(m-1)j}\,, \tag{3.11}$$

$$\Gamma = \frac{1}{n}\left\{\frac{2}{m} H^i{}_{,(m-1)i} - \frac{2}{m-1} G^{ij} H^l{}_{,(m-1)\,l\,(m-1)\,j\,(m-2)i}\right\} \tag{3.12}$$

behave as required. Thus we can determine the covariant derivative along a path of a vector field of the second kind.

4. Covariant differential of the line-element. To derive a covariant differential of the line-element, we begin with the

THEOREM. *If the pfaffian forms*

$$P^i = \sum_{r=0}^{M} P^i_{(r)k}\, \mathfrak{d}x^{(r)k} \tag{4.1}$$

are transformed as a vector of weight p, then the pfaffians

$$\delta_t P^i = \sum_{r=0}^{M} P^i_{(r)k}\, \mathfrak{d}x^{(r+1)k} + \sum_{r=0}^{M} \{D_t P^i_{(r)k} + \Gamma^i_j P^j_{(r)k} + p\Gamma P^i_{(r)k}\}\mathfrak{d}x^{(r)k} \tag{4.2}$$

are also transformed as a vector of weight $p+1$.

According to our assumption, we have

$$\sum_{r=0}^{M} \bar{P}^\alpha_{[r]\gamma}\, \bar{\mathfrak{d}}\xi^{[r]\gamma} = \sigma^p \frac{\partial\xi^\alpha}{\partial x^i} \sum_{r=0}^{M} P^i_{(r)k} \bar{\mathfrak{d}}x^{(r)k}.$$

By substitution of $\bar{\mathfrak{d}}\xi^{[r]\gamma}$ from (2.5) in the left-hand side, we get the following transformation law of $P^i_{(r)j}$:

$$\sum_{r=s}^{M} \frac{\partial\xi^{[r]\beta}}{\partial x^{(s)j}} \bar{P}^\alpha_{[r]\beta} = \sigma^p \frac{\partial\xi^\alpha}{\partial x^i} P^i_{(s)j} \qquad (s=0, 1, \cdots, M). \tag{4.3}$$

After these preparations, substituting $\bar{\mathfrak{d}}\xi^{[r]\gamma}$ from (2.5) into

$$\bar{\delta}_\tau \bar{P}^\alpha = \sum_{r=0}^{M} \bar{P}^\alpha_{[r]\gamma}\, \bar{\mathfrak{d}}\xi^{[r+1]\gamma} + \sum_{r=0}^{M} \{\bar{D}_\tau \bar{P}^\alpha_{[r]\gamma} + \bar{\Gamma}^\alpha_\beta \bar{P}^\beta_{[r]\gamma} + p\bar{\Gamma}\bar{P}^\alpha_{[r]\gamma}\}\bar{\mathfrak{d}}\xi^{[r]\gamma}$$

and making use of (2.7), (2.6), (4.3), (3.2) and (3.3), we can establish our theorem by long but rather easy calculations.

It is evident from (2.5) that $\delta x^i \equiv \mathfrak{d}x^i$ are pfaffians with vector character of weight 0; hence by virtue of the theorem we see that $\delta x^{(r+1)i} \equiv \delta_t(\delta x^{(r)i})$ $(r=0, 1, \cdots, m-2)$ are pfaffians with vector character of weight $r+1$. Therefore we may use these $\delta x^{(r)i}$ $(r=0, 1, \cdots, m-1)$ as desired covariant differential of the line-element. They will be written in terms of $\mathfrak{d}x^{(r)i}$ explicitly as

$$(4.4)\qquad \begin{cases} \delta x^i = \mathfrak{d}x^i\,, \\ \delta x^{(r)i} = \mathfrak{d}x^{(r)i} + \sum\limits_{s=0}^{r-1} \Lambda^{(r)i}_{(s)j}\, \mathfrak{d}x^{(s)j} & (r=1, 2, \cdots, m-1), \end{cases}$$

where $\Lambda^{(r)i}_{(s)j}$ are defined by recurrence:

$$(4.5)\qquad \begin{cases} \Lambda^{(r+1)i}_{(r)j} = (r+1)\Gamma^i_j + \dfrac{r(r+1)}{2}\,\Gamma\delta^i_j & (r=0, 1, \cdots, m-2)\,, \\ \Lambda^{(r+1)i}_{(s)j} = D_t \Lambda^{(r)i}_{(s)j} + \Lambda^{(r)i}_{(s-1)j} + (\Gamma^i_h + r\Gamma\delta^i_h)\,\Lambda^{(r)h}_{(s)j} & \begin{pmatrix} r=2, 3, \cdots, m-2 \\ s=1, 2, \cdots, r-1 \end{pmatrix}, \\ \Lambda^{(r+1)i}_{(0)j} = D_t \Lambda^{(r)i}_{(0)j} + (\Gamma^i_h + r\Gamma\delta^i_h)\,\Lambda^{(r)h}_{(0)j} & (r=1, 2, \cdots, m-2)\,. \end{cases}$$

We can solve (4.4) in $\mathfrak{d}x^{(r)i}$ as

$$(4.6)\qquad \begin{cases} \mathfrak{d}x^i = \delta x^i\,, \\ \mathfrak{d}x^{(r)i} = \delta x^{(r)i} - \sum\limits_{s=0}^{r-1} \Omega^{(r)i}_{(s)j}\, \delta x^{(s)j} & (r=1, 2, \cdots, m-1)\,, \end{cases}$$

where

$$(4.7)\qquad \begin{cases} \Omega^{(r)i}_{(r-1)j} = \Lambda^{(r)i}_{(r-1)j} & (r=1, 2, \cdots, m-1)\,, \\ \Omega^{(r)i}_{(s)j} = \Lambda^{(r)i}_{(s)j} - \sum\limits_{t=s+1}^{r-1} \Lambda^{(r)i}_{(t)h}\, \Omega^{(t)h}_{(s)j} & \begin{pmatrix} r=2, 3, \cdots, m-1 \\ s=0, 1, \cdots, r-2 \end{pmatrix}. \end{cases}$$

We derive the transformation laws of $\Omega^{(r)i}_{(s)j}$ and $\Lambda^{(r)i}_{(s)j}$, using (4.6), (4.4), (2.5) and the vector character of $\delta x^{(r)i}$, as follows:

$$(4.8)\qquad \frac{\partial \xi^{[r]\alpha}}{\partial x^{(s)j}} = \sum_{t=s+1}^{r} \frac{\partial \xi^{[r]\alpha}}{\partial x^{(t)i}}\, \Omega^{(t)i}_{(s)j} - \sigma^s \frac{\partial \xi^\beta}{\partial x^j}\, \bar{\Omega}^{[r]\alpha}_{[s]\beta} \qquad \begin{pmatrix} r=1, 2, \cdots, m-1 \\ s=0, 1, \cdots, r-1 \end{pmatrix},$$

$$(4.9)\qquad \frac{\partial \xi^{[r]\alpha}}{\partial x^{(s)j}} = \sigma^r \frac{\partial \xi^\alpha}{\partial x^i}\, \Lambda^{(r)i}_{(s)j} - \sum_{t=s}^{r-1} \frac{\partial \xi^{[t]\beta}}{\partial x^{(s)j}}\, \bar{\Lambda}^{[r]\alpha}_{[t]\beta} \qquad \begin{pmatrix} r=1, 2, \cdots, m-1 \\ s=0, 1, \cdots, r-1 \end{pmatrix}.$$

It is evident that (4.8) and (4.9) are equivalent to each other.

5. Connections of a vector field of the second kind. Let us determine a covariant differential Dv^i of a vector field v^i of weight p by

$$(5.1)\qquad Dv^i = dv^i + (\Gamma^i_j\, v^j + p\Gamma v^i)\, dt + \Gamma^i_{jk}\, v^j\, \mathfrak{d}x^k\,.$$

In this case Dv^i is a vector of weight p. Making use of (3.2) and (3.3), we see that Γ^i_{jk} are subject to

$$\frac{\partial \xi^\alpha}{\partial x^i}\Gamma^i_{jk}=\frac{\partial \xi^\beta}{\partial x^j}\frac{\partial \xi^\gamma}{\partial x^k}\bar{\Gamma}^\alpha_{\beta\gamma}+\frac{\partial^2 \xi^\alpha}{\partial x^j \partial x^k}. \tag{5.2}$$

If f is a scalar field of weight p, then we can easily verify that

$$Df=df+p\Gamma f\,dt$$

is also a scalar pfaffian of weight p. By decomposition of this in terms of dt, $\delta x^{(r)i}$ $(r=0, 1, \cdots, m-1)$ as

$$Df=(\tilde{\nabla} f)\,dt+\sum_{r=0}^{m-1}(\tilde{\nabla}_{(r)i} f)\,\delta x^{(r)i}, \tag{5.3}$$

we have the covariant derivatives $\tilde{\nabla} f$ and $\tilde{\nabla}_{(r)i} f$:

$$\begin{cases} \tilde{\nabla} f=D_t f+p\Gamma f, \\ \tilde{\nabla}_{(m-1)i} f=f_{,(m-1)i}, \\ \tilde{\nabla}_{(r)i} f=f_{,(r)i}+\sum_{s=r+1}^{m-1}\Omega^{(s)j}_{(r)i} f_{,(s)j} \qquad (r=0, 1, \cdots, m-2). \end{cases} \tag{5.4}$$

Then we can find that Γ^i_{jk} is defined by

$$\Gamma^i_{jk}=\tilde{\nabla}_{(1)k}\Gamma^i_j\equiv\Gamma^i_{j,(1)k}-\sum_{s=2}^{m-1}\Omega^{(s)h}_{(1)k}\Gamma^i_{j,(s)h} \tag{5.5}$$

or

$$\Gamma^i_{jk}=\frac{1}{m}\tilde{\nabla}_{(1)k}H^i{}_{,(m-1)j}\equiv\frac{1}{m}\left\{H^i{}_{,(m-1)j(1)k}-\sum_{s=2}^{m-1}\Omega^{(s)h}_{(1)k}H^i{}_{,(m-1)j(s)h}\right\}. \tag{5.6}$$

We can obtain the covariant derivatives of a vector field decomposing (5.1) in terms of dt, $\delta x^{(r)i}$ $(r=0, 1, \cdots, m-1)$ as

$$Dv^i=(\nabla v^i)\,dt+\sum_{r=0}^{m-1}(\nabla_{(r)j}\, v^i)\,\delta x^{(r)j}. \tag{5.7}$$

They are given as follows:

$$(5.8)\quad \begin{cases} \nabla v^i = D_t v^i + \Gamma'^i_j v^j + p\Gamma' v^i\,, \\ \nabla_{(m-1)j}\, v^i = v^i{}_{,(m-1)j}\,, \\ \nabla_{(r)j} = v^i{}_{,(r)j} - \sum_{s=r+1}^{m-1} \Omega^{(s)h}_{(r)j}\, v^i{}_{,(s)h} & (r=1,2,\cdots,m-2)\,, \\ \nabla_{(0)j} v^i = v^i{}_{,(0)j} + \Gamma^i_{kj} v^k - \sum_{s=1}^{m-1} \Omega^{(s)h}_{(0)j} v^i{}_{,(s)h}\,. \end{cases}$$

The covariant derivatives ∇v^i and $\nabla_{(r)j} v^i$ constitute the components of a vector of weight $p+1$ and those of an affinor of weight $p-r$.

6. Curvature and torsion tensors. As is well-known, the curvature and torsion tensors of the connection can be obtained, on the one hand, by parallel displacement of a vector along a closed infinitesimal circuit and, on the other hand, by construction of all the commutators of the operators ∇, $\nabla_{(r)j}$ $(r=0, 1, \cdots, m\text{-}1)$. We shall use the latter method.

Let v^i be a vector field of the second kind of weight p in $X_{n+1}^{(m-1)}$. If we put

$$(6.1)\quad \begin{cases} 2\,[\nabla_{(r)j}\, \nabla]\, v^i = -R^i_{h(r)j} v^h + pR_{(r)j} v^i + \nabla_{(r-1)j} v^i + S^{(m-1)h}_{(r)j} \nabla_{(m-1)h} v^i \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (r=1,2,\cdots,m-1)\,, \\ 2\,[\nabla_{(0)j} \nabla] v^i = -R^i_{h(0)j} v^h + pR_{(0)j} v^i + S^{(m-1)h}_{(0)j} \nabla_{(m-1)h} v^i\,, \\ 2\,[\nabla\ \nabla] v^i = 2\,[\nabla_{(m-1)j} \nabla_{(m-1)k}]\, v^i = 0, \\ 2\,[\nabla_{(r)j} \nabla_{(s)k}]\, v^i = \sum_{t=s+1}^{m-1} S^{(t)h}_{(r)j(s)k}\, \nabla_{(t)h} v^i & \begin{pmatrix} r=1,2,\cdots,m-1 \\ s=1,2,\cdots,r,\ s\neq m-1 \end{pmatrix}, \\ 2\,[\nabla_{(r)j} \nabla_{(0)k}]\, v^i = -R^i_{h(r)j(0)k} v^h + \sum_{t=1}^{m-1} S^{(t)h}_{(r)j(0)k} \nabla_{(t)h} v^i\ (r=1,2,\ldots,m-1)\,, \\ 2\,[\nabla_{(0)j} \nabla_{(0)k}\,]\, v^i = -R^i_{h(0)j(0)k}\, v^h + \sum_{t=0}^{m-1} S^{(t)h}_{(0)j(0)k}\, \nabla_{(t)h} v^i\,, \end{cases}$$

then we can get the curvature tensors $R^i_{h(r)j}$, $R_{(r)j}$, $R^i_{h(r)j(0)k}$ and the torsion tensors $S^{(m-1)h}_{(r)j}$, $S^{(t)h}_{(r)j(s)k}$ which are however omitted here.

7. Connections of a vector field of the first kind. We have determined the covariant derivatives of a vector field v^i of the second kind. In this paragraph, let us determine the covariant derivatives of a vector field v^I of the first kind. Suppose that the covariant differential D^*v^I be given in the form

$$D^*v^I = dv^I + {}^*\Gamma^I_{JK} v^J dy^K . \tag{7.1}$$

The unknown parameters of connection ${}^*\Gamma^I_{JK}$ must be defined by the functions H^i and their derivatives. Under the transformation (1.1) ${}^*\Gamma^I_{JK}$ are subject to

$$\frac{\partial\eta^A}{\partial y^I}\,{}^*\Gamma^I_{JK} = \frac{\partial\eta^B}{\partial y^J}\,\frac{\partial\eta^C}{\partial y^k}\,{}^*\overline{\Gamma}^A_{BC} + \frac{\partial^2\eta^A}{\partial y^J \partial y^K} . \tag{7.2}$$

Now it is convenient to write this separately into the following eight cases:

(i) $A=0,\ J=j,\ K=k.$ Rewriting (7.2) we have

$$\frac{1}{\sigma}\,{}^*\Gamma^0_{jk} = \frac{\partial\xi^\beta}{\partial x^j}\,\frac{\partial\xi^\gamma}{\partial x^k}\,{}^*\overline{\Gamma}^0_{\beta\gamma} ,$$

from which we see that ${}^*\Gamma^0_{jk}$ is an affinor of weight -1. Hence we can put

$${}^*\Gamma^0_{jk} = 0 . \tag{7.3}$$

(ii) $A=0,\ J=0,\ K=k.$ Noticing (7.3) we have from (7.2)

$${}^*\Gamma^0_{0k} = \frac{\partial\xi^\gamma}{\partial x^k}\,{}^*\overline{\Gamma}^0_{0\gamma}$$

which shows that ${}^*\Gamma^0_{0k}$ is a vector. Hence we may put

$${}^*\Gamma^0_{0k} = 0 . \tag{7.4}$$

(iii) $A=0,\ J=j,\ K=0.$ Similarly by virtue of (7.3) and (7.2) we find that ${}^*\Gamma^0_{j0}$ is also a vector: hence we can put

$${}^*\Gamma^0_{j0} = 0. \tag{7.5}$$

(iv) $A=0,\ J=0,\ K=0.$ On account of (7.3), (7.4) and (7.5), (7.2) reduces to

$$\sigma\,(-{}^*\overline{\Gamma}^0_{00}) = \sigma^2\,(-{}^*\Gamma^0_{00}) - \sigma^{[1]} .$$

Comparing this with (3.3), it is evident that we can put

$${}^*\Gamma^0_{00} = -\Gamma . \tag{7.6}$$

(v) $A=\alpha,\ J=j,\ K=0.$ By virtue of (7.5), we have from (7.2)

$$\frac{\partial\xi^\alpha}{\partial x^i}\,{}^*\Gamma^i_{j0}=\frac{\partial\xi^\beta}{\partial x^j}\frac{\partial\xi^\gamma}{\partial t}\,{}^*\overline{\Gamma}^\alpha_{\beta\gamma}+\frac{1}{\sigma}\frac{\partial\xi^\beta}{\partial x^j}\,{}^*\overline{\Gamma}^\alpha_{\beta 0}+\frac{\partial^2\xi^\alpha}{\partial t\partial x^j}\,.$$

(vi) $A=\alpha,\ J=0,\ K=k$. By virtue of (7.4), we have from (7.2)

$$\frac{\partial\xi^\alpha}{\partial x^i}\,{}^*\Gamma^i_{0k}=\frac{\partial\xi^\beta}{\partial t}\frac{\partial\xi^\gamma}{\partial x^k}\,{}^*\overline{\Gamma}^\alpha_{\beta\gamma}+\frac{1}{\sigma}\frac{\partial\xi^\gamma}{\partial x^k}\,{}^*\overline{\Gamma}^\alpha_{0\gamma}+\frac{\partial^2\xi^\alpha}{\partial t\partial x^k}\,.$$

(vii) $A=\alpha,\ J=j,\ K=k$. By virtue of (7.3), we have from (7.2)

$$\frac{\partial\xi^\alpha}{\partial x^i}\,{}^*\Gamma^i_{jk}=\frac{\partial\xi^\beta}{\partial x^j}\frac{\partial\xi^\gamma}{\partial x^k}\,{}^*\overline{\Gamma}^\alpha_{\beta\gamma}+\frac{\partial^2\xi^\alpha}{\partial x^j\partial x^k}\,.$$

Comparing this with (5.2), it follows that we may put

(5.5) $${}^*\Gamma^i_{jk}=\Gamma^i_{jk}=\nabla_{(1)k}\Gamma^i_j\,,$$

or

(5.6) $${}^*\Gamma^i_{jk}=\Gamma^i_{jk}=\frac{1}{m}\nabla_{(1)k}H^i{}_{,(m-1)j}\,.$$

(viii) $A=\alpha,\ J=0,\ K=0$. In this case (7.2) reduces to

$$\frac{\partial\xi^\alpha}{\partial x^i}\,{}^*\Gamma^i_{00}+\frac{\partial\xi^\alpha}{\partial t}\,{}^*\Gamma^0_{00}=\frac{1}{\sigma^2}\,{}^*\overline{\Gamma}^\alpha_{00}+\frac{\partial\xi^\beta}{\partial t}\frac{\partial\xi^\gamma}{\partial t}\,{}^*\overline{\Gamma}^\alpha_{\beta\gamma}$$
$$+\frac{1}{\sigma}\frac{\partial\xi^\gamma}{\partial t}\,{}^*\overline{\Gamma}^\alpha_{0\gamma}+\frac{1}{\sigma}\frac{\partial\xi^\beta}{\partial t}\,{}^*\overline{\Gamma}^\alpha_{\beta 0}+\frac{\partial^2\xi^\alpha}{\partial t^2}\,.$$

Making use of these transformation laws, let us determine the remaining unknown parameters. In the first place, eliminating $D_t(\partial\xi^\alpha/\partial x^j)$ from (3.2), (vi) and (vii) and noticing (2.1 ii), we see that

$$P^i_k=\Gamma^i_k-\Gamma^i_{jk}\,x^{(1)j}-{}^*\Gamma^i_{0k}$$

is an affinor. Hence putting this equal to zero: $P^i_k=0$, we get

(7.7) $${}^*\Gamma^i_{0k}=\Gamma^i_k-\Gamma^i_{jk}x^{(1)j}\,.$$

In the next, from (3.2), (v), (vii) and (2.1 ii), we see similarly that

$$Q^i_j=\Gamma^i_j-\Gamma^i_{jk}\,x^{(1)k}-{}^*\Gamma^i_{j0}$$

is also an affinor. Hence we can take

(7.8) $${}^*\Gamma^i_{j0}=\Gamma^i_j-\Gamma^i_{jk}\,x^{(1)k}\,.$$

Finally, eliminating $\partial^2\xi^\alpha/\partial x^j\partial x^k$, $\partial^2\xi^\alpha/\partial t\partial t^j$ and $\partial^2\xi^\alpha/\partial t^2$ from (2.1 iii), (v), (vi), (vii) and (viii), and noticing (3.3), (2.1 ii), it is found that

$$R^i = x^{(2)i} - \Gamma^i_{jk} x^{(1)j} x^{(1)k} + 2\Gamma^i_j x^{(1)j} + \Gamma x^{(1)i} + {}^*\Gamma^i_{00}$$

is a vector. Hence putting at last $R^i = 0$, we get

$$(7.9) \qquad {}^*\Gamma^i_{00} = -\{x^{(2)i} - \Gamma^i_{jk} x^{(1)j} x^{(1)k} + 2\Gamma^i_j x^{(1)j} + \Gamma x^{(1)i}\} .$$

Thus we have determined all the parameters of connection.

The covariant derivatives are obtainable by decomposition of covariant differential (7.1) in terms of dt and $\delta x^{(r)i}$:

$$(7.10) \qquad D^* v^I = (\nabla^* v^I) dt + \sum_{r=0}^{m-1} (\nabla^*_{(r)j} v^I) \delta x^{(r)j} .$$

The covariant derivatives $\nabla^* v^I$ and $\nabla^*_{(r)j} v^I$ $(r = 0, 1, \cdots, m-1)$ thus defined are given as follows:

$$(7.11) \qquad \begin{cases} \nabla^* v^I = D_t v^I + {}^*\Gamma^I_J v^J , \\ \nabla^*_{(m-1)j} v^I = v^I{}_{,(m-1)j} , \\ \nabla^*_{(r)j} v^I = v^I{}_{,(r)j} - \sum_{t=r+1}^{m-1} \Omega^{(t)k}_{(r)j} v^I{}_{,(t)k} \qquad (r = 1, 2, \cdots, m-2) , \\ \nabla^*_{(0)j} v^I = v^I{}_{,(0)j} + {}^*\Gamma^I_{Kj} v^K - \sum_{t=1}^{m-1} \Omega^{(t)k}_{(0)j} v^I{}_{,(t)k} , \end{cases}$$

where we have put

$${}^*\Gamma^I_J = {}^*\Gamma^I_{J0} + {}^*\Gamma^I_{Jk} x^{(1)k} .$$

Furthermore, let us denote by $\tilde{\nabla}^*_{(0)j} v^I$ the expressions obtained from $\nabla^*_{(0)j} v^I$ by omitting the term ${}^*\Gamma^I_{Kj} v^K$, i.e.

$$(7.12) \qquad \tilde{\nabla}^*_{(0)j} v^I = v^I{}_{,(0)j} - \sum_{t=1}^{m-1} \Omega^{(t)k}_{(0)j} v^I{}_{,(t)k} .$$

$\nabla^*_{(r)j} v^I$ is a vector of the second kind of weight $-r$ with respect to j.

8. Mixed curvature and torsion tensors. The invariants of the connection defined by (7.1) are obtainable in the same manner as in § 6, i.e., we get the curvature and torsion tensors by constructing all the commutators of the operators ∇^* and $\nabla^*_{(r)j}$.

If we put

$$
(8.1)\left\{\begin{array}{ll}
2[\nabla^*_{(r)j}\nabla^*]v^I=-R^{*I}_{H(r)j}v^H+\nabla^*_{(r-1)j}v^I+S^{*(m-1)k}_{(r)j}\nabla^*_{(m-1)k}v^I & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (r=1,2,\cdots,m-1), & \\
2[\nabla^*_{(0)j}\nabla^*]v^I=-R^{*I}_{H(0)j}v^H+S^{*(m-1)h}_{(0)j}\nabla^*_{(m-1)h}v^I, & \\
2[\nabla^*\nabla^*]v^I=2[\nabla^*_{(m-1)j}\nabla^*_{(m-1)k}]v^I=0, & \\
2[\nabla^*_{(r)j}\nabla^*_{(s)k}]v^I=\sum_{t=s+1}^{m-1}S^{*(t)h}_{(r)j(s)k}\nabla^*_{(t)h}v^I & \begin{pmatrix} r=1,2,\cdots,m-1\\ s=1,2,\cdots,r,\ s\neq m-1\end{pmatrix}, \\
2[\nabla^*_{(r)j}\nabla^*_{(0)k}]v^I=-R^{*I}_{H(r)j(0)k}v^H+\sum_{t=1}^{m-1}S^{*(t)h}_{(r)j(0)k}\nabla^*_{(t)h}v^I & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (r=1,2,\cdots,m-1), & \\
2[\nabla^*_{(0)j}\nabla^*_{(0)k}]v^I=-R^{*I}_{H(0)j(0)k}v^H+\sum_{t=0}^{m-1}S^{*(t)h}_{(0)j(0)k}\nabla^*_{(t)h}v^I, &
\end{array}\right.
$$

then the curvature tensors $R^{*I}_{H(r)j}$, $R^{*I}_{H(r)j(s)k}$ and the torsion tensors $S^{*(m-1)h}_{(r)j}$, $S^{*(t)h}_{(r)j(s)k}$ are calculated as follows:

$$
(8.2)\left\{\begin{array}{ll}
R^{*I}_{H(0)j}=D_t\,{}^*\Gamma^I_{Hj}-\nabla^*_{(0)j}\,{}^*\Gamma^I_H-\Gamma^k_j\,{}^*\Gamma^I_{Hk}, & \\
R^{*I}_{H(1)j}={}^*\Gamma^I_{Hj}-\nabla^*_{(1)j}\,{}^*\Gamma^I_H, & \\
R^{*I}_{H(r)j}=-\nabla^*_{(r)j}\,{}^*\Gamma^I_H & (r=2,3,\cdots,m-1),
\end{array}\right.
$$

$$
(8.3)\left\{\begin{array}{ll}
R^{*I}_{H(0)j(0)k}=\tilde{\nabla}^*_{(0)k}\,{}^*\Gamma^I_{Hj}-\tilde{\nabla}^*_{(0)j}\,{}^*\Gamma^I_{Hk}+{}^*\Gamma^I_{Lk}\,{}^*\Gamma^L_{Hj}-{}^*\Gamma^I_{Lj}\,{}^*\Gamma^L_{Hk}, & \\
R^{*I}_{H(r)j(0)k}=-\nabla^*_{(r)j}\,{}^*\Gamma^I_{Hk} & (r=1,2,\cdots,m-1),
\end{array}\right.
$$

$$
(8.4)\qquad S^{*(m-1)h}_{(r)j}=S^{(m-1)h}_{(r)j} \qquad\qquad (r=0,1,\cdots,m-1),
$$

$$
(8.5)\qquad S^{*(t)h}_{(r)j(s)k}=S^{(t)h}_{(r)j(s)k} \qquad \begin{pmatrix} r=0,1,\cdots,m-1\\ s=0,1,\cdots,r,\ s\neq m-1\\ t=s+1,s+2,\cdots,m-1\end{pmatrix},
$$

in (8.5) we admit $t=0$ when and only when $r=0$ and $s=0$.

Chapter II. Relations between two kinds of connections.

9. Two kinds of covariant differentials. Suppose that the covariant differentials of a vector field v^i, a scalar field f of weight

p, and a vector field $v^I \equiv (v^0, v^i)$ are defined respectively by

(9.1) $$Dv^i = dv^i + \Gamma^i_{jk} v^j \, \delta x^k + (\Gamma^i_j v^j + p\Gamma v^i)\, dt\,,$$

(9.2) $$Df = df + p\, \Gamma f\, dt\,,$$

(9.3) $$D^* v^I = dv^I + {}^*\Gamma^I_{JK} v^J dy^K\,.$$

In this paragraph I intend to show that the parameters of connection ${}^*\Gamma^I_{JK}$ determined by suitable geometrical conditions coincide precisely with those in § 7.

(i) v^0 is a scalar field of weight -1. We have from (9.3)

$$(D^*v^I)_{I=0} = dv^0 + {}^*\Gamma^0_{0k} v^0 dx^k + {}^*\Gamma^0_{00} v^0 dt + {}^*\Gamma^0_{jk} v^j dx^k + {}^*\Gamma^0_{j0} v^j dt\,.$$

On the other hand, we have from (9.2) putting $p = -1$

$$Dv^0 = dv^0 - \Gamma v^0 dt\,.$$

If we suppose that

(9.4) $$(D^* v^I)_{I=0} \equiv Dv^0$$

holds, we get

(9.5) $${}^*\Gamma^0_{0k} = 0\,, \quad {}^*\Gamma^0_{j0} = 0\,, \quad {}^*\Gamma^0_{jk} = 0\,, \quad {}^*\Gamma^0_{00} = -\Gamma\,.$$

(ii) As we know that $V^i = v^i - x^{(1)i} v^0$ is a vector field of weight 0, we have from (9.1) putting $p = 0$

$$D(v^i - x^{(1)i} v^0) = dv^i - x^{(1)i} dv^0 - v^0 dx^{(1)i}$$
$$+ \Gamma^i_{jk} (v^j - x^{(1)j} v^0)(dx^k - x^{(1)k} dt) + \Gamma^i_j \,(v^j - x^{(1)j} v^0) dt\,.$$

On the other hand, noticing the covariant differential of the line-element we have

$$\delta x^{(1)i} = dx^{(1)i} - x^{(2)i} dt + \Gamma^i_j (dx^j - x^{(1)j} dt)\,,$$

or

(9.6) $$dx^{(1)i} = \delta x^{(1)i} + x^{(2)i} dt - \Gamma^i_j (dx^j - x^{(1)j} dt)\,.$$

Hence we have

$$D(v^i - x^{(1)i} v^0) = -v^0 \delta x^{(1)i} + dv^i - x^{(1)i} dv^0 - v^0 \{x^{(2)i} dt - \Gamma^i_j (dx^j - x^{(1)j} dt)\}$$
$$+ \Gamma^i_{jk} (v^j - x^{(1)j} v^0)(dx^k - x^{(1)k} dt) + \Gamma^i_j (v^j - x^{(1)j} v^0) dt\,.$$

By virtue of (9.3) we also have

$$(D^*v^I)_{I=i}-x^{(1)i}\,(D^*v^I)_{I=0}=dv^i-x^{(1)i}dv^0+({}^*\Gamma^i_{00}-{}^*\Gamma^0_{00}\,x^{(1)i})v^0dt$$
$$+({}^*\Gamma^i_{0k}-{}^*\Gamma^0_{0k}x^{(1)i})\,v^0dx^k+({}^*\Gamma^i_{j0}-{}^*\Gamma^0_{j0}\,x^{(1)i})\,v^jdt$$
$$+({}^*\Gamma^i_{jk}-{}^*\Gamma^0_{jk}x^{(1)i})\,v^jdx^k\,.$$

Therefore if we suppose that

$$(D^*v^I)_{I=i}-x^{(1)i}\,(D^*v^I)_{I=0}\equiv D\,(v^i-x^{(1)i}v^0)+v^0\,\delta x^{(1)i}\,, \tag{9.7}$$

then we have, noticing (9.5),

$$\left\{\begin{aligned} &{}^*\Gamma^i_{jk}=\Gamma^i_{jk} \\ &{}^*\Gamma^i_{j0}=\Gamma^i_j-\Gamma^i_{jk}\,x^{(1)k}\,, \\ &{}^*\Gamma^i_{0k}=\Gamma^i_k-\Gamma^i_{jk}\,x^{(1)j}\,, \\ &{}^*\Gamma^i_{00}=-\{x^{(2)i}-\Gamma^i_{jk}\,x^{(1)j}\,x^{(1)k}+2\,\Gamma^i_j\,x^{(1)j}+\Gamma x^{(1)i}\}\,. \end{aligned}\right. \tag{9.8}$$

Thus we know that the parameters of connection ${}^*\Gamma^I_{JK}$ (9.5) and (9.8) determined by the geometrical conditions (9.4) and (9.7) are nothing but those already defined in § 7.

(iii) The following fact is noteworthy. The components v^i constitute a contravariant vector field of weight 0 when and only when $v^0\equiv 0$. Therefore we have from (9.3)

$$(D^*v^I)_{I=i}=dv^i+{}^*\Gamma^i_{jk}v^jdx^k+{}^*\Gamma^i_{j0}\,v^jdt\,,$$
$$(D^*v^I)_{I=0}={}^*\Gamma^0_{jk}v^jdx^k+{}^*\Gamma^0_{j0}v^jdt\,.$$

On the other hand, we have from (9.1) putting $p=0$

$$Dv^i=dv^i+\Gamma^i_{jk}\,v^j\,(dx^k-x^{(1)k}dt)+\Gamma^i_j\,v^jdt\,.$$

Hence by virtue of (9.5) and (9.8), it is evident that

$$(D^*v^I)_{I=0}\equiv 0,\quad (D^*v^I)_{I=i}=Dv^i\ \text{ for } v^0\equiv 0\,. \tag{9.9}$$

The preceding results are also obtained using a covariant vector field instead of contravariant.

10. Covariant derivatives. We can easily obtain the relations between two kinds of covariant derivatives using the results in § 9.

(i) In the first place from (9.4) we have

$$\left\{\begin{aligned} &(\nabla^*v^I)_{I=0}\equiv\nabla v^0\,, \\ &(\nabla^*_{(r)j}v^I)_{I=0}\equiv\nabla_{(r)j}v^0 \end{aligned}\right. \qquad (r=0,1,\cdots,m-1)\,. \tag{10.1}$$

Since v^0 is a scalar field of weight -1, ∇v^0 is a scalar field of weight 0, and $\nabla_{(r)j}v^0$ is a vector field of the second kind of weight $-(1+r)$.

(ii) From (9.7) we have

$$(10.2)\quad \begin{cases} (\nabla^* v^I)_{I=i} - x^{(1)i}(\nabla^* v^I)_{I=0} \equiv \nabla(v^i - x^{(1)i}v^0), \\ (\nabla^*_{(r)j}v^I)_{I=i} - x^{(1)i}(\nabla^*_{(r)j}v^I)_{I=0} \equiv \nabla_{(r)j}(v^i - x^{(1)i}v^0) + \delta^1_r\,\delta^i_j\,v^0 \end{cases}$$
$$(r=0, 1, \cdots, m-1).$$

Since $v^i - x^{(1)i}v^0$ is a vector field of weight 0 and v^0 is a scalar field of weight -1, the first of (10.2) is a vector field of weight $+1$ and the second is a tensor field of weight $-$r.

(iii) From (9.9) we have, for $v^0 \equiv 0$,

$$(10.3)\quad \begin{cases} (\nabla^* v^I)_{I=0} \equiv 0, \quad (\nabla^* v^I)_{I=i} \equiv \nabla v^i, \\ (\nabla^*_{(r)j}v^I)_{I=0} \equiv 0, \quad (\nabla^*_{(r)j}v^I)_{I=i} \equiv \nabla_{(r)j}v^i \end{cases} \qquad (r=0, 1, \cdots, m-1).$$

Since $v^0 \equiv 0$, v^i is a vector field of the second kind of weight 0. Therefore, ∇v^i is a vector field of weight $+1$, and $\nabla_{(r)j}v^i$ is a tensor field of weight $-r$.

11. Curvatures and torsions. We can get, making use of the results in the preceding paragraph, all the relations which hold between curvatures and torsions obtained in §§6, 8.

(i) From (10.1) it is clear that

$$(2[\nabla^*_{(r)j}\nabla^*]v^I)_{I=0} \equiv 2[\nabla_{(r)j}\nabla]v^0 \qquad (r=0, 1, \cdots, m-1),$$

$$(2[\nabla^*_{(r)j}\nabla^*_{(s)k}]v^I)_{I=0} \equiv 2[\nabla_{(r)j}\nabla_{(s)k}]v^0 \qquad \begin{pmatrix} r=0, 1, \cdots, m-1 \\ s=0, 1, \cdots, r \end{pmatrix}.$$

Noticing that v^0 is a scalar field of weight -1, and comparing the left-hand side with the right by use of (8.1) and (6.1), we get, besides (8.4) and (8.5),

$$(11.1)\quad R^{*0}_{h(r)j} \equiv 0, \; R^{*0}_{H(r)j(0)k} \equiv 0, \; R^{*0}_{0(r)j} \equiv R_{(r)j} \qquad (r=0, 1, \cdots, m-1).$$

(ii) From (10.3) we have for $v^0 = 0$

$$(2[\nabla^*_{(r)j}\nabla^*]v^I)_{I=0} \equiv 0 \qquad (r=0, 1, \cdots, m-1),$$

$$(2[\nabla^*_{(r)j}\nabla^*]v^I)_{I=i} \equiv 2[\nabla_{(r)j}\nabla]v^i \qquad (r=0, 1, \cdots, m-1),$$

$$(2[\nabla^*_{(r)j}\nabla^*_{(s)k}]v^I)_{I=0} \equiv 0 \qquad \begin{pmatrix} r=0, 1, \cdots, m-1 \\ s=0, 1, \cdots, r \end{pmatrix},$$

$$(2[\nabla^*_{(r)j}\nabla^*_{(s)k}]v^I)_{I=i}\equiv 2[\nabla_{(r)j}\nabla_{(s)k}]v^i \qquad \begin{pmatrix} r=0,\ 1,\ \cdots,\ m-1 \\ s=0,\ 1,\ \cdots,\ r \end{pmatrix}.$$

In this case, noticing that v^i is a vector of weight 0 and comparing the left-hand side with the right, we get the following relations in addition to (8.4), (8.5) and (11.1):

$$\text{(11.2)} \quad R^{*i}_{h(r)j}\equiv R^i_{h(r)j}, \quad R^{*i}_{h(r)j(0)k}\equiv R^i_{h(r)j(0)k} \qquad (r=0,\ 1,\ \cdots,\ m-1).$$

(iii) From (10.2) we can prove that there exist the following relations between two kinds of Poisson operators:

$$(2[\nabla^*_{(r)j}\nabla^*]v^I)_{I=i}-x^{(1)i}(2[\nabla^*_{(r)j}\nabla^*]v^I)_{I=0}\equiv 2[\nabla_{(r)j}\nabla](v^i-x^{(1)i}v^0)$$
$$(r=0,\ 1,\ \cdots,\ m-1),$$

$$(2[\nabla^*_{(r)j}\nabla^*_{(s)k}]v^I)_{I=i}-x^{(1)i}(2[\nabla^*_{(r)j}\nabla^*_{(s)k}]v^I)_{I=0}\equiv 2[\nabla_{(r)j}\nabla_{(s)k}](v^i-x^{(1)i}v^0)$$
$$\begin{pmatrix} r=0,\ 1,\ \cdots,\ m-1 \\ s=0,\ 1,\ \cdots,\ r \end{pmatrix}.$$

Noticing that $v^i-x^{(1)i}v^0$ is a vector field of weight 0 and using these relations, we obtain the following relations in addition to the already obtained:

$$\text{(11.3)} \quad \begin{cases} R^*_{0(r)j}\equiv x^{(1)i}R_{(r)j}-x^{(1)h}R^i_{h(r)j}+\delta^2_r\delta^i_j \\ R^{*i}_{0(r)j(0)k}\equiv S^{(1)i}_{(r)j(0)k}-x^{(1)h}R^i_{h(r)j(0)k} \end{cases} \qquad (r=0,\ 1,\ \cdots,\ m-1).$$

Hence all the relations desired are given by (8.4), (8.5), (11.1), (11.2) and (11.3). Thus, we arrive at the following

THEOREM. *The torsion tensors of the first kind coincide identically with those of the second kind. The curvature tensors of the first kind are expressible by those of the second kind and the torsion tensors* $S^{(1)i}_{(r)j(0)k}$ $(r=0,\ 1,\ \cdots,\ m-1)$.

Chapter III. The other geometries of paths.

The method hitherto employed in the case of the generalized rheonomic geometry of paths, after some suitable modifications, is available to geometries of paths such as ordinary, intrinsic and rheonomic. We give in each case the concrete modifications and show the relations between the results obtained by our method and those known for each geometry.

12. Ordinary geometry.

(i) Under the transformation group of coordinates

$$\xi^{\alpha}=\xi^{\alpha}(x^i), \qquad \tau=t \tag{12.1}$$

we are familiar with the notions of scalar, vector, tensor and others.

(ii) It is easily verified that, under the group (12.1), the line-element of the $(m-1)$-th order, its differential and the functions H^i are subject to the transformation laws which are obtained from those in § 2 by putting specially $\sigma=1$ and $\partial\xi^{[r]\alpha}/\partial t=0$ $(r=0, 1, \cdots, m-1)$ and writting $\xi^{(r)\alpha}$ in place of $\xi^{[r]\alpha}$. As a consequence it suffices to use the ordinary differential of the line-element $dx^{(r)i}$ instead of $\mathfrak{d}x^{(r)i}$.

(iii) The covariant derivative along a path of a vector field v^i in $X_{n+1}^{(m-1)}$ can be defined by[4)]

$$\delta_t v^i=D_t v^i+\Gamma^i_j v^j. \tag{12.2}$$

The parameters of connection Γ^i_j are transformed as

$$\frac{\partial\xi^{\beta}}{\partial x^j}\bar{\Gamma}^{\alpha}_{\beta}=\frac{\partial\xi^{\alpha}}{\partial x^i}\Gamma^i_j-\frac{\partial^2\xi^{\alpha}}{\partial x^i\partial x^j}x^{(1)i}, \tag{12.3}$$

hence in the same manner as in § 3, we can see that Γ^i_j are given by (3.11). However in the present case it is already known[5)] that it may be defined by

$$\Gamma^i_j=\frac{1}{m}H^i_{,(m-1)j} \tag{12.4}$$

which is simpler than (3.11).

(iv) The analogue of the theorem in § 4 can be proved without much difficulty:

THEOREM. *If the pfaffian forms*

$$P^i=\sum_{r=0}^{M}P^i_{(r)k}dx^{(r)k}$$

are transformed as a vector, then the pfaffians

$$\delta_t P^i=\sum_{r=0}^{M}P^i_{(r)k}dx^{(r+1)k}+\sum_{r=0}^{M}\{D_t P^i_{(r)k}+\Gamma^i_j P^j_{(r)k}\}dx^{(r)k}$$

4) The term pTv^i in (3.1) is a vector in an ordinary geometry.

5) A. Kawaguchi and H. Hombu [8].

are also transformed as a vector.

By virtue of this theorem we derive, as in the same way as in § 4, the covariant differential of the line-element

$$(12.5)\qquad \begin{cases}\delta x^i = dx^i,\\ \delta x^{(r)i} = dx^{(r)i} + \sum_{s=0}^{r-1} \Lambda_{(s)j}^{(r)i}\, dx^{(s)j}\end{cases} \qquad (r=1,\ 2,\ \cdots,\ m-1),$$

where $\Lambda_{(s)j}^{(r)i}$ are defined by recurrent formulae analogous[6] to (4.5). However it is known[7] that they may be defined simply by

$$(12.6)\qquad \Lambda_{(s)j}^{(r)i} = \frac{r!(m+s-r)!}{m!\,s!} H^i{}_{,(m+s-r)j} \qquad \begin{pmatrix} r=1,\ 2,\ \cdots,\ m-1 \\ s=0,\ 1,\ \cdots,\ r-1 \end{pmatrix}.$$

They are subject to the law analogous to (4.9):

$$(12.7)\qquad \frac{\partial \xi^{(r)\alpha}}{\partial x^{(s)i}} = \frac{\partial \xi^\alpha}{\partial x^j} \Lambda_{(s)i}^{(r)j} - \sum_{t=s}^{r-1} \frac{\partial \xi^{(t)\beta}}{\partial x^{(s)i}} \bar{\Lambda}_{(t)\beta}^{(r)\alpha} \qquad \begin{pmatrix} r=1,\ 2,\ \cdots,\ m-1 \\ s=0,\ 1,\ \cdots,\ r-1 \end{pmatrix}.$$

(v) The covariant differential of a vector field v^i can be defined by[8]

$$(12.8)\qquad Dv^i = dv^i + \Gamma^i_{jk} v^j dx^k .$$

The parameters of connection Γ^i_{jk} are subject to (5.2) and can be defined by (5.5) or (5.6). In the present case it is rather simple to use (5.6)[9]. The covariant derivatives of v^i and the fundamental invariants of the connection can be easily written.

13. Intrinsic geometry.

(i) For the transformation group of coordinates and parameter

$$(13.1)\qquad \xi^\alpha = \xi^\alpha(x^i), \quad \tau = \tau(t)$$

we can also define contravariant and covariant vectors and scalar of weight p by (1.4) and (1.4)′.

(ii) The transformation laws, under the group (13.1), of the line-element, the functions H^i and the differential of the line-element are almost the same to those in § 2. It is necessary to put $\partial\xi^\alpha/\partial t = 0$,

6) Omit the term containing Γ.

7) A. Kawaguchi and H. Hombu [8].

8) The terms $p\Gamma v^i\, dt$ and $(\Gamma^i_j - \Gamma^i_{jk} x^{(1)k})\, v^j\, dt$ in (5.1) are both vectors in our case.

9) The foundation of the theory was given by A. Kawaguchi and H. Hombu [8], Satz 7 on pp. 35, 42 and 44.

which is the only modification. In spite of this, we must take care that $\partial\xi^{[r]\alpha}/\partial t$ ($r=1, 2, \cdots, m-1$) do not vanish. Hence we must use, also in this case, the pfaffians $\mathfrak{d}x^{(r)i}$.

(iii) The covariant derivative along a path of a vector field v^i of weight p can also be defined by (3.1). The transformation laws of Γ^i_j and Γ are given by (3.2) and (3.3) too. Hence in the same manner as in § 3, we can see that these are given by (3.11) and (3.12). These results are already known by A. Kawaguchi and H. Hombu [8]. On the other hand S. Hokari [4] gave another result by use of the method of variations founded by D. D. Kosambi [6], [7]. T. Ohkubo [9] also studied the case of the third order using the so-called method of eliminations, but he did not discuss the covariant derivative along the path.

(iv) As to the covariant differential of the line-element, we have the same results as those of A. Kawaguchi and H. Hombu. As already mentioned in (ii), we must use the pfaffians $\mathfrak{d}x^{(r)i}$, instead of $dx^{(r)i}$ in the fundamental theorem[10] concerning the covariant differential of the line-element. Therefore the covariant differential[11] given there must be modified to ours given by (4.4). Concerning the results of S. Hokari, the circumstance is the same. The results of T. Ohkubo are somewhat different from ours.

(v) The covariant differential of a vector field v^i of weight p can be defined by (5.1) or by[12]

$$Dv^i = dv^i + p\Gamma v^i dt + \Gamma^i_{jk} v^j dx^k. \tag{13.2}$$

The parameters of connection Γ^i_{jk} are subject to the transformation laws (5.2) and may be defined by (5.5) or (5.6). For general m, A. Kawaguchi and H. Hombu used (5.5) and S. Hokari used both (5.5) and (5.6). For $m=3$, various formulae for Γ^i_{jk} are given in [8], [4] and [9]. Some modifications for the covariant derivatives given by A. Kawaguchi and H. Hombu and by S. Hokari are necessary.

14. Rheonomic geometry.

(i) Under the rheonomic transformation group

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10) A. Kawaguchi and H. Hombu [8], Satz 16 on p. 58.

11) A. Kawaguchi and H. Hombu [8], Formula (3.18) on p. 59.

12) In the present case the term $(\Gamma^i_j - \Gamma^i_{jk} x^{(1)k}) v^j\, dt$ is a vector of weight p, by omitting it we have (13.2). A. Kawaguchi and H. Hombu [8] and S. Hokari [4] used also (13.2).

(14.1) $$\xi^\alpha = \xi^\alpha(t, x^i), \qquad \tau = t$$

we can define, as in the same manner as in § 1, two kinds of vectors, and discuss the relations between them.

(ii) Under the group (14.1), the transformation laws of the line-element, and the functions H^i are obtainable from (2.1)′ by putting $\sigma = 1$.

(iii) The covariant derivative along a path of a vector field v^i can be defined by (12.2)[13]. The parameters of connection Γ^i_j are transformed as

(14.2) $$\frac{\partial \xi^\beta}{\partial x^j}\bar{\Gamma}^\alpha_\beta = \frac{\partial \xi^\alpha}{\partial x^i}\Gamma^i_j - D_t\frac{\partial \xi^\alpha}{\partial x^j},$$

hence we see that Γ^i_j can be given by (3.11). However in the rheonomic geometry it is already known[14] that they may be defined by the simpler expression (12.4).

(iv) The analogue of the theorem in § 4 may be easily proved:

THEOREM. *If the pfaffian forms* (4.1) *are transformed as a vector, then the new pfaffians*

$$\delta_t P^i = \sum_{r=0}^{M} P^i_{(r)k}\mathfrak{d}x^{(r+1)k} + \sum_{r=0}^{M}\{D_t P^i_{(r)k} + \Gamma^i_j P^j_{(r)k}\}\mathfrak{d}x^{(r)k}$$

are also transformed as a vector.

By virtue of this theorem we can define the covariant differential of the line-element in the form (4.4), where $\Lambda^{(r)i}_{(s)j}$ are defined by the analogous recurrent formulae as (4.5)[15]. In the rheonomic case, it is already known[16] that they can be defined simply by (12.6).

(v) The covariant differential of a vector field can be defined by[17]

$$Dv^i = dv^i + \Gamma^i_j v^j dt + \Gamma^i_{jk} v^j \mathfrak{d}x^k.$$

The parameters of connection Γ^i_{jk} are subject to (5.2); thus they can be defined by (5.5) or (5.6). However the latter is better than the former. These results were obtained by H. Hombu [5] and T. Suguri [11].

13) The term pTv^i in (3.1) is a vector for the rheonomic geometry.

14) See H. Hombu [5] and T. Suguri [11].

15) Omit the term containing T.

16) See H. Hombu [5] and T. Suguri [11].

17) The term $pTv^i\, dt$ in (5.1) is a vector for the rheonomic geometry.

Chapter IV. The equivalence problems.

In the pressent chapter we discuss the equivalence problem in each geometry of paths. This problem is one of the most fundamental but unsolved problems. We use the well-known theorem concerning the mixed system of partial differential equations of the first order[18].

15. Ordinary geometry. Under the transformation group of coordinates

$$\xi^{\alpha}=\xi^{\alpha}(x^{i}), \qquad \tau=t \tag{15.1}$$

the line-elements of the $(m-1)$-th order are subject to

$$\begin{cases} \text{(i)} & \xi^{\alpha}=\xi^{\alpha}(x^{i}), \\ \text{(ii)} & \xi^{(1)\alpha}=\dfrac{\partial\xi^{\alpha}}{\partial x^{i}}x^{(1)i}, \\ \text{(iii)} & \xi^{(r+1)\alpha}=\dfrac{\partial\xi^{\alpha}}{\partial x^{i}}x^{(r+1)i}+\displaystyle\sum_{s=0}^{r-1}\dfrac{\partial\xi^{(r)\alpha}}{\partial x^{(s)i}}x^{(s+1)i} \qquad (r=1,2,\cdots,m-1). \end{cases} \tag{15.2}$$

If we consider a system of paths of the m-th order defined by

$$x^{(m)i}+H^{i}(t,\ x,\ x^{(1)},\ \cdots,\ x^{(m-1)})=0, \tag{15.3}$$

then the functions H^{i} are subject to

$$\overline{H}^{\alpha}=\frac{\partial\xi^{\alpha}}{\partial x^{i}}H^{i}-\sum_{s=0}^{m-2}\frac{\partial\xi^{(m-1)\alpha}}{\partial x^{(s)i}}x^{(s+1)i}. \tag{15.4}$$

In other words, if the system of equations (15.3) is transformed, under a transformation of the group (15.1), into

$$\xi^{(m)\alpha}+\overline{H}^{\alpha}(t,\ \xi,\ \xi^{(1)},\ \cdots,\ \xi^{(m-1)})=0, \tag{15.5}$$

then there exist relations (15.2) and (15.4) between two sets of quantities

$$\begin{cases} \text{(i)} & \{x^{i},x^{(1)i},\cdots,x^{(m-1)i},H^{i}(t,x,x^{(1)},\cdots,x^{(m-1)})\}, \\ \text{(ii)} & \{\xi^{\alpha},\xi^{(1)\alpha},\cdots,\xi^{(m-1)\alpha},\overline{H}^{\alpha}(t,\xi,\xi^{(1)},\cdots,\xi^{(m-1)})\}. \end{cases} \tag{15.6}$$

Conversely, for any given systems (15.3) and (15.5), if there exists a transformation of the group such that (15.2) and (15.4) hold, we

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18) See e.g. T. Y. Thomas [14] or O. Veblen [15].

say that *the two systems of paths are equivalent under the group.* Our problem is to obtain a necessary and sufficient condition for the equivalence of the two systems.

According to the method in § 12, we construct the parameters of connection $\Lambda^{(r)i}_{(s)j}$, Γ^i_{jk} defined by (12.6), (5.6) using the functions H^i and those for $\bar{H}^\alpha$. When (15.4) hold, there exist the relations (12.7) and (5.2). From (15.2), it is evident that we can put

$$\frac{\partial \xi^\alpha}{\partial x^i}=\frac{\partial \xi^{(1)\alpha}}{\partial x^{(1)i}}=\cdots\cdots=\frac{\partial \xi^{(m-1)\alpha}}{\partial x^{(m-1)i}}=u^\alpha_i,$$

hence we can express each

$$\frac{\partial x^{(r)\alpha}}{\partial x^{(s)i}} \qquad \begin{pmatrix} r=1,2,\cdots,m-1 \\ s=0,1,\cdots,\ r-1 \end{pmatrix}$$

successively as polynomial of

$$u^\alpha_i,\ \Lambda^{(t)j}_{(s)i},\ \bar{\Lambda}^{(t)\beta}_{(s)\alpha} \qquad \begin{pmatrix} t=1,2,\cdots,r \\ s=0,1,\cdots,t-1 \end{pmatrix}.$$

Thus we find that $n(n+m)$ functions

(15.7) $$\xi^\alpha,\ \xi^{(1)\alpha},\ \cdots,\ \xi^{(m-1)\alpha},\ u^\alpha_i=\partial\xi^\alpha/\partial x^i$$

of $mn+1$ independent variables

(15.8) $$t,\ x^i,\ x^{(1)i},\ \cdots,\ x^{(m-1)i}$$

satisfy the following differential equations:

$$(15.9)\quad \begin{cases} \text{(i)} & \dfrac{\partial \xi^\alpha}{\partial t}=0,\ \dfrac{\partial \xi^\alpha}{\partial x^i}=u^\alpha_i,\ \dfrac{\partial \xi^\alpha}{\partial x^{(s)i}}=0 \qquad (s=1,2,\cdots,m-1), \\ \text{(ii)} & \begin{cases} \dfrac{\partial \xi^{(r)\alpha}}{\partial t}=0,\ \dfrac{\partial \xi^{(r)\alpha}}{\partial x^{(r)i}}=u^\alpha_i & (r=1,2,\cdots,m-1), \\ \dfrac{\partial \xi^{(r)\alpha}}{\partial x^{(s)i}}=0 & \begin{pmatrix} r=1,2,\cdots,m-2 \\ s=r+1,r+2,\cdots,m-1 \end{pmatrix}, \\ \dfrac{\partial \xi^{(r)\alpha}}{\partial x^{(s)i}}=u^\alpha_j\Lambda^{(r)j}_{(s)i}-\sum_{t=s}^{r-1}\bar{\Lambda}^{(r)\alpha}_{(t)\beta} & \begin{pmatrix} r=1,2,\cdots,m-1 \\ s=0,1,\cdots,\ r-1 \end{pmatrix}, \end{cases} \\ \text{(iii)} & \begin{cases} \partial u^\alpha_j/\partial t=0,\ \partial u^\alpha_j/\partial x^{(r)k}=0 & (r=1,2,\cdots,m-1), \\ \partial u^\alpha_j/\partial x^k=u^\alpha_i\Gamma^i_{jk}-u^\beta_j u^\gamma_k\bar{\Gamma}^\alpha_{\beta\gamma}. \end{cases} \end{cases}$$

For the later convenience we rewrite (15.4) as follows:

$$\overline{H}^{\alpha}=u_i^{\alpha}H^i-\sum_{s=0}^{m-1}\frac{\partial\xi^{(m-1)\alpha}}{\partial x^{(s)i}}x^{(s+1)i}, \tag{15.10}$$

where we consider that the right-hand side is a polynomial expressed in terms of

$$u_i^{\alpha},\ H^i,\ x^{(t)i},\ \Lambda^{(t)h}_{(u)j},\ \overline{\Lambda}^{(t)\beta}_{(u)\gamma} \qquad \begin{pmatrix} t=1,2,\cdots,m-1\\ u=0,1,\cdots,t-1\end{pmatrix}.$$

Thus we know that *if* (15.2) *and* (15.4) *hold, then* (15.9) *and* (15.10) *are satisfied.* Conversely, we prove the following: *A set of solutions of* (15.9) *satisfying equations* (15.10), *conforms to* (15.2) *and* (15.4), *under a suitable initial condition.*

For this purpose we use the relations

$$\frac{\partial\xi^{(r)\alpha}}{\partial x^{(s)i}}=\binom{r}{s}\frac{\partial\xi^{(r-s)\alpha}}{\partial x^i} \qquad \begin{pmatrix} r=1,2,\cdots,m-1\\ s=0,1,\cdots,\ r-1\end{pmatrix} \tag{15.11}$$

which are easily verified by virtue of (15.9 ii) and (12.6).

It is evident that a set of solutions now considering is of the form:

$$\xi^{\alpha}=\xi^{\alpha}(x^i), \qquad u_i^{\alpha}=u_i^{\alpha}(x^j),$$

$$\xi^{(r)\alpha}=\xi^{(r)\alpha}(x,\ x^{(1)},\ \cdots,\ x^{(r)}) \qquad (r=\ 1,2,\cdots,m-1).$$

In the first place, for $\xi^{(1)}(x,x^{(1)})$ we have from (15.9 ii),

$$\xi^{(1)\alpha}=\frac{\partial\xi^{\alpha}}{\partial x^i}x^{(1)i}+c^{(1)\alpha}(x), \tag{15.12}$$

hence

$$\frac{\partial\xi^{(1)\alpha}}{\partial x^j}=\frac{\partial^2\xi^{\alpha}}{\partial x^i\partial x^j}x^{(1)i}+\frac{\partial c^{(1)\alpha}}{\partial x^j}. \tag{15.13}$$

On the other hand by differentiating (15.10) in $x^{(m-1)j}$, we have

$$\frac{\partial\xi^{\beta}}{\partial x^j}\overline{H}^{\alpha},_{(m-1)\beta}=\frac{\partial\xi^{\alpha}}{\partial x^i}H^i,_{(m-1)j}-\frac{\partial\xi^{(m-1)\alpha}}{\partial x^{(m-2)j}}-\frac{\partial^2\xi^{\alpha}}{\partial x^i\partial x^j}x^{(1)i},$$

which is equivalent to

$$\frac{\partial\xi^{(1)\alpha}}{\partial x^j}=\frac{\partial^2\xi^{\alpha}}{\partial x^i\partial x^j}x^{(1)i}$$

by virtue of (12.6), (15.11) and (15.9). Comparing this with (15.12), we have

$$\partial c^{(1)\alpha}/\partial x^j=0, \text{ i. e. } c^{(1)\alpha}=\text{const.}$$

Hence under the initial condition

$$(\xi^{(1)\alpha})_0=0 \text{ when } (x^{(1)i})_0=0,$$

we have, from (15.12), the relations (15.2 ii).

In the second place, noticing (15.9 ii), (15.11) and (15.13), we have

$$\xi^{(2)\alpha}=\frac{\partial \xi^\alpha}{\partial x^i}x^{(2)i}+\frac{\partial^2\xi^\alpha}{\partial x^i \partial x^j}x^{(1)i}x^{(1)j}+c^{(2)\alpha}(x)\,, \tag{15.14}$$

hence,

$$\frac{\partial \xi^{(2)\alpha}}{\partial x^j}=\frac{\partial^2\xi^\alpha}{\partial x^i\partial x^j}x^{(2)i}+\frac{\partial^3\xi^\alpha}{\partial x^i\partial x^j\partial x^k}x^{(1)i}x^{(1)k}+\frac{\partial c^{(2)\alpha}}{\partial x^j}\,. \tag{15.15}$$

On the other hand, differentiating (15.10) in $x^{(m-2)j}$ we have

$$\begin{aligned}\frac{\partial\xi^{(m-1)\beta}}{\partial x^{(m-2)j}}\overline{H}^\alpha{}_{,\,(m-1)\beta}+\frac{\partial \xi^\beta}{\partial x^j}\overline{H}^\alpha{}_{,\,(m-2)\beta}\\ =u_i^\alpha H^i{}_{,\,(m-2)j}-\frac{\partial\xi^{(m-1)\alpha}}{\partial x^{(m-3)j}}-\sum_{s=1}^{m-2}\frac{\partial^2\xi^{(m-1)\alpha}}{\partial x^{(m-2)j}\partial x^{(s)i}}x^{(s+1)i}\,.\end{aligned} \tag{15.16}$$

On account of (15.11), (15.15), (15.16), (15.9 ii) and (12.6), we can conclude that

$$\partial c^{(2)\alpha}/\partial x^j=0, \text{ i. e. } \qquad c^{(2)\alpha}=\text{const.}$$

Hence under the initial condition

$$(\xi^{(2)\alpha})_0=0, \text{ when } (x^{(1)i})_0=(x^{(2)i})_0=0\,,$$

we see that (15.2 iii) holds for $r=1$. We can accomplish the general proof of (15.2 iii) by induction.

Let (15.2 iii) hold for $r=1, 2, \cdots, p-1$, and consider the solution

$$\xi^{(p+1)\alpha}=\xi^{(p+1)\alpha}(x,\ x^{(1)},\ \cdots,\ x^{(p+1)})$$

and the function $F^{(p+1)\alpha}(x, x^{(1)}, \cdots, x^{(p+1)})$ defined by

$$F^{(p+1)\alpha}=\frac{\partial\xi^\alpha}{\partial x^i}x^{(p+1)i}+\sum_{u=0}^{p-1}\frac{\partial\xi^{(p)\alpha}}{\partial x^{(u)i}}x^{(u+1)i}\,.$$

Then we can easily verify that

$$\frac{\partial}{\partial x^{(s)i}}(\xi^{(p+1)\alpha}-F^{(p+1)\alpha})=0 \qquad (s=1,2,\cdots,p+1)$$

by virtue of (15.11), (15.9 ii) and the assupmtions of the induction. Hence we have

$$\xi^{(p+1)\alpha}=\frac{\partial\xi^{\alpha}}{\partial x^{i}}x^{(p+1)i}+\sum_{u=0}^{p-1}\frac{\partial\xi^{(p)\alpha}}{\partial x^{(u)i}}x^{(u+1)i}+c^{(p+1)\alpha}(x). \tag{15.17}$$

Accordingly

$$\frac{\partial\xi^{(p+1)\alpha}}{\partial x^{j}}=\frac{\partial^2\xi^{\alpha}}{\partial x^{i}\partial x^{j}}x^{(p+1)i}+\sum_{u=0}^{p-1}\frac{\partial^2\xi^{(p)\alpha}}{\partial x^{j}\partial x^{(u)i}}x^{(u+1)i}+\frac{\partial c^{(p+1)\alpha}}{\partial x^{j}}. \tag{15.18}$$

On the other hand, differentiating (15.10) with respect to $x^{(m-p-1)j}$ we have

$$\begin{aligned}&\sum_{t=m-p-1}^{m-1}\frac{\partial\xi^{(t)\beta}}{\partial x^{(m-p-1)j}}\overline{H}^{\alpha}{}_{,(t)\beta}\\ &=u_i^{\alpha}H^{i}{}_{,(m-p-1)j}-\frac{\partial\xi^{(m-1)\alpha}}{\partial x^{(m-p-2)j}}-\sum_{s=0}^{m-2}\frac{\partial^2\xi^{(m-1)\alpha}}{\partial x^{(m-p-1)j}\partial x^{(s)i}}x^{(s+1)i}.\end{aligned} \tag{15.19}$$

Using these relations (15.18) and (15.19), we can conclude that

$$\partial c^{(p+1)\alpha}/\partial x^{j}=0,\ \text{i.e.}\quad c^{(p+1)\alpha}=\text{const.},$$

from which we see that under the initial condition

$$(\xi^{(p+1)\alpha})_0=0\ \text{ when }\ (x^{(1)i})\ =(x^{(2)i})_0=\cdots=(x^{(p+1)i})_0=0,$$

(15.2 iii) holds for $r=p$. Hence in general we have (15.2 iii). As for the relation (15.4), it is evident from (15.10).

Now (15.2 iii) and (15.4) are rewritten in the form

$$\xi^{(r+1)\alpha}+\sum_{s=0}^{r-1}\overline{A}{}^{(r)\alpha}_{(s)\beta}\,\xi^{(s+1)\beta}=u_i^{\alpha}(x^{(r+1)i}+\sum_{s=0}^{r-1}A^{(r)i}_{(s)j}\,x^{(s+1)j}) \qquad (r=1,2,\cdots,m-2),$$

$$\overline{H}^{\alpha}-\sum_{s=0}^{m-2}\overline{A}{}^{(m-1)\alpha}_{(s)\beta}\xi^{(s+1)\beta}=u_i^{\alpha}(H^{i}-\sum_{s=0}^{m-2}A^{(m-1)i}_{(s)j}\,x^{(s+1)j})$$

on account of (15.9). Therefore, to solve (15.9) under the conditions (15.10) is equivalent to solve it under the conditions

$$\overline{K}{}^{\alpha}_{(r)}=u_i^{\alpha}K^{i}_{(r)} \qquad (r=1,2,\cdots,m), \tag{15.20}$$

where we put

$$
(15.21) \qquad \begin{cases} K^i_{(1)}=x^{(1)i}, \\ K^i_{(r)}=\delta_t K^i_{(r-1)}=x^{(r)i}+\sum_{s=0}^{r-2}\Lambda^{(r-1)i}_{(s)j}x^{(s+1)j} & (r=2, 3, \dots, m-1), \\ K^i_{(m)}=\delta_t K^i_{(m-1)}=-H^i+\sum_{s=0}^{m-2}\Lambda^{(m-1)i}_{(s)j}x^{(s+1)j}. \end{cases}
$$

Hence we have the following conclusion:

The equivalence problem of the systems of paths (15.3) *and* (15.5) *under the group* (15.1) *is reduced to the problem for a necessary and sufficient condition that* (15.9) *may have a set of solutions satisfying the conditions* (15.20).

Now according to T. Y. Thomas [14] or O. Veblen [15], we construct for (15.9) and (15.20) the sequence of sets of equations concerning the variables (15.7) and (15.8):

$$
(15.22) \qquad F^{(1)}=0, F^{(2)}=0, \dots, F^{(N)}=0, \dots,
$$

where $F^{(1)}=0$ is the set of equations consisting of (15.20), the equations of integrability of (15.9) and the equations obtained by differentiating (15.20) with respect to the arguments (15.8) and eliminating the derivatives of the functions (15.7) by means of (15.9); and $F^{(p+1)}=0$ for $p\geqq 1$ is the set of equations obtained by differentiating the set $F^{(p)}=0$ with respect to the arguments (15.8) and eliminating the derivatives of the functions (15.7) by means of (15.9). Then by the well-known theorem for the existence of a solution of the mixed system, we see that *a necessary and sufficient condition that the two systems of paths be equivalent is that there exists a positive integer N such that the first N sets of equations of* (15.22) *are algebraically consistent considered as equations for the determination of the variables* (15.7) *as functions of the independent variables* (15.8), *and that all their solutions satisfy the* $(N+1)$*-th set of equations of the sequence.*

The first set of the sequence (15.22) gives the transformation laws of the set of invariants $K^i_{(r)}$, curvature and torsion tensors and the p-th set for $p>1$ gives the transformation laws of the covariant derivatives of the $(p-1)$-th order of them. Therefore we have the

THEOREM. *The curvature and torsion tensors and the set of invariants $K^i_{(r)}$ $(r=1, 2, \dots, m)$ and their successive covariant derivatives* *constitute the complete system of differential invariants for the ordinary geometry of paths.*

16. Intrinsic geometry. Consider the two sets of equations

$$(16.1)\quad \begin{cases} \text{(i)} & x^{(m)i}+H^i(t, x, x^{(1)}, \dots, x^{(m-1)})=0\,, \\ \text{(ii)} & \xi^{[m]\alpha}+\bar{H}^{\alpha}(\tau, \xi, \xi^{[1]}, \dots, \xi^{[m-1]})=0 \end{cases}$$

and the transformation group of coordinates and parameter

$$(16.2)\qquad \xi^{\alpha}=\xi^{\alpha}(x^i)\,, \qquad \tau=\tau(t)\,.$$

The transformation laws of $(x^i, x^{(1)i}, \cdots, x^{(m-1)i})$ and H^i are given by

$$(16.3)\quad \begin{cases} \text{(i)} & \xi^{[1]\alpha}=\sigma\dfrac{\partial\xi^{\alpha}}{\partial x^i}x^{(1)i}\,, \\ \text{(ii)} & \xi^{[r]\alpha}=\sigma^r\dfrac{\partial\xi^{\alpha}}{\partial x^i}x^{(r)i}+\sigma\displaystyle\sum_{s=0}^{r-2}\dfrac{\partial\xi^{[r-1]\alpha}}{\partial x^{(s)i}}x^{(s+1)i}+\sigma\dfrac{\partial\xi^{[r-1]\alpha}}{\partial t} \\ & \qquad (r=2, 3, \cdots, m-1)\,, \\ \text{(iii)} & -\bar{H}^{\alpha}=-\sigma^m\dfrac{\partial\xi^{\alpha}}{\partial x^i}H^i+\sigma\displaystyle\sum_{s=0}^{m-2}\dfrac{\partial\xi^{[m-1]\alpha}}{\partial x^{(s)i}}x^{(s+1)i}+\sigma\dfrac{\partial\xi^{[m-1]\alpha}}{\partial t}\,. \end{cases}$$

If there exists a transformation belonging to (16.2) such that (16.3) hold between two sets of quantities

$$\{t, x^i, x^{(1)i}, \cdots, x^{(m-1)i}, H^i(t, x, x^{(1)}, \cdots, x^{(m-1)})\}\,,$$

$$\{\tau, \xi^{\alpha}, \xi^{[1]\alpha}, \cdots, \xi^{[m-1]\alpha}, \bar{H}^{\alpha}(\tau, \xi, \xi^{[1]}, \cdots, \xi^{[m-1]})\}\,,$$

we say that the *two systems of paths are equivalent under the group* (16.2). According to the method in § 13, we construct $\Gamma, \Gamma^i_j, \Lambda^{(r)i}_{(s)j}$ and Γ^i_{jk} and the corresponding functions using H^i and $\bar{H}^{\alpha}$. If the two systems (16.1) are equivalent, then the relations (3.2), (3.3) (4.9) and (5.2) hold.

As is already known that

$$K^i_{(1)}\equiv x^{(1)i}$$

is a vector of weight $+1$, we define successively[19]

$$(16.4)\qquad K^i_{(r+1)}=\delta_t K^i_{(r)}=D_t K^i_{(r)}+\Gamma^i_j K^j_{(r)}+r\Gamma K^i_{(r)}$$
$$(r=1, 2, \cdots, m-1)\,,$$

19) The $\mathfrak{K}^i$ defined by A. Kawaguchi and H. Hombu [8], Formula (3.22) on p. 60 is not a vector.

then $K^i_{(r)}$ is a vector of weight r. If we put

$$(16.5)\quad \begin{cases} K^i_{(r)}=x^{(r)i}+\sum_{s=1}^{r-1} U^{(r)i}_{(s)j}\, x^{(s)j} & (r=2,3,\cdots,m-1), \\ K^i_{(m)}=-H^i+\sum_{s=1}^{m-1} U^{(m)i}_{(s)j}\, x^{(s)j}, & \end{cases}$$

then $U^{(r)i}_{(s)j}$ are defined by

$$(16.6)\quad \begin{cases} U^{(r+1)i}_{(r)j}=r\Gamma^i_j+\dfrac{r(r+1)}{2}\Gamma\delta^i_j & (r=1,2,\cdots,m-1), \\ U^{(r+1)i}_{(s)j}=D_t U^{(r)i}_{(s)j}+U^{(r)i}_{(s-1)j}+(\Gamma^i_h+r\Gamma\delta^i_h)U^{(r)h}_{(s)j} & \begin{pmatrix} r=3,4,\cdots,m-1 \\ s=2,3,\cdots,\ r-1 \end{pmatrix}, \\ U^{(r+1)i}_{(1)j}=D_t U^{(r)i}_{(1)j}+(\Gamma^i_h+r\Gamma\delta^i_h)\,U^{(r)h}_{(1)j} & (r=2,3,\cdots,m-1). \end{cases}$$

We can see that (16.3 ii) and (16.3 iii) are rewritten as follows:

$$(16.7)\qquad \bar{K}^\alpha_{(r)}=\sigma^r\frac{\partial\xi^\alpha}{\partial x^i}K^i_{(r)} \qquad (r=2,3,\cdots,m).$$

After these considerations we find that n^2+nm+2 functions

$$(16.8)\qquad \tau,\ \xi^\alpha,\ \xi^{[1]\alpha},\cdots,\ \xi^{[m-1]\alpha},\ d\tau/dt=1/\sigma,\ \partial\xi^\alpha/\partial x^i=u^\alpha_i$$

of $nm+1$ independent variables

$$(16.9)\qquad t,\ x^i,\ x^{(1)i},\ \cdots,\ x^{(m-1)i}$$

satisfy the following differential equations:

$$(16.10)\quad \begin{cases} \text{(i)}\quad d\tau/dt=1/\sigma,\ \ \partial\tau/\partial x^{(s)j}=0 & (s=0,1,\cdots,m-1), \\ \text{(ii)}\quad d\sigma/dt=\sigma\Gamma-\bar{\Gamma},\ \ \partial\sigma/\partial x^{(s)j}=0 & (s=0,1,\cdots,m-1), \\ \text{(iii)}\quad \begin{cases} \dfrac{\partial\xi^{[r]\alpha}}{\partial x^{(r)j}}=\sigma^r u^\alpha_j & (r=0,1,\cdots,m-1), \\ \dfrac{\partial\xi^{[r]\alpha}}{\partial x^{(s)j}}=0 & \begin{pmatrix} r=0,1,\cdots,m-2 \\ s=r+1,r+2,\cdots,m-1 \end{pmatrix}, \\ \dfrac{\partial\xi^{[r]\alpha}}{\partial x^{(s)j}}=\sigma^r u^\alpha_i\Lambda^{(r)i}_{(s)j}-\sum_{t=s}^{r-1}\dfrac{\partial\xi^{[t]\beta}}{\partial x^{(s)j}}\bar{\Lambda}^{[r]\alpha}_{[t]\beta} & \begin{pmatrix} r=1,2,\cdots,m-1 \\ s=0,1,\cdots,\ r-1 \end{pmatrix}, \end{cases} \end{cases}$$

$$
\left\{
\begin{aligned}
&\text{(iv)} \left\{
\begin{aligned}
&\frac{\partial \xi^{[r]\alpha}}{\partial t} = \frac{1}{\sigma}\,\xi^{[r+1]\alpha} - \sigma^r u^\alpha_i x^{(r+1)i} - \sum_{s=0}^{r-1} \frac{\partial \xi^{[r]\alpha}}{\partial x^{(s)i}}\, x^{(s+1)i} \qquad (r=1, 2, \cdots, m-2), \\
&\frac{\partial \xi^{[m-1]\alpha}}{\partial t} = -\frac{1}{\sigma}\,\bar{H}^\alpha + \sigma^{m-1} u^\alpha_i H^i - \sum_{s=0}^{m-2} \frac{\partial \xi^{[m-1]\alpha}}{\partial x^{(s)i}}\, x^{(s+1)i}
\end{aligned}
\right. \\
&\text{(v)} \left\{
\begin{aligned}
&\partial u^\alpha_j/\partial t = 0, \quad \partial u^\alpha_j/\partial x^{(s)k} = 0 \qquad (s=1, 2, \cdots, m-1), \\
&\partial u^\alpha_j/\partial x^k = u^\alpha_i \Gamma^i_{jk} - u^\beta_j u^\gamma_k \bar{\Gamma}^\alpha_{\beta\gamma},
\end{aligned}
\right.
\end{aligned}
\right.
$$

and the identity

$$
\bar{K}^\alpha_{(r)} = \sigma^r u^\alpha_i K^i_{(r)} \qquad (r=1, 2, \cdots, m). \tag{16.11}
$$

Conversely if (16.10) admits a solution satisfying the conditions (16.11), we can conclude that (16.1) are equivalent. Hence we have:

The equivalence problem is reduced to the problem for a necessary and sufficient condition that (16.10) *may have a solution satisfying the conditions* (16.11).

Noticing that a solution of (16.10) satisfies the relations

$$
\left\{
\begin{aligned}
&\text{(i)} && \frac{\partial \xi^{[r+1]\alpha}}{\partial x^i} = \sigma D_t \frac{\partial \xi^{[r]\alpha}}{\partial x^i} && (r=0, 1, \cdots, m-2), \\
&\text{(ii)} && \frac{\partial \xi^{[r+1]\alpha}}{\partial x^{(s)i}} = \sigma\left\{D_t \frac{\partial \xi^{[r]\alpha}}{\partial x^{(s)i}} + \frac{\partial \xi^{[r]\alpha}}{\partial x^{(s-1)i}}\right\} && \left(\begin{aligned}&r=1, 2, \cdots, m-2\\ &s=1, 2, \cdots, r\end{aligned}\right), \\
&\text{(iii)} && \frac{\partial \xi^{[r+1]\alpha}}{\partial x^{(r)j}} = (r+1)\sigma^r \frac{\partial \xi^{[1]\alpha}}{\partial x^j} + \frac{r(r+1)}{2}\,\sigma^r(\sigma\Gamma - \bar{\Gamma})u^\alpha_j && (r=0, 1, \cdots, m-2),
\end{aligned}
\right. \tag{16.12}
$$

we can construct a sequence of sets of equations analogous to (15.22) concerning the variables (16.8) and (16.9). Hence we get the following conclusion:

A necessary and sufficient condition for the equivalence of the two systems of paths (16.1) *under the group* (16.2) *is expressible algebraically by use of curvature and torsion tensors, a set of invariants* $K^i_{(r)}$ ($r=1, 2, \cdots, m$) *and their successive covariant derivatives.*

17. Rheonomic geometry. Let us consider the two systems of paths defined by

$$(17.1)\quad \begin{cases} \text{(i)} & x^{(m)i}+H^i(t, x, x^{(1)}, \cdots, x^{(m-1)})=0, \\ \text{(ii)} & \xi^{(m)\alpha}+\overline{H}^\alpha(t, \xi, \xi^{(1)}, \cdots, \xi^{(m-1)})=0 \end{cases}$$

and the rheonomic transformation group

$$(17.2)\qquad \xi^\alpha=\xi^\alpha(t, x^i), \quad \tau=t.$$

The trnasformation laws of $(x^i, x^{(1)i}, \cdots, x^{(m-1)i})$ and H^i are given by

$$(17.3)\quad \begin{cases} \text{(i)} & \xi^{(1)\alpha}=\dfrac{\partial\xi^\alpha}{\partial x^i}x^{(1)i}+\dfrac{\partial\xi^\alpha}{\partial t}, \\ \text{(ii)} & \xi^{(r+1)\alpha}=\dfrac{\partial\xi^\alpha}{\partial x^i}x^{(r+1)i}+\sum\limits_{s=0}^{r-1}\dfrac{\partial\xi^{(r)\alpha}}{\partial x^{(s)i}}x^{(s+1)i}+\dfrac{\partial\xi^{(r)\alpha}}{\partial t} \\ & \qquad (r=1, 2, \cdots, m-2), \\ \text{(iii)} & -\overline{H}^\alpha=-\dfrac{\partial\xi^\alpha}{\partial x^i}H^i+\sum\limits_{s=0}^{m-2}\dfrac{\partial\xi^{(m-1)\alpha}}{\partial x^{(s)i}}x^{(s+1)i}+\dfrac{\partial\xi^{(m-1)\alpha}}{\partial t}. \end{cases}$$

We can define the equivalence of the two systems of paths analogously to those in §§ 15, 16. We construct Γ'^i_j, $\Lambda^{(r)i}_{(s)j}$ and Γ'^i_{jk} and the corresponding functions using H^i and $\overline{H}^\alpha$. Then if the two systems (17.1) are equivalent, there exist relations (14.2), (4.9) (put $\sigma=1$) and (5.2).

Hence we know that $n(n+m)$ functions

$$(17.4)\qquad \xi^\alpha, \ \xi^{(1)\alpha}, \cdots, \xi^{(m-1)\alpha}, \ u^\alpha_i=\partial\xi^\alpha/\partial x^i$$

of $mn+1$ independent variables

$$(17.5)\qquad t, \ x^i, \ x^{(1)i}, \cdots, x^{(m-1)i}$$

satisfy the following:

$$\text{(i)}\quad \begin{cases} \dfrac{\partial\xi^{(r)\alpha}}{\partial x^{(r)j}}=u^\alpha_j & (r=0, 1, \cdots, m-1), \\ \dfrac{\partial\xi^{(r)\alpha}}{\partial x^{(s)j}}=0 & \begin{pmatrix} r=0, 1, \cdots, m-2 \\ s=r+1, r+2, \cdots, m-1 \end{pmatrix}, \\ \dfrac{\partial\xi^{(r)\alpha}}{\partial x^{(s)j}}=u^\alpha_i\Lambda^{(r)i}_{(s)j}-\sum\limits_{t=s}^{r-1}\dfrac{\partial\xi^{(t)\beta}}{\partial x^{(s)j}}\overline{\Lambda}^{(r)\alpha}_{(t)\beta} & \begin{pmatrix} r=1, 2, \cdots, m-1 \\ s=0, 1, \cdots, r-1 \end{pmatrix}, \end{cases}$$

$$
(17.6)\quad \begin{cases} \text{(ii)} & \begin{cases} \dfrac{\partial \xi^{\alpha}}{\partial t} = \xi^{(1)\alpha} - u_i^{\alpha} x^{(1)i}, \\[2mm] \dfrac{\partial \xi^{(r)\alpha}}{\partial t} = \xi^{(r+1)\alpha} - u_i^{\alpha} x^{(r+1)i} - \sum_{s=0}^{r-1} \dfrac{\partial \xi^{(r)\alpha}}{\partial x^{(s)i}} x^{(s+1)i} \\ \qquad\qquad (r=1, 2, \cdots, m-2), \\[2mm] \dfrac{\partial \xi^{(m-1)\alpha}}{\partial t} = -\overline{H}^{\alpha} + u_i^{\alpha} H^i - \sum_{s=0}^{m-2} \dfrac{\partial \xi^{(m-1)\alpha}}{\partial x^{(s)i}} x^{(s+1)i}, \end{cases} \\[4mm] \text{(iii)} & \begin{cases} \partial u_j^{\alpha} / \partial x^{(r)k} = 0 \qquad (r=1, 2, \cdots, m-1), \\ \partial u_j^{\alpha} / \partial x^k = u_i^{\alpha} \Gamma^i_{jk} - u_j^{\beta} u_k^{\gamma} \overline{\Gamma}^{\alpha}_{\beta\gamma}, \\ \partial u_j^{\alpha} / \partial t = u_i^{\alpha} \Gamma^i_j - u_j^{\beta} \overline{\Gamma}^{\alpha}_{\beta} - (\partial u_j^{\alpha} / \partial x^k) x^{(1)k}. \end{cases} \end{cases}
$$

Conversely, if (17.6) admits a solution, (17.1) are equivalent under (17.2).

Using the relations

$$
(17.7)\quad \begin{cases} \text{(i)} & \dfrac{\partial \xi^{(r+1)\alpha}}{\partial x^i} = D_t \dfrac{\partial \xi^{(r)\alpha}}{\partial x^i} \qquad (r=0, 1, \cdots, m-2), \\[2mm] \text{(ii)} & \dfrac{\partial \xi^{(r+1)\alpha}}{\partial x^{(s)i}} = D_t \dfrac{\partial \xi^{(r)\alpha}}{\partial x^{(s)i}} + \dfrac{\partial \xi^{(r)\alpha}}{\partial x^{(s+1)i}} \qquad \begin{pmatrix} r=1, 2, \cdots, m-2 \\ s=1, 2, \cdots, r \end{pmatrix}, \\[2mm] \text{(iii)} & \dfrac{\partial \xi^{(r+1)\alpha}}{\partial x^{(r)i}} = (r+1) \dfrac{\partial \xi^{(1)\alpha}}{\partial x^i} \qquad (r=0, 1, \cdots, m-2), \end{cases}
$$

satisfied by a solution of (17.6), we can get the following conclusion in the same manner as in § 15.

A necessary and sufficient condition for the equivalence of the two systems of paths (17.1) *under the group* (17.2) *is expressible algebraically in terms of curvature and torsion tensors and their successive covariant derivatives.*

18. Generalized rheonomic geometry. We can also define the equivalence of two systems of paths defined by

$$
(18.1)\quad \begin{cases} \text{(i)} & x^{(m)i} + H^i(t, x, x^{(1)}, \cdots, x^{(m-1)}) = 0, \\ \text{(ii)} & \xi^{[m]\alpha} + \overline{H}^{\alpha}(\tau, \xi, \xi^{[1]}, \cdots, \xi^{[m-1]}) = 0 \end{cases}
$$

under the generalized rheonomic transformation group

$$
(18.2)\qquad \xi^{\alpha} = \xi^{\alpha}(t, x^i), \quad \tau = \tau(t).
$$

According to our method, we construct Γ, Γ'^{i}_{j}, $\Lambda^{(r)i}_{(s)j}$ and Γ'^{i}_{jk} and the corresponding functions from H^i and $\overline{H}^{\alpha}$. Then if two systems of paths are equivalent, there exist (3.2), (3.3), (4.9) and (5.2). Therefore we know that n^2+mn+2 functions

$$\tau\,,\ \xi^{\alpha},\ \xi^{[1]\alpha},\cdots,\xi^{[m-1]\alpha},\ d\tau/dt=1/\sigma\,,\ \partial\xi^{\alpha}/\partial x^{i}=u^{\alpha}_{i} \tag{18.3}$$

of $mn+1$ independent variables

$$t,\ x^{i},\ x^{(1)i},\cdots,x^{(m-)i} \tag{18.4}$$

satisfy the following differential equations:

$$\left\{\begin{array}{lll}
\text{(i)} & d\tau/dt=1/\sigma,\ \partial\tau/\partial x^{(s)j}=0 & (s=0,1,\cdots,m-1)\,, \\
\text{(ii)} & d\sigma/dt=\sigma\Gamma'-\overline{\Gamma'}\,,\ \partial\sigma/\partial x^{(s)j}=0 & (s=0,1,\cdots,m-1)\,, \\
\text{(iii)} & \left\{\begin{array}{l}
\dfrac{\partial\xi^{[r]\alpha}}{\partial x^{[r]j}}=\sigma^{r}u^{\alpha}_{j} \\
\dfrac{\partial\xi^{[r]\alpha}}{\partial x^{[s)j}}=0 \\
\dfrac{\partial\xi^{[r]\alpha}}{\partial x^{(s)j}}=\sigma^{r}u^{\alpha}_{i}\Lambda^{(r)i}_{(s)j}-\displaystyle\sum_{t=s}^{r-1}\frac{\partial\xi^{[t]\beta}}{\partial x^{(s)j}}\overline{\Lambda}^{[r]\alpha}_{[t]\beta}
\end{array}\right. & \begin{array}{l}
(r=0,1,\cdots,m-1)\,, \\
\begin{pmatrix} r=0,1,\cdots,m-2 \\ s=r+1,r+2,\cdots,m-1\end{pmatrix}, \\
\begin{pmatrix} r=1,2,\cdots,m-1 \\ s=0,1,\cdots,\ r-1\end{pmatrix},
\end{array} \\
\text{(iv)} & \left\{\begin{array}{l}
\dfrac{\partial\xi^{[r]\alpha}}{\partial t}=\dfrac{1}{\sigma}\,\xi^{[r+1]\alpha}-\sigma^{r}u^{\alpha}_{i}x^{(r+1)i}-\displaystyle\sum_{s=0}^{r-1}\frac{\partial\xi^{[r]\alpha}}{\partial x^{(s)i}}\,x^{(s+1)i} \\
\dfrac{\partial\xi^{[m-1]\alpha}}{\partial t}=-\dfrac{1}{\sigma}\,\overline{H}^{\alpha}+\sigma^{m-1}u^{\alpha}_{i}\,H^{i}-\displaystyle\sum_{s=0}^{m-2}\frac{\partial\xi^{[m-1]\alpha}}{\partial x^{(s)i}}\,x^{(s+1)i}\,,
\end{array}\right. & (r=0,1,\cdots,m-2)\,, \\
\text{(v)} & \left\{\begin{array}{l}
\partial u^{\alpha}_{j}/\partial x^{(r)k}=0 \\
\partial u^{\alpha}_{j}/\partial x^{k}=u^{\alpha}_{i}\,\Gamma'^{i}_{jk}-u^{\beta}_{j}\,u^{\gamma}_{k}\,\overline{\Gamma'}^{\alpha}_{\beta\gamma}\,, \\
\partial u^{\alpha}_{j}/\partial t=u^{\alpha}_{i}\Gamma'^{i}_{j}-(1/\sigma)\,u^{\beta}_{j}\overline{\Gamma'}^{\alpha}_{\beta}-(\partial u^{\alpha}_{j}/\partial x^{k})\,x^{(1)k}\,.
\end{array}\right. & (r=1,2,\cdots,m-1)\,,
\end{array}\right. \tag{18.5}$$

Conversely, if (18.5) admits a solution, we can conclude that (18.1) are equivalent.

By using the relations

$$\left\{\begin{array}{lll}
\text{(i)} & \dfrac{\partial\xi^{[r+1]\alpha}}{\partial x^{j}}=\sigma D_{t}\dfrac{\partial\xi^{[r]\alpha}}{\partial x^{j}} & (r=0,1,\cdots,m-2)\,, \\
\text{(ii)} & \dfrac{\partial\xi^{[r+1]\alpha}}{\partial x^{(s)j}}=\sigma\left\{D_{t}\,\dfrac{\partial\xi^{[r]\alpha}}{\partial x^{(s)j}}+\dfrac{\partial\xi^{[r]\alpha}}{\partial x^{(s-1)j}}\right\} & \begin{pmatrix} r=1,2,\cdots,m-2 \\ s=1,2,\cdots,r \end{pmatrix},
\end{array}\right. \tag{18.6}$$

$$\left\{ \text{(iii)} \quad \frac{\partial \xi^{[r+1]\alpha}}{\partial x^{(r)j}} = (r+1)\,\sigma^{r+1} D_t \frac{\partial \xi^\alpha}{\partial x^j} + \frac{r(r+1)}{2}\,\sigma^r (\sigma \Gamma - \bar{\Gamma})\, u^\alpha_j \right.$$
$$(r=0, 1, \cdots, m-2),$$

which hold for a solution of (18.5), we get a conclusion analogous to that of § 17.

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