

A PARAMETRIZED DE RHAM DECOMPOSITION THEOREM FOR CR MANIFOLDS

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Abstract. The CR equivalence problem between CR manifolds with slice structure is studied. Let N be a connected holomorphically nondegenerate real analytic hypersurface and $M(p)$ a finitely nondegenerate real analytic hypersurface parametrized by $p \in N$. Let \tilde{M} be a totality of N and $M(p)$ with moving p in N . Assume that M and \tilde{M} (with a same structure as M) are CR equivalent and that N and \tilde{N} are also CR equivalent. Then we prove that, for any $p \in N$, there exists $\tilde{p} \in \tilde{N}$ such that $M(p)$ is CR equivalent to $\tilde{M}(\tilde{p})$.

1. Introduction. In 1952, de Rham proved the decomposition theorem of Riemannian manifolds, which states that a simply connected and complete Riemannian manifold M is isometric to the direct product $M_0 \times M_1 \times \cdots \times M_k$, where M_0 is a Euclidean space (possibly of dimension 0) and M_1, \dots, M_k are all simply connected, complete, irreducible Riemannian manifolds. Such a decomposition is unique up to ordering.

The purpose of this article is to study a parameter version of this theorem in CR geometry.

We denote $z' = (z_1, \dots, z_m) \in \mathbb{C}^m$, $z = (z', z_{m+1})$ and $w' = (w_1, \dots, w_n) \in \mathbb{C}^n$, $w = (w', w_{n+1})$. Now we shall define a CR manifold with slices $M(p)$ over a base space N . Let $N \subset \mathbb{C}^{n+1}$ be a real analytic hypersurface. Then, by the implicit function theorem, we may assume that it is defined by the equation

$$(1) \quad w_{n+1} = Q^2(w', \bar{w}', \bar{w}_{n+1})$$

in the normal coordinate for a real analytic function Q^2 . Take $w \in \mathbb{C}^{n+1}$ and let $M(w) \subset \mathbb{C}^{m+1}$ be a real analytic hypersurface with real analytic parameters w and \bar{w} . Then by the coordinate change which is biholomorphic in z and is real analytic in w and \bar{w} , we may assume that its defining function is of the form

$$(2) \quad z_{m+1} = Q^1(z', \bar{z}', \bar{z}_{m+1}; w, \bar{w})$$

in the normal coordinate in z and \bar{z} , where Q^1 is a real analytic function.

DEFINITION 1.1. Let $M \subset \mathbb{C}^{m+n+2}$ be a CR manifold. We say that M is a CR manifold with slices $M(p) \subset \mathbb{C}^{m+1}$ over a base space $N \subset \mathbb{C}^{n+1}$ for $p \in N$ if it is constructed by

$$(3) \quad M = \bigcup_{p \in N} (M(p) \times \{p\}).$$

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Here N is defined by the equation (1) and $M(p)$ is by (2) with $w = p \in N$.

We will consider the case where there exists a point $q \in \mathbb{C}^{m+1}$ such that, for any $p \in N$, the point q is contained in $M(p)$. In this case, we have $M = \bigcup_{p \in N} (M(p) \times \{p\}) \supset \bigcup_{p \in N} (\{q\} \times \{p\}) = \{q\} \times N$.

In this paper, we assume that all objects are in C^ω smooth. Let A be a certain object defined on the source space. Then we denote by \tilde{A} a corresponding object on the target space with the same properties as A .

The main theorem of this article is the following. The definition of terms in the statement are given in Section 2.

THEOREM 1.2. *Let $M \subset \mathbb{C}^{m+n+2}$ be a CR manifold with slices $M(p)$ over a base space N , where $N \subset \mathbb{C}^{n+1}$ is a holomorphically nondegenerate connected real hypersurface defined by (1) and $M(p) \subset \mathbb{C}^{m+1}$ is a real hypersurface defined by (2) with C^ω parametrization by $p \in N$. Assume that there exists $q \in \mathbb{C}^{m+1}$ such that, for any $p \in N$, q is contained in $M(p)$ and $M(p)$ is finitely nondegenerate at q . Assume M (resp. N) is CR equivalent to \tilde{M} (resp. \tilde{N}) by a CR diffeomorphism H (resp. $H|_{\{q\} \times N}$). Then, for any $p \in N$, there exists $\tilde{p} \in \tilde{N}$ such that $M(p)$ is CR equivalent to $\tilde{M}(\tilde{p})$.*

There are two comments.

1. This theorem claims that if the total manifold consists of a parameter space and slices on it and if such two total manifolds are CR equivalent and parameter spaces are also CR equivalent, then we can conclude that a slice of a manifold is mapped CR equivalently into a slice of the other.
2. Since we work under the hypothesis of a holomorphically nondegeneracy and use the theorem of identity in Lemmas 3.4 and 3.6, the regularity of real analyticity can not be weakened to be infinitely differentiable.

If all slices do not depend on $p \in N$, the total manifold M is just the direct product of the base space and a slice. We have already proved the case in [9].

THEOREM 1.3. *Let (M, p) be a germ of a smooth CR-manifold which is finitely nondegenerate and of finite type on a dense subset. Then, up to permutations, there exists a unique decomposition*

$$(4) \quad (M, p) \cong (M_1, p_1) \times \cdots \times (M_m, p_m),$$

where each germ (M_j, p_j) is irreducible. Furthermore, if f is (the germ of) a smooth local CR-diffeomorphism between (M, p) and (the germ of) another smooth CR-manifold (M', p') and if $(M', p') \cong (M'_1, p'_1) \times \cdots \times (M'_{m'}, p'_{m'})$ is the corresponding decomposition into irreducible factors, then $m = m'$ and, after a permutation of the factors (M'_j, p'_j) , f is factorized as a direct product of the form $f = f^1 \times \cdots \times f^m$, where $f^j: (M_j, p_j) \rightarrow (M'_j, p'_j)$ are (germs of) local CR-diffeomorphisms for $j = 1, \dots, m$.

A parameter version of this decomposition theorem was also obtained by the author. The worm hypersurface W_h is defined by

$$(5) \quad W_h = \{(z, w) \in \mathbb{C}^2 : |z - \exp(ih(w, \bar{w}))|^2 - 1 = 0, h \text{ is a harmonic function}\}.$$

This is parametrized by a harmonic function $h(w, \bar{w})$ and is of infinite type along $\{z = 0\}$ in the sense of Bloom and Graham [4]. When we fix $w = w_0$, a part of a circle centered at $\exp(ih(w_0, \bar{w}_0))$ and radius one, which we denote by $W_h(w_0)$, appears. $W_h(w_0)$ goes around along complex variety $\{z = 0\}$ as w_0 moves. Therefore the hypersurface W_h consists of a base space $\{z = 0\}$ and slices $W_h(w)$. About a CR equivalence problem for such hypersurfaces, the author proved the following theorem in [8].

THEOREM 1.4. *Let $(f, g) : W_h \rightarrow W_{h'}$ be a real analytic CR equivalence mapping near the origin between worm hypersurfaces parametrized by harmonic functions h and h' . Then f and g are expanded in a standard coordinate in \mathbb{C}^2 as*

$$(6) \quad f(z, w) = z, \quad g(z, w) = \sum_{q \geq 1} b_q w^q.$$

Since $\{z = 0\}$ is mapped to $\{z = 0\}$ and $W_h(w)$ to $W_{h'}(g(w))$, this theorem is regarded as a parameter version of [9].

This condition in Theorem 1.2 is called “slice preserving” or “leaf preserving” and they appear in other subjects. For example, the Rudin theorem and the Baracco-Zampieri theorem in complex analysis, which we mention in Section 4, are given with such condition. This situation also appears in differential geometry and the theory of differential equation. Let \mathcal{F} be a C^∞ -foliation on a compact C^∞ -manifold. Fukui [7] studied the group of all leaf preserving C^∞ -diffeomorphisms of (M, \mathcal{F}) which are isotopic to the identity through leaf preserving C^∞ -diffeomorphisms. He obtained the necessary and sufficient condition for such a group to be simple. As a problem of differential equation, the group-invariant solutions have been studied. We say that the group G acts projectably on the open set where independent and dependent variables belong if the action of G is of the form $(\tilde{u}, \tilde{x}) = (\varphi(x, u), \phi(x))$. In view of Idea 3.1, this is the form that we want to prove. Construct a system of partial differential equations whose solutions are the complexified defining functions of M in Theorem 1.2. Then a CR automorphism group of the manifold under consideration is a totally real part of the symmetry group of the system. Therefore by comparing this case with ours, Theorem 1.2 shows that the action of the symmetry group of such a system is projectable.

Here is an outline of this paper. In Section 2, we give some definition in the general situation and specify them for our setting in Subsection 3.1. In Subsection 3.2, we prove the main theorem along the line on Idea 3.1. Finally in Section 4, we pose two questions related to “slice preserving” and “level set preserving” situations.

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2. Nondegeneracy conditions and Segre mappings. Let M be a real analytic CR submanifold of codimension d in \mathbb{C}^N . We take a coordinate $Z = (z, w)$ in \mathbb{C}^N near the origin with $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_d)$, $n + d = N$. Let $\rho = (\rho_1, \dots, \rho_d)$ be a defining function for M . By implicit function theorem, we may assume that M is defined by the equations $w_j = Q_j(z, \bar{z}, \bar{w})$, $j = 1, \dots, d$.

2.1. Nondegeneracy conditions. In this subsection, we give definitions of nondegeneracy conditions in the general setting. See [1] for more detail.

By a holomorphic vector field, we mean a $(1, 0)$ -type vector field X of the form

$$(7) \quad X = \sum_{j=1}^N a_j(Z) \frac{\partial}{\partial Z_j}$$

with holomorphic functions a_j .

DEFINITION 2.1. A real analytic CR manifold M is holomorphically nondegenerate at p if there is no holomorphic vector field X tangent to M near p such that $X|_M \neq 0$.

Note that if M is connected and is holomorphically nondegenerate at a point p , then it is holomorphically nondegenerate at every point. So we shall omit the reference point.

Let $\{L_1, \dots, L_n\}$ be a local basis for the CR vector fields of type $(0, 1)$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, we use the notation

$$(8) \quad L^\alpha = L_1^{\alpha_1} \cdots L_n^{\alpha_n}.$$

We also use the notation

$$(9) \quad \frac{\partial \rho_j}{\partial Z} = \left(\frac{\partial \rho_j}{\partial Z_1}, \dots, \frac{\partial \rho_j}{\partial Z_N} \right).$$

DEFINITION 2.2. M is k -nondegenerate at p if

$$(10) \quad \text{span} \left\{ L^\alpha \left(\frac{\partial \rho_j}{\partial Z} \right) (p, \bar{p}) : j \in \{1, \dots, d\}, |\alpha| \leq k \right\} = \mathbb{C}^N.$$

If there exists such a finite integer k , M is said to be finitely nondegenerate at p .

In the hypersurface case, we have the following relations.

FACT 2.3. Let M be a real analytic connected hypersurface. Consider the following three conditions.

- (a) M is holomorphically nondegenerate.
- (b) There exist $p \in M$ and $k > 0$ such that M is k -nondegenerate at p .
- (c) M is of finite type at p (for definition, see [4]).

Then (a) is equivalent of (b), and (b) implies (c).

2.2. Segre mapping. In this subsection, we define a Segre mapping. The reader is referred to [2], [3] and [6] for the definition and properties of Segre mappings.

Let $M, \tilde{M} \subset \mathbb{C}^N$ be real analytic CR submanifolds, and $H : M \rightarrow \tilde{M}$ a real analytic CR mapping. We write $H = (F, G)$ for the coordinate $(\tilde{z}, \tilde{w}) \in \mathbb{C}^n \times \mathbb{C}^d$. Since the mapping

(F, G) sends M into \tilde{M} , they satisfy

$$(11) \quad G(z, w) = \tilde{Q}(F(z, w), \bar{F}(\chi, \tau), \bar{G}(\chi, \tau))$$

and its complex conjugate

$$(12) \quad \bar{G}(\chi, \tau) = \overline{\tilde{Q}}(\bar{F}(\chi, \tau), F(z, w), G(z, w)),$$

for $(z, w, \chi, \tau) \in \mathbb{C}^N \times \mathbb{C}^N$ with

$$(13) \quad w = Q(z, \chi, \tau).$$

For a positive integer k , we define a mapping $u^k : \mathbb{C}^{kn} \rightarrow \mathbb{C}^d$ inductively by

$$(14) \quad u^1(t^1) = 0, \quad u^k(t^1, \dots, t^k) = Q(t^k, t^{k-1}, \overline{u^{k-1}}(t^1, \dots, t^{k-1})), \quad k \geq 2,$$

for $t^j = (t_1^j, \dots, t_n^j) \in \mathbb{C}^n$. For simplicity, we use the notation $t^{[k]} = (t^1, \dots, t^k) \in \mathbb{C}^{kn}$.

DEFINITION 2.4. The k th Segre mapping of M at 0 is the mapping $v^k : \mathbb{C}^{kn} \rightarrow \mathbb{C}^N$ defined by

$$(15) \quad \mathbb{C}^{kn} \ni t^{[k]} \mapsto v^k(t^{[k]}) = (t^k, u^k(t^{[k]})) \in \mathbb{C}^N.$$

This mapping has a close relation to the finite type condition.

FACT 2.5. *Let M be of finite type in the sense of Bloom-Graham and v^k its Segre mapping. Then, for a sufficiently large k , v^k has a full rank. Therefore, together with Fact 2.3, the image under the Segre mapping v^k of the finitely nondegenerate hypersurface contains an open subset of \mathbb{C}^N .*

For example, if $k \geq 3$ is odd, we have

$$(16) \quad \begin{aligned} u^k(t^{[k]}) &= Q(t^k, t^{k-1}, \overline{Q}(t^{k-1}, t^{k-2}, Q(t^{k-2}, t^{k-3}, \dots, \overline{Q}(t^2, t^1, 0), \dots))) \\ &= Q(t^k, \overline{v^{k-1}}(t^{[k-1]})). \end{aligned}$$

By the definition of Segre mapping, (11) and (12) hold for

$$(17) \quad (z, w; \chi, \tau) = (v^k(t^{[k]}); \overline{v^{k-1}}(t^{[k-1]}))$$

and

$$(18) \quad (z, w; \chi, \tau) = (v^{k-2}(t^{[k-2]}); \overline{v^{k-1}}(t^{[k-1]}))$$

for any $k \geq 2$. Substituting (17) into (11), and (18) into (12), we have identities

$$(19) \quad G \circ v^k = \tilde{Q}(F \circ v^k, \overline{F \circ v^{k-1}}, \overline{G \circ v^{k-1}}),$$

$$(20) \quad \overline{G \circ v^{k-1}} = \overline{\tilde{Q}}(\overline{F \circ v^{k-1}}, F \circ v^{k-2}, G \circ v^{k-2}).$$

Then substitute (20) into (19). Continuing this process for $l = k-2, k-3, \dots, 1$, we obtain, for any $k \geq 1$ odd,

$$(21) \quad \begin{aligned} G \circ v^k &= \tilde{Q}(F \circ v^k, \overline{F \circ v^{k-1}}, \overline{\tilde{Q}}(\overline{F \circ v^{k-1}}, F \circ v^{k-2}, \\ &\quad \overline{\tilde{Q}}(\dots, \overline{\tilde{Q}}(F \circ v^2, F \circ v^1, G \circ v^1), \dots))), \end{aligned}$$

and for $k \geq 2$ even,

$$(22) \quad \begin{aligned} G \circ v^k &= \widetilde{Q}(F \circ v^k, \overline{F \circ v^{k-1}}, \widetilde{Q}(\overline{F \circ v^{k-1}}, F \circ v^{k-2}, \\ &\quad \widetilde{Q}(\dots, \widetilde{Q}(F \circ v^2, \overline{F \circ v^1}, \overline{G \circ v^1}), \dots)), \end{aligned}$$

where $F \circ v^j = (F \circ v^j)(t^{[j]})$ and $\overline{F \circ v^j} = \overline{(F \circ v^j)(t^{[j]})}$. We shall omit the variables of v^k if there is no confusion.

3. Proof of Theorem 1.2. Let H be a CR diffeomorphism between M and \widetilde{M} as in Theorem 1.2, and decompose it as $H(z, w) = (F_1, \dots, F_{m+1}, G_1, \dots, G_{n+1})(z, w)$ corresponding to the coordinate for \widetilde{M} . The following is the main idea to prove the theorem.

IDEA 3.1. If we can prove that G_1, \dots, G_{n+1} are independent of the variables z_1, \dots, z_{m+1} , then Theorem 1.2 is true.

Substitute the defining function (1) of N into (2) of $M(w)$ for $w \in N$. Then we can eliminate w_{n+1} from (2) and we may assume that the defining function of $M(w)$ is of the form:

$$(23) \quad z_{m+1} = Q^1(z', \bar{z}', \bar{z}_{m+1}; w', \bar{w}', \bar{w}_{n+1}).$$

3.1. Segre mappings in our setting. First, we fix the notation for our purpose. We correspond the variable t^k (resp. s^k) of the Segre mapping to $z' = (z_1, \dots, z_m)$ (resp. $w' = (w_1, \dots, w_n)$). Since N is holomorphically nondegenerate, there exists a point where N is finitely nondegenerate (cf. Fact 2.3). We take the origin as the reference point.

We shall denote by $v_N^k, v_{M(p)}^k$ and V^k the k th Segre mappings of $N, M(p)$ and M , respectively which are defined inductively as follows:

$$(24) \quad v_N^1(s^1) = (s^1, 0),$$

$$(25) \quad v_N^k(s^{[k]}) = (s^k, Q^2(s^k, \overline{v_N^{k-1}(s^{[k-1]})})),$$

$$(26) \quad v_{M(p)}^1(t^1, s^1) = (t^1, 0),$$

$$(27) \quad v_{M(p)}^k(t^{[k]}, s^{[k]}) = (t^k, Q^1(t^k, \overline{v_{M(p)}^{k-1}(t^{[k-1]}, s^{[k-1]})}; s^k, \overline{v_N^{k-1}(s^{[k-1]})})),$$

$$(28) \quad V^k(t^{[k]}, s^{[k]}) = (v_{M(p)}^k(t^{[k]}, s^{[k]}), v_N^k(s^{[k]})).$$

Now we need a remark on this notation. Since $v_{M(p)}^k$ depends on $p \in N$, we should write $v_{M(p)}^k(t^{[k]}, s_p^{[k]})$ instead of $v_{M(p)}^k(t^{[k]}, s^{[k]})$. But since the point $p \in N$ is arbitrary, we may omit p from $s_p^{[k]}$. This may not give any confusion. Since M is finitely nondegenerate, the image under V^k contains an open subset of \mathbb{C}^{n+m+2} .

First we write down the equation (21) or (22) for $(G_1, \dots, G_{n+1}) : M \rightarrow \widetilde{N}$ as

$$(29) \quad \begin{aligned} G_{n+1} \circ V^k &= \widetilde{Q}^2(G' \circ V^k, \overline{G' \circ V^{k-1}}, \widetilde{Q}^2(\overline{G' \circ V^{k-1}}, G' \circ V^{k-2}, \\ &\quad \widetilde{Q}^2(G' \circ V^{k-2}, \overline{G' \circ V^{k-3}}, \widetilde{Q}^2(\dots))). \end{aligned}$$

for $G' = (G_1, \dots, G_n)$. We restrict this equation to $t^{k-3} = t^{k-1}$ and $s^{k-3} = s^{k-1}$. Then the k th and the $(k-1)$ st restricted Segre mappings, on which we add a $\widehat{}$, are written as

$$(30) \quad \overline{\widehat{v}_{M(p)}^{k-1}} = (t^{k-1}, \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4})),$$

$$(31) \quad \overline{\widehat{v}_N^{k-1}} = (s^{k-1}, \overline{Q^2}(s^{k-1}, v_N^{k-4})),$$

$$(32) \quad \widehat{v}_{M(p)}^k = (t^k, Q^1(t^k, t^{k-1}, \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4}); s^k, \overline{\widehat{v}_N^{k-1}})),$$

$$(33) \quad \widehat{v}_N^k = (s^k, Q^2(s^k, s^{k-1}, \overline{Q^2}(s^{k-1}, v_N^{k-4}))).$$

Therefore, the equation (29) restricted to $t^{k-3} = t^{k-1}$ and $s^{k-3} = s^{k-1}$ is written as

$$(34) \quad G_{n+1} \circ \widehat{V}^k = \widetilde{Q^2}(G' \circ \widehat{V}^k, \overline{G' \circ \widehat{V}^{k-1}}, \overline{\widetilde{Q^2}(G' \circ \widehat{V}^{k-1}}, G' \circ V^{k-4}}, \overline{\widetilde{Q^2}(G' \circ V^{k-4}}, \overline{G' \circ V^{k-5}}, \overline{\widetilde{Q^2}(\dots)}})).$$

We introduce the variables T and S as

$$(35) \quad T = (G' \circ \widehat{V}^k, \overline{G' \circ \widehat{V}^{k-1}}, \overline{\widetilde{Q^2}(G' \circ \widehat{V}^{k-1}}, G' \circ V^{k-4}}, \overline{\widetilde{Q^2}(G' \circ V^{k-4}}, \overline{G' \circ V^{k-5}}, \overline{\widetilde{Q^2}(\dots)}})),$$

$$(36) \quad S = (\overline{G' \circ \widehat{V}^{k-1}}, G' \circ V^{k-4}, \widetilde{Q^2}(G' \circ V^{k-4}, \overline{G' \circ V^{k-5}}, \overline{\widetilde{Q^2}(\dots)})).$$

Take any $j = 1, \dots, m$ and fix it. Some indices which appear later may depend on j , but we shall not refer it. Differentiate (34) with respect to t_j^k and t_j^{k-1} . Then we obtain the identities

$$(37) \quad \begin{aligned} & \frac{\partial G_{n+1}}{\partial z_j} \circ \widehat{V}^k + \frac{\partial G_{n+1}}{\partial z_{m+1}} \circ \widehat{V}^k \\ & \quad \times \frac{\partial Q^1(t^k, t^{k-1}, \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4}); s^k, \overline{\widehat{v}_N^{k-1}})}{\partial t_j^k} \\ & = \sum_{l=1}^n \frac{\partial \widetilde{Q^2}}{\partial w_l}(T) \left\{ \frac{\partial G_l}{\partial z_j} \circ \widehat{V}^k + \frac{\partial G_l}{\partial z_{m+1}} \circ \widehat{V}^k \right. \\ & \quad \times \left. \frac{\partial Q^1(t^k, t^{k-1}, \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4}); s^k, \overline{\widehat{v}_N^{k-1}})}{\partial t_j^k} \right\}, \end{aligned}$$

$$\begin{aligned}
(38) \quad & \frac{\partial G_{n+1}}{\partial z_{m+1}} \circ \widehat{V}^k \frac{\partial Q^1(t^k, t^{k-1}, \overline{Q}^1(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4}); s^k, \overline{\widehat{v}_N^{k-1}})}{\partial t_j^{k-1}} \\
&= \sum_{l=1}^n \frac{\partial \widetilde{Q}^2}{\partial w_l}(T) \frac{\partial G_l}{\partial z_{m+1}} \circ \widehat{V}^k \\
&\quad \times \frac{\partial Q^1(t^k, t^{k-1}, \overline{Q}^1(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4}); s^k, \overline{\widehat{v}_N^{k-1}})}{\partial t_j^{k-1}} \\
&\quad + \sum_{l=1}^n \frac{\partial \overline{G_l \circ \widehat{V}^{k-1}}}{\partial t_j^{k-1}} \left\{ \frac{\partial \widetilde{Q}^2}{\partial \bar{w}_l}(T) + \frac{\partial \widetilde{Q}^2}{\partial \bar{w}_{n+1}}(T) \frac{\partial \widetilde{Q}^2}{\partial \bar{w}_l}(S) \right\}.
\end{aligned}$$

3.2. Independency of z . In this subsection, we shall prove that $G_1 \circ \widehat{V}^k(t^{[k]}, s^{[k]}, \dots, G_{n+1} \circ \widehat{V}^k(t^{[k]}, s^{[k]}))$ are independent of t^k , which will lead to the independency of z . Restrict the identity (38) to the variety

$$\begin{aligned}
(39) \quad & W_j = \left\{ (t^k, t^{k-1}, t^{[k-4]}, s^k, s^{k-1}, s^{[k-4]}) : \right. \\
& \left. \frac{\partial Q^1(t^k, t^{k-1}, \overline{Q}^1(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4}); s^k, \overline{\widehat{v}_N^{k-1}})}{\partial t_j^{k-1}} = 0 \right\}.
\end{aligned}$$

Then it becomes

$$(40) \quad \sum_{l=1}^n \frac{\partial \overline{G_l \circ \widehat{V}^{k-1}}}{\partial t_j^{k-1}} \left\{ \frac{\partial \widetilde{Q}^2}{\partial \bar{w}_l}(T) + \frac{\partial \widetilde{Q}^2}{\partial \bar{w}_{n+1}}(T) \frac{\partial \widetilde{Q}^2}{\partial \bar{w}_l}(S) \right\} = 0.$$

There exists j_0 depending on j such that

$$(41) \quad \frac{\partial^2 Q^1(t^k, t^{k-1}, \overline{Q}^1(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4}); s^k, \overline{\widehat{v}_N^{k-1}})}{\partial t_{j_0}^k \partial t_j^{k-1}} \neq 0$$

near the points where all variables are zero except for t^{k-1} . We may assume that $j_0 = 1$. We define vector fields $X^i, i = 1, \dots, n$, on W_j as

$$(42) \quad X^i = \frac{\partial^2 Q^1}{\partial t_1^k \partial t_j^{k-1}} \frac{\partial}{\partial s_i^k} - \frac{\partial^2 Q^1}{\partial s_i^k \partial t_j^{k-1}} \frac{\partial}{\partial t_1^k}.$$

We introduce the following notation.

$$(43) \quad A_{\alpha_1, \dots, \alpha_\lambda}^{i_1, \dots, i_\lambda} = X^{i_1} G_{\alpha_1} \circ \widehat{V}^k \times \dots \times X^{i_\lambda} G_{\alpha_\lambda} \circ \widehat{V}^k,$$

$$(44) \quad q_l = \frac{\partial \widetilde{Q}^2}{\partial \bar{w}_l}(T) + \frac{\partial \widetilde{Q}^2}{\partial \bar{w}_{n+1}}(T) \frac{\partial \widetilde{Q}^2}{\partial \bar{w}_l}(S),$$

$$(45) \quad q_{l;\alpha_1,\dots,\alpha_\lambda} = \frac{\partial^{\lambda+1} \widetilde{Q}^2}{\partial \bar{w}_l \partial w_{\alpha_1} \cdots \partial w_{\alpha_\lambda}}(T) + \frac{\partial^{\lambda+1} \widetilde{Q}^2}{\partial \bar{w}_{n+1} \partial w_{\alpha_1} \cdots \partial w_{\alpha_\lambda}}(T) \frac{\partial \overline{\widetilde{Q}^2}}{\partial \bar{w}_l}(S).$$

Let $A(\lambda)$ be an $n^\lambda \times n^\lambda$ matrix with element $A_{\alpha_1,\dots,\alpha_\lambda}^{i_1,\dots,i_\lambda}$ as the $((i_1 - 1)n^{\lambda-1} + (i_2 - 1)n^{\lambda-2} + \cdots + (i_{\lambda-1} - 1)n + i_\lambda, (\alpha_1 - 1)n^{\lambda-1} + (\alpha_2 - 1)n^{\lambda-2} + \cdots + (\alpha_{\lambda-1} - 1)n + \alpha_\lambda)$ -th component for $i_1, \dots, i_\lambda, \alpha_1, \dots, \alpha_\lambda = 1, \dots, n$. If we apply X^1, \dots, X^n to the vector (q_1, \dots, q_n) K times, then we have n^K vectors. We order them in the following way. The $((a_1 - 1)n^{K-1} + (a_2 - 1)n^{K-2} + \cdots + (a_{K-1} - 1)n + a_K)$ -th vector is $(X^{a_1} \cdots X^{a_K} q_1, \dots, X^{a_1} \cdots X^{a_K} q_n)$. We order the n^λ vectors $\{(q_{1;\alpha_1,\dots,\alpha_\lambda}, \dots, q_{n;\alpha_1,\dots,\alpha_\lambda})\}_{\alpha_1,\dots,\alpha_\lambda}$ for $\alpha_1, \dots, \alpha_\lambda = 1, \dots, n$ in the same way. Then we obtain the matrix equation below by applying X^1, \dots, X^n to (q_1, \dots, q_n) L times for sufficiently large L . The first matrix in the right-hand side is a lower triangle matrix whose diagonals are the matrices defined above.

$$(46) \quad \begin{pmatrix} X^1 q_1 & \cdots & X^1 q_n \\ \vdots & \ddots & \vdots \\ X^n q_1 & \cdots & X^n q_n \\ X^1 X^1 q_1 & \cdots & X^1 X^1 q_n \\ \vdots & \ddots & \vdots \\ X^n X^n q_1 & \cdots & X^n X^n q_n \\ \vdots & \ddots & \vdots \\ X^1 \cdots X^1 q_1 & \cdots & X^1 \cdots X^1 q_n \\ \vdots & \ddots & \vdots \\ X^n \cdots X^n q_1 & \cdots & X^n \cdots X^n q_n \end{pmatrix} = \begin{pmatrix} A(1) & & 0 \\ & A(2) & \\ * & & \ddots \\ & & & A(L) \end{pmatrix} \begin{pmatrix} q_{1;1} & \cdots & q_{n;1} \\ \vdots & \ddots & \vdots \\ q_{1;n} & \cdots & q_{n;n} \\ q_{1;11} & \cdots & q_{n;11} \\ \vdots & \ddots & \vdots \\ q_{1;nn} & \cdots & q_{n;nn} \\ \vdots & \ddots & \vdots \\ q_{1;1\dots 1} & \cdots & q_{n;1\dots 1} \\ \vdots & \ddots & \vdots \\ q_{1;n\dots n} & \cdots & q_{n;n\dots n} \end{pmatrix}.$$

We denote by (Xq) the matrix in the left-hand side and by (A) and by (q) the first and the second matrices in the right-hand side. It follows from $\det(A) = \det A(1) \times \cdots \times \det A(L)$ and $\det A(\lambda + 1) = (\det A(1))^{n^\lambda} (\det A(\lambda))^n$ that $\det(A) \neq 0$ is equivalent to $\det A(1) \neq 0$.

LEMMA 3.2.

$$(47) \quad \det A(1) = \det \begin{pmatrix} X^1 G_1 \circ \widehat{V}^k & \dots & X^1 G_n \circ \widehat{V}^k \\ \vdots & \ddots & \vdots \\ X^n G_1 \circ \widehat{V}^k & \dots & X^n G_n \circ \widehat{V}^k \end{pmatrix} \neq 0$$

near the origin.

PROOF. Evaluate the matrix (47) at the points where all variables are zero except for t^{k-1} . Then it becomes

$$(48) \quad \frac{\partial^2 Q^1}{\partial t_j^{k-1} \partial t_1^k} \begin{pmatrix} \frac{\partial G_1}{\partial w_1}(0) & \dots & \frac{\partial G_n}{\partial w_1}(0) \\ \vdots & \ddots & \vdots \\ \frac{\partial G_1}{\partial w_n}(0) & \dots & \frac{\partial G_n}{\partial w_n}(0) \end{pmatrix}.$$

Since $G|_N$ is a CR diffeomorphism between N and \widetilde{N} near the origin, and since we have (41), the determinant of (48) does not vanish identically. \square

We shall claim that $\text{rank}(Xq) = n$. Obviously, we have $\text{rank}(Xq) \leq n$. Next, we multiply the inverse matrix $(A)^{-1}$ to the both sides of (46), then we get

$$(49) \quad n = \text{rank}(q) = \text{rank}(A)^{-1}(Xq) \leq \text{rank}(Xq).$$

The first equality comes from the finite nondegeneracy of N at the origin. Therefore, we conclude that $\text{rank}(Xq) = n$.

LEMMA 3.3. For any k , $\overline{G_l \circ \widehat{V}^{k-1}}$ is independent of t_j^{k-1} on W_j for $l = 1, \dots, n$.

PROOF. By applying X^1, \dots, X^n to (40) step by step, we get

$$(50) \quad \sum_{l=1}^n \frac{\partial \overline{G_l \circ \widehat{V}^{k-1}}}{\partial t_j^{k-1}} X^{i_1} q_l = 0,$$

$$\vdots$$

$$(51) \quad \sum_{l=1}^n \frac{\partial \overline{G_l \circ \widehat{V}^{k-1}}}{\partial t_j^{k-1}} X^{i_1} \dots X^{i_\lambda} q_l = 0$$

for sufficiently large λ . Write down these in a matrix equation as

$$(52) \quad \begin{pmatrix} X^{i_1} q_1 & \dots & X^{i_1} q_n \\ X^{i_1} X^{i_2} q_1 & \dots & X^{i_1} X^{i_2} q_n \\ \vdots & \ddots & \vdots \\ X^{i_1} \dots X^{i_\lambda} q_1 & \dots & X^{i_1} \dots X^{i_\lambda} q_n \end{pmatrix} \begin{pmatrix} \overline{\frac{\partial G_1 \circ \widehat{V}^{k-1}}{\partial t_j^{k-1}}} \\ \vdots \\ \overline{\frac{\partial G_n \circ \widehat{V}^{k-1}}{\partial t_j^{k-1}}} \end{pmatrix} = 0.$$

The rank condition of (Xq) implies that any $\overline{G_l \circ \widehat{V}^{k-1}}$ is independent of t_j^{k-1} on W_j for any fixed j . \square

LEMMA 3.4. G_1, \dots, G_n are independent of z_1, \dots, z_{m+1} .

PROOF. By Lemma 3.3, we have

$$(53) \quad \frac{\partial \overline{G_l \circ \widehat{V}^{k-1}}}{\partial t_j^{k-1}} = \frac{\partial \overline{G_l}}{\partial \bar{z}_j} \circ \widehat{V}^{k-1} + \frac{\partial \overline{G_l}}{\partial \bar{z}_{n+1}} \circ \widehat{V}^{k-1} \frac{\partial \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4})}{\partial t_j^{k-1}} = 0$$

on W_j . By restricting this equality to the variety

$$(54) \quad Z_j = W_j \cap \left\{ \frac{\partial \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4})}{\partial t_j^{k-1}} = 0 \right\},$$

we get

$$(55) \quad \frac{\partial \overline{G_l}}{\partial \bar{z}_j} \circ \widehat{V}^{k-1} = 0$$

on Z_j . The following claim shows that $\widehat{V}^{k-1}(Z_j)$ contains an open set, which implies that G_l does not depend on z_j .

CLAIM 3.5. *The gradients by $t^{[k]}$ and $s^{[k]}$ of defining functions of Z_j and each component of \widehat{V}^{k-1} are linearly independent.*

To show the claim, we prove that the gradients of $\partial \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4})/\partial t_j^{k-1}$ and $\overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4})$ are linearly independent. Assume that they are not linearly independent, then there exists a function R such that

$$(56) \quad \left(\frac{\partial^2 \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4})}{\partial t_j^{k-1} \partial t_1^{k-4}}, \dots, \frac{\partial^2 \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4})}{\partial t_j^{k-1} \partial t_m^{k-4}} \right) = R \left(\frac{\partial \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4})}{\partial t_1^{k-4}}, \dots, \frac{\partial \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4})}{\partial t_m^{k-4}} \right).$$

Since $M(p)$ is finitely nondegenerate, we can choose variables $t_{\alpha_1}^{k-4}, \dots, t_{\alpha_\lambda}^{k-4}$ such that

$$(57) \quad \frac{\partial^{\lambda+1} \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4})}{\partial t_j^{k-1} \partial t_{\alpha_1}^{k-4}, \dots, \partial t_{\alpha_\lambda}^{k-4}}(0) \neq 0.$$

By differentiating the equality

$$(58) \quad \frac{\partial^2 \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4})}{\partial t_j^{k-1} \partial t_{\alpha_1}^{k-4}} = R \frac{\partial \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}; s^{k-1}, v_N^{k-4})}{\partial t_{\alpha_1}^{k-4}}$$

by $t_{\alpha_2}^{k-4}, \dots, t_{\alpha_\lambda}^{k-4}$, we obtain

$$(59) \quad \frac{\partial^{\lambda+1} \overline{Q^1}(t^{k-1}, v_{M(p)}^{k-4}, s^{k-1}, v_N^{k-4})}{\partial t_j^{k-1} \partial t_{\alpha_1}^{k-4}, \dots, \partial t_{\alpha_\lambda}^{k-4}} \\ = (\text{the summation of the differentials of } \overline{Q^1} \text{ by } t_{\alpha_1}^{k-4}, \dots, t_{\alpha_\lambda}^{k-4} \text{ of order less than or equal to } \lambda),$$

from which we reach a contradiction by evaluating at the origin. Therefore there does not exist such a function R , which shows the claim and G_l is independent of z_j . Since j is fixed arbitrary, we conclude that G_l is independent of z_j for $l = 1, \dots, n$, $j = 1, \dots, m$. Returning to the relation (53), we obtain that G_l is independent of z_{m+1} for $l = 1, \dots, n$. \square

LEMMA 3.6. G_{n+1} is independent of z_1, \dots, z_{m+1} .

PROOF. By the identity (38) together with Lemma 3.4, we obtain that

$$(60) \quad \frac{\partial G_{n+1}}{\partial z_{m+1}} \circ \widehat{V}^k = 0.$$

Then by the identity (37), we get

$$(61) \quad \frac{\partial G_{n+1}}{\partial z_j} \circ \widehat{V}^k = 0.$$

These show that G_{n+1} is independent of z_1, \dots, z_{m+1} . \square

Now we have proved that G_1, \dots, G_{n+1} are independent of z_1, \dots, z_{m+1} , which completes the proof of Theorem 1.2.

4. Some questions related to other topics.

4.1. The Rudin theorem. Rudin proved the following theorem in [10].

THEOREM 4.1. Suppose $n \geq 1$, $f : \mathbb{C}^n \rightarrow \mathbb{C}$ is a nonconstant polynomial, $V_\alpha = \{z \in \mathbb{C}^n : f(z) = \alpha\}$, and Φ is a biholomorphic map from \mathbb{C}^n onto \mathbb{C}^n . If $\alpha_1 \neq \alpha_2$ and $\Phi(V_{\alpha_i}) = V_{\alpha_i}$ for $i = 1, 2$, then $\Phi(V_\beta) = V_\beta$ for every $\beta \in \mathbb{C}$, and $f = f \circ \Phi$.

This theorem claims that if a biholomorphic map preserves two level sets of a polynomial, then it preserves all level sets. He shows that “polynomial” can not be replaced by “entire function”. Compare this theorem with ours. Let $H : M \rightarrow \widetilde{M}$ be an algebraic CR diffeomorphism between algebraic CR manifolds whose slices and base spaces are all algebraic. Assume that $H|_N : N \rightarrow \widetilde{N}$ is an algebraic CR diffeomorphism. Then we have the following question.

QUESTION 4.2. If there exist finite number of points $p_1, \dots, p_\lambda \in N$ and $\tilde{p}_1, \dots, \tilde{p}_\lambda \in \widetilde{N}$ such that $H(M(p_j)) \subset \widetilde{M}(\tilde{p}_j)$ for $j = 1, \dots, \lambda$, then, for any $p \in N$, does there exist $\tilde{p} \in \widetilde{N}$ such that $H(M(p)) = \widetilde{M}(\tilde{p})$ holds? If the statement is true for some λ , find the minimal number of λ .

4.2. The Baracco-Zampieri theorem. Baracco and Zampieri proved, in some sense, CR geometry version of Hartogs' theorem in [5]. They consider a real analytic foliation of \mathbb{C}^n by complex analytic manifolds of dimension m issued transversally from a generic manifold $M \subset \mathbb{C}^n$ of codimension m . Then they proved that any continuous function which is a CR function on M and has separate holomorphic extension along each leaf is holomorphic. Let M and \tilde{M} be the same as in Theorem 1.2. We have the following question.

QUESTION 4.3. Assume that $F : N \rightarrow \tilde{N}$ is a CR homeomorphism and that, for each $p \in N$, there exists $\tilde{p} \in \tilde{N}$ such that F has a CR homeomorphic extension $F : M(p) \rightarrow \tilde{M}(\tilde{p})$. Then does F have a CR homeomorphic extension $F : M \rightarrow \tilde{M}$?

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