Tôhoku Math. Journ.
Vol. 19, No. 3, 1967

WEIGHTED AVERAGES OF SUBMARTINGALES

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(Received April 20, 1967)

Let $\{x_n, \mathfrak{F}_n, n \geqq 1\}$ be a submartingale. For a given sequence of positive numbers $w_1, w_2, \cdots$, we consider the weighted averages

$$s_n = (w_1 x_1 + \cdots + w_n x_n)/W_n \qquad (n = 1, 2, \cdots)$$

where $W_n = w_1 + \cdots + w_n$. Although the sequence $\{s_n\}$ need not be a submartingale, we may expect some similar properties to the original submartingale.

THEOREM 1. *For the submartingale* $\{x_n, \mathfrak{F}_n, n \geqq 1\}$, *using the above notations, suppose that* $\lim_{n\to\infty} W_n = \infty$. *Then the following two conditions are equivalent to each other*:

$$(1) \qquad \sup_n \mathrm{E}\{|s_n|\} < \infty ,$$

$$(2) \qquad \sup_n \mathrm{E}\{|x_n|\} < \infty .$$

By the classical submartingale convergence theorem, the condition (2) *is sufficient to insure the almost sure convergence of* $\{x_n\}$, *hence so is the condition* (1).

PROOF. It is easy to get (1) from (2), in fact,

$$\mathrm{E}\{|s_n|\} \leqq \mathrm{E}\left\{\frac{w_1|x_1| + \cdots + w_n|x_n|}{W_n}\right\} \leqq \sup_j \mathrm{E}\{|x_j|\} .$$

To show that (1) implies (2) we consider the two cases of martingale and submartingale.

(i) Let $\{x_n, \mathfrak{F}_n, n \geqq 1\}$ be a martingale. If $m < n$, we have by the definition of conditional expectations and the martingale equality

$$\begin{aligned}\mathrm{E}\{|s_n|\} &= \mathrm{E}\{\mathrm{E}\{|s_n|\,|\mathfrak{F}_m\}\} \\ &\geqq \mathrm{E}\{|\mathrm{E}\{s_n|\mathfrak{F}_m\}|\}\end{aligned}$$

$$= \mathrm{E}\left\{\left|\frac{w_1x_1+\cdots+w_mx_m}{W_n}+\frac{w_{m+1}+\cdots+w_n}{W_n}x_m\right|\right\}$$

$$\geqq \frac{w_{m+1}+\cdots+w_n}{W_n}\mathrm{E}\{|x_m|\} - \frac{w_1\mathrm{E}\{|x_1|\}+\cdots+w_m\mathrm{E}\{|x_m|\}}{W_n}.$$

Making $n\to\infty$, m being fixed,

$$\sup_n \mathrm{E}\{|s_n|\} \geqq \mathrm{E}\{|x_m|\}$$

since $W_n\to\infty$ and $(w_{m+1}+\cdots+w_n)/W_n\to 1$ as $n\to\infty$. Thus we get (2) in this case.

(ii) Now let $\{x_n, \mathfrak{F}_n, n\geqq 1\}$ be a submartingale. Then we can write ([1] p. 296)

$$x_n = x_n' + \sum_{j=1}^{n}\Delta_j,$$

$$\Delta_1 = 0, \quad \Delta_j = \mathrm{E}\{x_j|\mathfrak{F}_{j-1}\} - x_{j-1} \qquad (j\geqq 2)$$

where $\{x_n', \mathfrak{F}_n, n\geqq 1\}$ is a martingale, and $\Delta_j\geqq 0$ $(j\geqq 1)$. Denote

$$\delta_n = \sum_{j=1}^{n}\Delta_j \qquad (n=1,2,\cdots),$$

then $\{\delta_n\}$ is a sequence of positive random variables and is monotone increasing with respect to n. Denote further by s_n, s_n' and t_n the weighted averages of x_n, x_n', and δ_n respectively, so that $s_n = s_n'+t_n$, and we find

$$\mathrm{E}\{s_n\} = \mathrm{E}\{s_n'\} + \mathrm{E}\{t_n\}$$

$$= \mathrm{E}\{x_1'\} + \mathrm{E}\{t_n\},$$

since by the martingale equality $\mathrm{E}\{x_n'\} = \mathrm{E}\{x_1'\}$ for all n.

Hence, from the assumption (1) we get

$$\sup_n \mathrm{E}\{t_n\} \leqq \sup_n \mathrm{E}\{|s_n|\} + \mathrm{E}\{|x_1'|\} < \infty$$

and again from the inequality

$$\mathrm{E}\{|s_n'|\} \leqq \mathrm{E}\{|s_n|\} + \mathrm{E}\{t_n\}$$

we find $\sup_n E\{|s'_n|\} < \infty$. Hence applying the case (i) to the martingale $\{x'_n, \mathfrak{F}_n, n \geqq 1\}$ we get $\sup_n E\{|x'_n|\} < \infty$. As $\{\delta_n\}$ is an increasing sequence,

$$\lim_{n\to\infty} E\{\delta_n\} = E\{\lim_{n\to\infty} \delta_n\} = E\{\lim_{n\to\infty} t_n\} = \sup_n E\{t_n\},$$

and then $\sup_n E\{\delta_n\} < \infty$, so that

$$\sup_n E\{|x_n|\} \leqq \sup_n E\{|x'_n|\} + \sup_n E\{\delta_n\} < \infty.$$

This completes the proof.

THEOREM 2. *Let $\{x_n, \mathfrak{F}_n, n \geqq 1\}$ be a submartingale. Let $w_1, w_2, \cdots$ be positive numbers such that $W_n = w_1 + \cdots + w_n \to \infty$ as $n \to \infty$ and let $\mathfrak{F}_\infty$ be the Borel field generated by the field $\bigcup_m \mathfrak{F}_m$. Suppose that any $\Lambda \in \mathfrak{F}_\infty$ satisfies the following condition:*
(H): for any $\varepsilon > 0$, there exist an integer m and a set $\Lambda_m \in \mathfrak{F}_m$ such that $\Lambda_m \subset \Lambda$ and $P\{\Lambda - \Lambda_m\} < \varepsilon$.
Then the sequence $\{x_n\}$ and the sequence of their weighted averages $s_n = (w_1 x_1 + \cdots + w_n x_n)/W_n$ are equiconvergent almost surely.

PROOF. It is enough to prove that if $\{s_n\}$ converges on Λ, then $\{x_n\}$ does almost surely on Λ, since the converse is evident. Further we shall show the almost sure convergence of $\{x_n\}$ on Λ under weaker assumption that the sequence $\{s_n\}$ is bounded almost surely on Λ.

Clearly we may suppose $\Lambda \in \mathfrak{F}_\infty$ and $P\{\Lambda\} > 0$. For any $\varepsilon > 0$, we can find a positive constant M and a set $\Lambda^* \in \mathfrak{F}_\infty$ such that

$$\Lambda^* = \Lambda \cap \{|s_n| < M \text{ for all } n\}$$

and $P\{\Lambda - \Lambda^*\} < \varepsilon/2$. By the assumption (H) we can find an integer m and a set $\Lambda_m \in \mathfrak{F}_m$ such that $\Lambda_m \subset \Lambda^*$ and $P\{\Lambda^* - \Lambda_m\} < \varepsilon/2$. Therefore we have $P(\Lambda - \Lambda_m\} < \varepsilon$, $\Lambda_m \in \mathfrak{F}_m$ and on Λ_m, $|s_n| < M$ for all n.

The process $\{x_n 1_{\Lambda_m}, \mathfrak{F}_n, n \geqq m\}$ is a submartingale, since for $n > m$,

$$E\{x_n 1_{\Lambda_m} | \mathfrak{F}_{n-1}\} = 1_{\Lambda_m} E\{x_n | \mathfrak{F}_{n-1}\} \geqq x_{n-1} 1_{\Lambda_m}.$$

From the preceding fact the weighted averages of this martingale $s_n 1_{\Lambda_m}$ satisfy that

$$M \geqq \int_{\Lambda_m} |s_n| dP = E\{|s_n 1_{\Lambda_m}|\}$$

for all $n \geqq m$. Hence by Theorem 1 $\sup_n \mathrm{E}\{|x_n 1_{\Lambda_m}|\} < \infty$, and by the classical theorem $\lim_{n\to\infty} x_n 1_{\Lambda_m}$ exists and finite almost surely, or so does $\lim_{n\to\infty} x_n$ on Λ_m. As ε is arbitrary we complete the proof.

REMARK. We give an example which shows that the condition (H) in Theorem 2 cannot be suppressed. Let $(\Omega, \mathfrak{B}, \mathrm{P})$ be the Wiener probability space with $\Omega=[0, 1[$, $\mathfrak{B}$ the class of all linear Borel sets in Ω and P the Lebesgue measure. Let $w_n=1$ for all n and define

$$x_1(\omega) = \begin{cases} -1 & \text{for } 0 \leqq \omega < 1/2 \\ 1 & \text{for } 1/2 \leqq \omega < 1 \end{cases}$$

$$x_2(\omega) = \begin{cases} -1 & \text{for } 0 \leqq \omega < 1/2 \\ 7 & \text{for } 1/2 \leqq \omega < 5/8 \\ -1 & \text{for } 5/8 \leqq \omega < 1 \end{cases}$$

and generally

$$x_{n+1}(\omega) = \begin{cases} x_n(\omega) & \text{for } 0 \leqq \omega < a_n \\ (-1)^{n+1}(2^{n+1}+3) & \text{for } a_n \leqq \omega < a_{n+1} \\ (-1)^n & \text{for } a_{n+1} \leqq \omega < 1 \end{cases}$$

where $a_1=1/2$, $a_n=1/2+\sum_{k=1}^{n-1} 1/2^{k+2}$ $(n=2, 3, \cdots)$. Let $\mathfrak{F}_n$ be the Borel field induced by $x_1, \cdots, x_n$, then $\{x_n, \mathfrak{F}_n, n \geqq 1\}$ is a martingale. The set $\Lambda = [3/4, 1[$ does not belong to the field $\bigcup_n \mathfrak{F}_n$, but is contained in $\mathfrak{F}_\infty$ and $\mathrm{P}\{\Lambda\} > 0$. Moreover on Λ, $x_n = (-1)^n$, and hence $s_n = (x_1 + \cdots + x_n)/n$ converges to zero whereas x_n diverges.

COROLLARY 1. *Let $p \geqq 1$ be a constant. Under the same assumption of Theorem 1, the two conditions $\sup_n \mathrm{E}\{|s_n|^p\} < \infty$ and $\sup_n \mathrm{E}\{|x_n|^p\} < \infty$ are equivalent to each other.*

PROOF. We get easily the conclusion along the same lines as in the proof of Theorem 1 after a suitable use of the Minkowski inequality.

Instead of the weighted averages we shall consider those of the Nörlund type

$$\bar{s}_n = (w_n x_1 + w_{n-1} x_2 + \cdots + w_1 x_n)/W_n \quad (n = 1, 2, \cdots),$$

then we can show the following corollary.

COROLLARY 2. *For a submartingale* $\{x_n, \mathfrak{F}_n, n \geqq 1\}$, *let* $p \geqq 1$ *be a constant and suppose that*

$$\lim_{n\to\infty} (\operatorname{Max}_{1\leqq j\leqq n} w_j)/W_n = 0$$

If $\sup_n \mathrm{E}\{|\bar{s}_n|^p\} < \infty$, *then* $\sup_n \mathrm{E}\{|x_n|^p\} < \infty$, *and conversely; under the condition (H), if* $\lim_{n\to\infty} \bar{s}_n$ *exists and is finite almost surely on* Λ, *then so does* $\lim_{n\to\infty} x_n$, *and conversely.*

PROOF. If we follow the corresponding lines of the proofs of Theorems 1 and 2, the only points to be noticed are:

$$(w_n + w_{n-1} + \cdots + w_{n-N+1})/W_n \to 0$$

and $(w_{n-N} + \cdots + w_1)/W_n \to 1$ as $n \to \infty$ for N fixed. These facts are immediate consequences of the additional assumption of $\{w_n\}$.

Finally we mention the Abel mean analogue to the above results. Denote the Abel mean of $\{x_n\}$ by

$$A_r = (1-r)\sum_{j=0}^{\infty} r^j x_j \qquad (0 < r < 1).$$

COROLLARY 3. *For a submartingale* $\{x_n, \mathfrak{F}_n, n \geqq 1\}$, *if* $\sup_{0<r<1} \mathrm{E}\{|A_r|^p\} < \infty$, *then* $\sup_n \mathrm{E}\{|x_n|^p\} < \infty$ *and conversely, where* $p \geqq 1$ *is constant. Under the condition (H) of Theorem 2, if* $\lim_{r\to 1-0} A_r$ *exists and is finite almost surely on* Λ, *then so does* $\lim_{n\to\infty} x_n$, *and conversely.*

PROOF. Let m be a fixed positive integer. If $\{x_n, \mathfrak{F}_n, n \geqq 1\}$ is a martingale, then

$$\begin{aligned} \mathrm{E}\{|A_r|^p\} &= \mathrm{E}\{\mathrm{E}\{|A_r|^p|\mathfrak{F}_m\}\} \\ &\geqq \mathrm{E}\left\{\left|\mathrm{E}\left\{(1-r)\sum_{j=0}^{m-1} r^j x_j + (1-r)\sum_{j=m}^{\infty} r^j x_j \middle| \mathfrak{F}_m\right\}\right|^p\right\} \\ &= \mathrm{E}\left\{\left|(1-r)\sum_{j=0}^{m-1} r^j x_j + (1-r)\, x_m \sum_{j=m}^{\infty} r^j\right|^p\right\} \end{aligned}$$

$$\geqq 2^{1-p} r^{mp} \mathrm{E}\{|x_m|^p\} - (1-r)^p \mathrm{E}\left\{\left|\sum_{j=0}^{m-1} r^j x_j\right|^p\right\}.$$

Make $r \to 1-0$ for fixed m, and we get

$$\sup_{0<r<1} \mathrm{E}\{|A_r|^p\} \geqq 2^{1-p}\mathrm{E}\{|x_m|^p\}.$$

In this case the first conclusion of Corollary was proved; the submartingale case is treated along the similar way to the proof of Theorem 1, and so are the other conclusions.

REFERENCE

[1] J. L. DOOB, Stochastic Processes, Wiley, New York, 1953.

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