Tôhoku Math. Journ.
21(1969), 144-151.

A NOTE ON INVARIANT SUBSPACES

KOH-ICHI KITANO

(Received October 28, 1968)

In this note the following theorem will be proved. We base our arguments on T. A. Gillespie's paper [4], F. F. Bonsall's lecture note [2] and P. Meyer-Nieberg's paper [5]. For the sake of completeness, the proof below repeats the relevant arguments in [4].

THEOREM. *Let X be a normed linear space over $\boldsymbol{C}$ (the complex number field) of dimension greater than one or over $\boldsymbol{R}$ (the real number field) of dimension greater than two and let T be a bounded linear operator in X such that* $\liminf \|T^n e\|^{1/n} = 0$ *for some non-zero vector e in X. If the uniformly closed algebra generated by T and the identity contains a non-zero compact operator S, then T has a proper closed invariant subspace.*

In [1] the present theorem is proved in the case X is a Hilbert space over $\boldsymbol{C}$.

We need the following notations and some results. X will denote a normed linear space over $\boldsymbol{K}$ ($=\boldsymbol{C}$ or $\boldsymbol{R}$); we assume if $\boldsymbol{K}=\boldsymbol{C}$, $\dim X > 1$ and if $\boldsymbol{K}=\boldsymbol{R}$, $\dim X > 2$; an operator means a bounded linear operator in X and a subspace means a closed linear manifold. In order to prove the existence of an invariant subspace, there is no loss of generality in assuming the existence of a unit vector e in X such that $\liminf \|T^n e\|^{1/n} = 0$ and the vectors $e, Te, T^2e, \cdots, T^ne, \cdots$ are linearly independent and have X for their closed linear span, i.e., if let E_n be the linear span of $\{e, Te, \cdots, T^{n-1}e\}$, then $X = \overline{\bigcup_{n=1}^{\infty} E_n}$ (cf. Lemma 3 of the present note).

(i) If E is a non-empty subset of X and $x \in X$, the distance from x to E, $d(x, E)$, is defined by $d(x,E) = \inf\{\|x-y\| : y \in E\}$.

(ii) Given a sequence E_n of subspaces of X, define $\liminf E_n$ to be

$$\liminf E_n = \{x \in X : \lim d(x, E_n) = 0\}.$$

It is clear that $\liminf E_n$ is a subspace of X and $\liminf E_{j(n)} = \liminf E_n$ for any subsequence $\{j(n)\}$ of $\{n\}$. If for every $n \geqq 1$, G_n is a subspace of F_n, then

$$\liminf G_n \subset \liminf F_n .$$

(iii) Given a finite dimensional subspace E of X and $x \in X$, there exists a point $u \in E$ such that $\|x-u\| = d(x, E)$. Each such u we call a nearest point of E to x.

(iv) Given finite dimensional subspaces E, F with $E \subset F$ and $E \neq F$, the canonical map $x \to x'$ of F onto F/E is a bounded linear map of norm 1 and attains its bound, since F has finite dimension; hence there exists $v \in F$ such that $\|v\| = 1 = \|v'\| = d(v, E)$. We call each such v a unit vector orthogonal to E. In the sequel, let e_n be a unit vector in E_n orthogonal to E_{n-1}.

LEMMA 1. *Let T be an operator given in the theorem. Then there exists a subsequence $\{j(n)\}$ of $\{n\}$ such that*

$$\lim_{n\to\infty} d(Te_{j(n)}, E_{j(n)}) = 0 .$$

PROOF. Since E_n is the linear span of $\{E_{n-1}, T^{n-1}e\}$, for each integer n we have

$$e_n = \alpha_n T^{n-1}e + f_n$$

with $\alpha_n \in \boldsymbol{K}$, $\alpha_n \neq 0$ and $f_n \in E_{n-1}$. Thus we have

$$(1) \qquad e_n \equiv \alpha_n T^{n-1} e \qquad (\text{mod } E_{n-1}) .$$

By the definition of E_n, $TE_{n-1} \subset E_n$, this gives

$$(2) \qquad T^r e_n \equiv \alpha_n T^{n+r-1} e \qquad (\text{mod } E_{n+r-1})$$

for $n \geqq 1$, $r \geqq 1$. Also, replacing n by $n+r$ in (1), we have

$$(3) \qquad e_{n+r} \equiv \alpha_{n+r} T^{n+r-1} e \qquad (\text{mod } E_{n+r-1}) ,$$

and so, by (2) and (3)

$$(4) \qquad T^r e_n \equiv \frac{\alpha_n}{\alpha_{n+r}} e_{n+r} \qquad (\text{mod } E_{n+r-1})$$

for $n \geqq 1$, $r \geqq 1$. We note that, since $d(e_n, E_{n-1})=1$, it follows from (4) that

$$d(T^r e_n, E_{n+r-1}) = \frac{|\alpha_n|}{|\alpha_{n+r}|} \tag{5}$$

for $n \geqq 1$, $r \geqq 1$. On the other hand, by (2)

$$d(T^r e_n, E_{n+r-1}) = d(\alpha_n T^{n+r-1} e, E_{n+r-1}),$$

that is,

$$d(T^{n+r-1} e, E_{n+r-1}) = \frac{1}{|\alpha_{n+r}|} \tag{6}$$

for $n \geqq 1$, $r \geqq 1$, therefore we have

$$d(T^n e, E_n) = \frac{1}{|\alpha_{n+1}|} \qquad n \geqq 1.$$

We also have

$$\|T^n e\| \geqq d(T^n e, E_n) = \frac{1}{|\alpha_{n+1}|} \qquad n \geqq 1. \tag{7}$$

By hypothesis, $\liminf \|T^n e\|^{1/n} = 0$, so we see that

$$\liminf d(Te_n, E_n) = \liminf \frac{|\alpha_n|}{|\alpha_{n+1}|}$$

$$\leqq \liminf \left(\frac{1}{|\alpha_{n+1}|}\right)^{1/n} \leqq \liminf \|T^n e\|^{1/n} = 0.$$

Thus, there exists a sequence $\{j(n)\} \subset \{n\}$ such that

$$\lim_{n\to\infty} d(Te_{j(n)}, E_{j(n)}) = 0.$$

This completes the proof of the Lemma 1.

The following lemma is proved in [5], but we give a proof for convenience' sake.

LEMMA 2. *Let* $\{F_n\}$ *and* $\{G_n\}$ *be two sequences of subspaces of* X *such that* $\liminf G_{j(n)} = \liminf G_n$ *for any subsequence* $\{j(n)\} \subset \{n\}$ *and* $G_n \subset F_n$, $\dim(F_n/G_n) \leqq m$ *for all* n, *then*

$$\dim(\liminf F_n/\liminf G_n) \leqq m.$$

PROOF. Let $v_0, v_1, \cdots, v_m$ be vectors of $\liminf F_n$. By the definition of $\liminf F_n$, there exist sequences $\{v_{n,p}\} \subset F_n$, with $v_{n,p} \to v_p$ as $n \to \infty$ for $p = 0, 1, \cdots, m$. Since $\dim(F_n/G_n) \leqq m$, we can choose scalars $\alpha_{n,0}, \alpha_{n,1}, \cdots, \alpha_{n,m} \in \boldsymbol{K}$ such that

$$\sum_{p=0}^{m} \alpha_{n,p} v_{n,p} \equiv 0 \qquad (\mathrm{mod}\ G_n)$$

and

$$\sum_{p=0}^{m} |\alpha_{n,p}| = 1.$$

Here, there exists a sequence $\{j(n)\} \subset \{n\}$ such that $\alpha_{j(n),p} \to \alpha_p$ as $n \to \infty$ for $p = 0, 1, \cdots, m$. Therefore it follows that

$$\lim_{n\to\infty} \sum_{p=0}^{m} \alpha_{j(n),p} v_{j(n),p} = \sum_{p=0}^{m} \alpha_p v_p$$

and

$$\sum_{p=0}^{m} \alpha_p v_p \equiv 0 \qquad (\mathrm{mod}\ \liminf G_{j(n)}),$$

$\sum_{p=0}^{m} |\alpha_p| = 1$ is valid. Thus $\dim(\liminf F_n/\liminf G_n) \leqq m$.

LEMMA 3. *If T is an operator in a finite dimensional normed linear space X over $\boldsymbol{K}$, then there exist subspaces $L_0, L_1, \cdots, L_n$ of X such that $(0) = L_0 \subset L_1 \subset \cdots \subset L_n = X, TL_j \subset L_j$ $(j = 0, 1, \cdots, n)$ and* (a) *if $\boldsymbol{K} = \boldsymbol{R}$, $\dim(L_j/L_{j-1}) = 2$ or 1 $(j = 1, 2, \cdots, n)$,* (b) *if $\boldsymbol{K} = \boldsymbol{C}$, $\dim(L_j/L_{j-1}) = 1$ $(j = 1, 2, \cdots, n)$.*

This lemma is well known results (see, for example, [3]). We turn now to the proof of the theorem.

PROOF OF THE THEOREM. We consider the operator T_n of E_n into itself $(n \geqq 1)$ defined by

$$T_n | E_{n-1} = T | E_{n-1}, \quad T_n e_n = u_n,$$

where u_n is a nearest point of E_n to Te_n. We show that

$$(8) \qquad \|Tx-T_nx\| \leqq d(Te_n, E_n)\|x\| \quad x \in E_n,\ n \geqq 1.$$

Let $x \in E_n$. Then $x = y+\lambda e_n$ for some $\lambda \in \boldsymbol{K}, y \in E_{n-1}$

$$\|Tx-T_nx\| = |\lambda|\,\|Te_n-u_n\| = |\lambda|\,d(Te_n, E_n).$$

On the other hand, for a unit vector e_n orthogonal to E_{n-1}

$$\|\lambda e_n+y\| \geqq |\lambda| \qquad \lambda \in \boldsymbol{K},\ y \in E_{n-1}.$$

In fact, this is trivial if $\lambda=0$. If $\lambda \neq 0$,

$$\|e_n + \frac{1}{\lambda}y\| \geqq d(e_n, E_{n-1}) = 1.$$

Therefore

$$\|Tx-T_nx\| \leqq d(Te_n, E_n)\|x\| \quad x \in E_n, n \geqq 1.$$

From Lemma 1 and (8), we see that if $\{x_n\}$ is a bounded sequence, $x_n \in E_{j(n)}$, then

$$(9) \qquad \lim_{n\to\infty} \|Tx_n-T_{j(n)}x_n\| = 0.$$

It follows from (9) that if H_n is a sequence of subspaces of $E_{j(n)}$ invariant for $T_{j(n)}$, then for every subsequence $\{H_{n_k}\}$ $\liminf H_{n_k}$ is an invariant subspace for T.

We prove next, by induction on k, that for each positive integer k there exists a constant M_k such that

$$(10) \qquad \|T^kx-T_n^kx\| \leqq M_kd(Te_n, E_n)\|x\| \quad x \in E_n, n \geqq 1.$$

The case $k=1$; given by (8), ($M_1 = 1$). We suppose that (10) holds for some k, and deduce there of that it holds for $k+1$. On this hypothesis, we have for all $x \in E_n$,

$$\begin{aligned}\|T_n^kx\| &\leqq \|T^kx\| + M_kd(Te_n, E_n)\|x\| \\ &\leqq (\|T^k\| + M_k\|T\|)\|x\| = A_k\|x\|, \text{ say.}\end{aligned}$$

Since $T_n^k E_n \subset E_n$, (8) gives that for all $x \in E_n$,

$$\|TT_n^k x - T_n^{k+1} x\| \leqq d(Te_n, E_n)\|T_n^k x\| \leqq A_k d(Te_n, E_n)\|x\| .$$

Thus for all $x \in E_n$

$$\begin{aligned}\|T^{k+1} x - T_n^{k+1} x\| &\leqq \|T^{k+1} x - TT_n^k x\| + \|TT_n^k x - T_n^{k+1} x\| \\ &\leqq \|T\|\,\|T^k x - T_n^k x\| + \|TT_n^k x - T_n^{k+1} x\| \\ &\leqq (\|T\| M_k + A_k)\, d(Te_n, E_n)\|x\| .\end{aligned}$$

Hence, by induction, (10) is now proved. It follows at once from (10) that, for a given polynomial $P(T)$ in T, there exists a constant K such that

$$\|P(T)\, x - P(T_n) x\| \leqq K\, d(Te_n, E_n)\|x\| \tag{11}$$

for $x \in E_n$, $n \geqq 1$. Hence we can find constants $\{K_r\}_{r \geqq 1}$ such that

$$\|P_r(T) x - P_r(T_n) x\| \leqq K_r d(Te_n, E_n)\|x\| \tag{12}$$

for $x \in E_n$, $n \geqq 1$, $r \geqq 1$, where $P_r(\ \cdot\ )$ are polynomials such that $P_r(T) \rightarrow S$ (in norm) as $r \rightarrow \infty$. Since $ST = TS$ and $S \neq 0$, we may assume that the null space of S is zero, for otherwise $S^{-1}(0)$ is a proper invariant subspace for T. Therefore $Se \neq 0$, and we can choose α with $0 < \alpha < 1$ and $\alpha\|S\| < \|Se\|$. Since $T_{j(n)}$ is an operator of $E_{j(n)}$ into itself, by Lemma 3 there exist subspaces E_n^i of $E_{j(n)}$ invariant for $T_{j(n)}$,

$$(0) = E_n^0 \subset E_n^1 \subset \cdots \subset E_n^{i(n)} = E_{j(n)}$$

and

$$\dim (E_n^{i+1}/E_n^i) \leqq 2 .$$

We have $d(e, E_n^0) = 1 > \alpha$, $d(e, E_n^{i(n)}) = 0 < \alpha$. Thus for each n there is a greatest i, i_n say, such that $d(e, E_n^i) \geqq \alpha$. Let $F_n = E_n^{i_n}$, $G_n = E_n^{i_n+1}$. Then

$$d(e, F_n) \geqq \alpha, \quad d(e, G_n) < \alpha \qquad (n \geqq 1) .$$

It follows at once from the first of these inequalities that, for any subsequence $\{n_k\} \subset \{n\}$,

$$e \notin \liminf F_{n_k} . \tag{13}$$

Since $d(e, G_n) < \alpha$, there exists a sequence $\{x_n\} \subset G_n$ is bounded, i.e., $\|x_n\| < \alpha + \|e\| = \alpha + 1$. Using the compactness of S, we have a subsequence $\{n_k\} \subset \{n\}$ such that

$$\lim_{k\to\infty} Sx_{n_k} = x \in X .$$

We show next, that x belongs to $\liminf G_{n_k}$. For any $\varepsilon > 0$, there exists n_0 such that

$$\|S - P_{n_0}(T)\| < \frac{\varepsilon}{\alpha+1} .$$

By Lemma 1, there exists k_0 such that

$$d(Te_{j(n_k)}, E_{j(n_k)}) < \frac{\varepsilon}{K_{n_0}(\alpha+1)} \qquad k \geqq k_0 .$$

By (12)

$$\|P_{n_0}(T)x_{n_k} - P_{n_0}(T_{j(n_k)})x_{n_k}\| \leqq K_{n_0} d(Te_{j(n_k)}, E_{j(n_k)})(\alpha+1)$$

for $k \geqq 1$. Therefore $k \geqq k_0$ implies that

$$\begin{aligned}\|Sx_{n_k} - P_{n_0}(T_{j(n_k)})x_{n_k}\| &\leqq \|Sx_{n_k} - P_{n_0}(T)x_{n_k}\| + \|P_{n_0}(T)x_{n_k} - P_{n_0}(T_{j(n_k)})x_{n_k}\| \\ &\leqq \|S - P_{n_0}(T)\|(\alpha+1) + K_{n_0} d(e_{j(n_k)}, E_{j(n_k)})(\alpha+1) \\ &< \varepsilon + \varepsilon = 2\varepsilon .\end{aligned}$$

Since $\lim\limits_{k\to\infty} Sx_{n_k} = x$, there exists $k_1 \geqq k_0$ such that

$$\|Sx_{n_k} - x\| < \varepsilon \qquad k \geqq k_1 .$$

Thus if $k \geqq k_1$,

$$\|x - P_{n_0}(T_{j(n_k)})x_{n_k}\| \leqq \|x - Sx_{n_k}\| + \|Sx_{n_k} - P_{n_0}(T_{j(n_k)})x_n\| < \varepsilon + 2\varepsilon = 3\varepsilon$$

Since G_{n_k} is invariant for $T_{j(n_k)}$, we have $P_{n_0}(T_{j(n_k)})x_{n_k} \in G_{n_k}$ and so

$$d(x, G_{n_k}) \leqq \|x - P_{n_0}(T_{j(n_k)})x_{n_k}\| < 3\varepsilon \qquad k \geqq k_1 .$$

Therefore $\lim d(x, G_{n_k}) = 0$, and $x \in \liminf G_{n_k}$. Now, on the other hand,

$$\|Se - x\| = \lim_{k\to\infty} \|Se - Sx_{n_k}\| \leqq \alpha \|S\| < \|Se\| .$$

Thus we have $x \neq 0$, and so $\liminf G_{n_k}$ will be a proper invariant subspace for T unless $\liminf G_{n_k} = X$. By (13) and (ii) $\liminf F_{m_k} \neq X$ for every subsequence $\{m_k\} \subset \{n\}$. Now, if $\liminf F_{m_k} = (0)$ for every subsequence $\{m_k\} \subset \{n\}$, by Lemma 2

$$\dim(\liminf G_{n_k}) \leqq 2 .$$

Therefore $\liminf G_{n_k} \neq X$. This completes the proof of the theorem.

REFERENCES

[1] W. B. ARVESON AND J. FELDMAN, A note on invariant subspaces, Michigan Math. J., 15 (1968), 61–64.
[2] F. F. BONSALL, Compact linear operators, Lectures notes, Yale Univ., 1967.
[3] I. M. GEL'FAND, Lectures on linear algebra, (Translated by A. Shenitzer from the Rivised Second Russian Edition), 1961, Interscience, New York.
[4] T. A. GILLESPIE, An invariant subspace theorem of J. Feldman, Pacific J. Math., 26 (1968), 67–72.
[5] P. MEYER-NIEBERG, Invariante Unterräume von polynomkompakten Operatoren, Arch. Math., 19(1968), 180–182.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN