WIRTINGER TYPE INEQUALITIES AND ELLIPTIC DIFFERENTIAL INEQUALITIES

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Riccati transformations are frequently encountered in the study of oscillatory behavior of ordinary differential equations. The purpose of this note is to extend such transformations to linear elliptic differential equations and inequalities and to generate certain integral relations and identities. Such a connection between a Riccati equation and the two-dimensional elliptic equation

$$u_{xx} + u_{yy} + p(x,y)u = 0$$

was first observed by Beesack [2]. In this paper we show how his method can be modified and extended. In addition to generating inequalities of the Wirtinger type, the integral relations obtained are also useful in establishing comparison and oscillation theorems for elliptic equations. This approach provides yet another connection between some of the methods used in ordinary differential equations and those employed by a number of authors [3, 5-8, 11-14] in the study of Sturmian theorems for elliptic differential equations and systems.

A variable point of n-dimensional Euclidean space R^n will be denoted by $x = (x_1, \cdots, x_n)$. Let G be a bounded domain of R^n with piecewise smooth boundary ∂G, and let G_1 be an open set of R^n such that $\overline{G} \subset G_1$. Throughout this paper we shall adopt the Einstein summation convention in which Latin indices i, j, k, etc. will take values $1, 2, \cdots, m$ while Greek indices α and β will take values $1, 2, \cdots, n$. All functions considered will be real-valued with domain $\overline{G}$ or G_1. We consider the system of linear second order differential inequalities

(1) $$\nabla \cdot (\widetilde{A} \nabla U) + 2\widehat{B} \cdot \nabla U + PU \leqslant 0,$$

where

$$(2) \quad \begin{cases} PU = \sum_{j=1}^{m} p^i{}_j u^j{}_k = p^i{}_j u^j{}_k \,, \\[2ex] \nabla U = \dfrac{\partial}{\partial x_\alpha}(u^j{}_k)\,, \\[2ex] \widehat{B} \cdot \nabla U = \sum_{\beta=1}^{n} \sum_{j=1}^{m} B^i{}_j{}^\beta \dfrac{\partial}{\partial x_\beta}(u^j{}_k) = B^i{}_j{}^\beta \dfrac{\partial}{\partial x_\beta}(u^j{}_k)\,, \\[2ex] \nabla \cdot (\widetilde{A}\nabla U) = \sum_{\alpha,\beta=1}^{n} \sum_{j=1}^{m} \dfrac{\partial}{\partial x_\alpha}\left\{ A_\alpha{}^i{}_j{}^\beta \dfrac{\partial}{\partial x_\beta}(u^j{}_k)\right\} = \dfrac{\partial}{\partial x_\alpha}\left\{ A_\alpha{}^i{}_j{}^\beta \dfrac{\partial}{\partial x_\beta}(u^j{}_k)\right\}\,, \\[2ex] i,k = 1,2,\cdots,m, \quad \alpha,\beta = 1,2,\cdots,n\,. \end{cases}$$

Here $P = (p^i{}_j)$ is a given $m \times m$ matrix of class $C(G_1)$, $\widehat{B} = B^i{}_j{}^\beta$ is an $m \times m \times n$ third order tensor of class $C(G_1)$, and $\widetilde{A} = A_\alpha{}^i{}_j{}^\beta$ is an $n \times m \times m \times n$ fourth order tensor of class $C^1(G_1)$. In addition we also suppose P to be symmetric and $\widetilde{A}$ to be symmetric and positive semidefinite throughout G_1, i. e.,

$$A_\alpha{}^i{}_j{}^\beta = A_\beta{}^i{}_j{}^\alpha = A_\beta{}^j{}_i{}^\alpha\,,$$

$$\widehat{Z}^T \widetilde{A} \widehat{Z} = Z^\alpha{}_i{}^k A_\alpha{}^i{}_j{}^\beta Z_\beta{}^j{}_k \geqslant 0\,,$$

for every $\widehat{Z} = Z_\beta{}^j{}_k$. The inequality in (1) is understood to mean the resulting $m \times m$ matrix function is negative semidefinite at each point of G_1.

We recall that if u is a solution of the scalar ordinary differential equation

$$u'' + p(x)u = 0$$

such that $u(x) \neq 0$ on some interval I, then setting $v = u' u^{-1}$, one finds v to be a solution of the Riccati equation

$$v' + v^2 + p(x) = 0\,.$$

In Theorem 1 below we shall establish the analog of this elementary fact for inequation (1) by means of an extended Riccati matrix transformation. Such Riccati matrix equations have been used extensively in ordinary differential equations, c.f. [1], [4], [9], and [10].

LEMMA A. *If U is any solution of the equation in* (1) *such that the $n \times n$ matrix function $U^T \widehat{B} \cdot \nabla U$ is symmetric in G_1, and if*

(3)
$$E = U^T \widetilde{A} \nabla U - \nabla U^T \widetilde{A} U,$$

then $\nabla \cdot E \equiv 0$.

PROOF. From the symmetry of $\widetilde{A}$ and P, a direct calculation using (1) shows that

$$\nabla \cdot E = \nabla U^T \widetilde{A} \nabla U + U^T \nabla \cdot (\widetilde{A} \nabla U) - \nabla \cdot (\nabla U^T \widetilde{A}) U - \nabla U^T \widetilde{A} \nabla U$$

$$= 2[\nabla U^T \widehat{B}^T U - U^T \widehat{B} \cdot \nabla U] + U^T (P^T - P) U$$

$$= -2[U^T \widehat{B} \cdot \nabla U - (U^T \widehat{B} \cdot \nabla U)^T].$$

Since $U \in C^2(G_1)$, $\nabla \cdot E \in C^1(G_1)$. By hypothesis $U^T \widehat{B} \cdot \nabla U$ is symmetric so that $\nabla \cdot E \equiv 0$.

When $n = 1$, this result is well known and it implies in particular that E is constant. In the special case where E is identically zero, the solution U is sometimes called self-conjugate, c.f. [4]. Following this usage we shall also refer to any solution for which $E \equiv 0$ self-conjugate, c.f. [8] and [14].

THEOREM 1. *Let U be a solution of* (1) *such that $U(x)$ is nonsingular at every $x \in G_1$. Define*

(4)
$$\widehat{Z} = (\nabla U) U^{-1}.$$

Then $\widehat{Z}$ satisfies the Riccati inequation

(5)
$$\nabla \cdot (\widetilde{A} \widehat{Z}) + 2\widehat{B} \cdot \widehat{Z} + \widetilde{A} \widehat{Z} * \widehat{Z} + P \leqslant 0,$$

where $\widetilde{A} \widehat{Z} * \widehat{Z}$ denotes the contracted multiplication

$$\sum_{\alpha=1}^{n} A_{\alpha}{}^{i}{}_{j}{}^{\beta} Z_{\beta}{}^{j}{}_{s} Z_{\alpha}{}^{s}{}_{k}.$$

Suppose $\widehat{Z} = Z_{\alpha}{}^{i}{}_{j}$ is an $n \times m \times m$ third order tensor function of class $C^1(G_1)$ which satisfies the Riccati equation

(6)
$$\nabla \cdot (\widetilde{A} \widehat{Z}) + 2\widehat{B} \cdot \widehat{Z} + \widetilde{A} \widehat{Z} * \widehat{Z} + P = 0,$$

and the compatibility conditions

$$(7) \qquad \frac{\partial}{\partial x_\alpha}(Z_{\beta}{}^i{}_j) = \frac{\partial}{\partial x_\beta}(Z_{\alpha}{}^i{}_j)$$

$$Z_{\beta}{}^i{}_j Z_{\alpha}{}^j{}_k = Z_{\alpha}{}^i{}_j Z_{\beta}{}^j{}_k .$$

Then the equation

$$(8) \qquad \nabla \cdot (\widetilde{A} \nabla U) + 2\widehat{B} \cdot \nabla U + PU = 0$$

has a solution U in G_1.

PROOF. The first statement of the theorem follows directly from the definition of $\widehat{Z}$ and (1). To show the second part we suppose that $\widehat{Z} = Z_{\alpha}{}^i{}_j$ is a solution of (6) for which (7) holds. Consider now the system of first order partial differential equations

$$(9) \qquad \nabla U = \widehat{Z} U ,$$

or in component form,

$$(9') \qquad \frac{\partial}{\partial x_\alpha}(u^i{}_k) = Z_{\alpha}{}^i{}_j u^j{}_k .$$

According to the theorem of Frobenius [4], a necessary and sufficient condition for (9′) to be solvable is that

$$(10) \qquad \frac{\partial}{\partial x_\beta}(Z_{\alpha}{}^i{}_j u^j{}_k) = \frac{\partial}{\partial x_\alpha}(Z_{\beta}{}^i{}_j u^j{}_k) ,$$

provided $\widehat{Z} \in C^1(G_1)$. Expanding both sides of (10) we see that (7) is indeed sufficient for (9′) to have solution $(u^i{}_k)$. A direct calculation using (6) and (9) shows that

$$\begin{aligned}
\nabla \cdot (\widetilde{A} \nabla U) &= \nabla \cdot [(\widetilde{A}\widehat{Z})U] \\
&= [\nabla \cdot (\widetilde{A}\widehat{Z})]U + (\widetilde{A}\widehat{Z}) \nabla U \\
&= -[2\widehat{B} \cdot \widehat{Z} + \widetilde{A}\widehat{Z} * \widehat{Z} + P]U + (\widetilde{A}\widehat{Z} * \widehat{Z})U \\
&= -2\widehat{B} \cdot (\widehat{Z}U) - PU \\
&= -2\widehat{B} \cdot (\nabla U) - PU
\end{aligned}$$

so that (8) follows.

COROLLARY 1.1. *Let U be a solution of* (1) *such that $U(x)$ is nonsingular and self-conjugate at each $x \in G_1$, i.e.,*

$$(11) \qquad\qquad U^T \widetilde{A} \nabla U \equiv \nabla U^T \widetilde{A} U .$$

If $\widehat{Z}$ is defined by (4), *then $\widehat{Z}$ satisfies the Riccati inequation*

$$(12) \qquad\qquad \nabla \cdot (\widetilde{A}\widehat{Z}) + 2\widehat{B} \cdot \widehat{Z} + \widehat{Z}^T \widetilde{A} \widehat{Z} + P \leqslant 0 .$$

PROOF. Since U^{-1} exists for every $x \in G_1$, (4) and (11) together imply that

$$\widetilde{A}\widehat{Z} = \widehat{Z}^T \widetilde{A}$$

so that (12) follows from (5).

In the particular case where $m = 1$ inequality (1) reduces to

$$(1') \qquad\qquad \nabla \cdot (A \nabla u) + 2B \cdot \nabla u + p(x) u \leqslant 0 ,$$

where $A = (A^{\alpha\beta})$ is a symmetric $n \times n$ matrix, $B = (B^1, \cdots, B^n)$ is an n-vector, and $p(x)$ and $u(x)$ are scalar-valued functions. Furthermore, formulas (4) and (5) can be written as

$$(4') \qquad\qquad z_\alpha = u^{-1} \frac{\partial u}{\partial x_\alpha}, \quad \alpha = 1, \cdots, n$$

and

$$(5') \qquad\qquad \frac{\partial}{\partial x_\alpha} (A^{\alpha\beta} z_\beta) + 2B^\alpha z_\alpha + A^{\alpha\beta} z_\alpha z_\beta + p(x) \leqslant 0$$

respectively. In this case we can strengthen the second half of Theorem 1 to read as follows:

COROLLARY 1.2. *Suppose $Z = (z_1, \cdots, z_n)$ is a vector field of class $C^1(G_1)$ for which* (5') *and the compatibility conditions*

$$(7') \qquad\qquad \frac{\partial}{\partial x_\alpha} (z_\beta) = \frac{\partial}{\partial x_\beta} (z_\alpha), \quad \alpha, \beta = 1, \cdots, n$$

are satified. Then there exists a scalar-valued function $u(x) \neq 0$ in G which is also a solution of (1').

To see this we merely let $u(x) = \exp f(x)$ in (4') and note that f must satisfy the conditions

$$\frac{\partial f}{\partial x_\alpha} = z_\alpha, \quad \alpha = 1, \cdots, n.$$

In other words we must find a scalar field f whose gradient is the given smooth vector field $Z = (z_1, \cdots, z_n)$. As before (7') is both necessary and sufficient for the existence of such an f. The result follows.

THEOREM 2. *Let U be a solution of* (1) *such that $U(x)$ is nonsingular for each $x \in G_1$ and that* (11) *holds. Denote by*

$$(13) \qquad \Omega = \left\{ W \in [C(\overline{G}) \cap C^1(G)] : \int_G \nabla W^T \widetilde{A} \nabla W \, dx < +\infty \right\}.$$

Then for every $W \in \Omega$,

$$(14) \qquad Q[W] \geqslant M[W] + 2 \int_G W^T(\widehat{B} \cdot \widehat{Z}) W \, dx + \int_{\partial G} W^T(\eta \cdot \widetilde{A}\widehat{Z}) W \, ds,$$

where

$$(15) \qquad \left| \begin{array}{l} Q[W] = \int_G [\nabla W^T \widetilde{A} \nabla W - W^T P W] \, dx, \\[2mm] M[W] = \int_G (\nabla W - \widehat{Z} W)^T \widetilde{A}(\nabla W - \widehat{Z} W) \, dx, \end{array} \right.$$

and $\eta = (\eta^1, \cdots, \eta^n)$ denotes the outward pointing unit normal at ∂G.

PROOF. From (15) we have

$$M[W] = \int_G [\nabla W^T \widetilde{A} \nabla W + (\widehat{Z} W)^T \widetilde{A}(\widehat{Z} W)] \, dx$$

$$- \int_G [\nabla W^T \widetilde{A}(\widehat{Z} W) + (\widehat{Z} W)^T \widetilde{A} \nabla W] \, dx.$$

To evaluate the last integral we note that

$$\nabla \cdot (W^T \widetilde{A} \widehat{Z} W) = \nabla W^T (\widetilde{A} \widehat{Z}) W + W^T \nabla \cdot (\widetilde{A} \widehat{Z}) W + W^T (\widetilde{A} \widehat{Z}) \nabla W$$

so that

$$(16) \qquad M[W] = \int_G [\nabla W^T \widetilde{A} \nabla W - \nabla \cdot (W^T \widetilde{A} \widehat{Z} W)]\, dx$$

$$+ \int_G W^T [\nabla \cdot (\widetilde{A} \widehat{Z}) + \widehat{Z}^T \widetilde{A} \widehat{Z}]\, W dx.$$

By hypothesis $U(x)$ is nonsingular and self-conjugate so that Cor. 1.1 and Gauss' Theorem can be applied to (16) to yield

$$M[W] \leqslant \int_G (\nabla W^T \widetilde{A} \nabla W - W^T P W)\, dx$$

$$- 2 \int_G W^T (\widehat{B} \cdot \widehat{Z})\, W dx - \int_{\partial G} W^T (\eta \cdot \widetilde{A} \widehat{Z})\, W ds$$

which is the desired result.

COROLLARY 2.1. *Suppose $\widetilde{A}$ is positive definite. Let U be a self-conjugate solution of the self-adjoint equation*

$$(17) \qquad \nabla \cdot (\widetilde{A} \nabla U) + PU = 0$$

such that $U(x)$ is nonsingular for every $x \in G_1$. Then for every $W \in \Omega$

$$(18) \qquad Q[W] \geqslant \int_{\partial G} W^T [\eta \cdot \widetilde{A}(\nabla U) U^{-1}]\, W ds,$$

with equality holding if, and only if, $W \equiv UK$, where K is constant matrix.

PROOF. In this case $\widehat{B} \equiv 0$ and inequality (1) is an equality so that (14) becomes

$$Q[W] = M[W] + \int_{\partial G} W^T [\eta \cdot \widetilde{A}(\nabla U) U^{-1}]\, W ds.$$

Since $\widetilde{A}$ is symmetric and positive definite, $M[W] \geqslant 0$ for all $W \in \Omega$ so that (18) follows. Moreover, equality can hold in (18) if, and only if, $M[W]=0$, i.e.,

$$\nabla W \equiv (\nabla U) U^{-1} W.$$

Rewriting this we get

$$[U^{-1}(\bigtriangledown U)U^{-1}]W - U^{-1}\bigtriangledown W \equiv 0\,,$$

or

$$\bigtriangledown(U^{-1}W) \equiv 0$$

so that we must have $U^{-1}W \equiv$ const.

We remark that the integral relation (14) contains as a special case an identity of Beesack ([2], p. 479, formula (1.4)) in which $m=n=1$ and $\widehat{B} \equiv 0$. Moreover, inequality (18) is recognized as an inequality of the Wirtinger type. In order to state this more explicitly we shall suppose a boundary condition for (17) of the form

(19)
$$U = 0 \qquad \text{on } \Gamma_1$$
$$U_\sigma = g(x)U \quad \text{on } \Gamma_2\,,$$

where Γ_1 and Γ_2 are disjoint sets whose union is ∂G. Here U_σ is the conormal derivative

$$U_\sigma = \eta \cdot \widetilde{A}\,\bigtriangledown U = \eta^\alpha A_\alpha{}^i{}_j{}^\beta \frac{\partial}{\partial x_\beta}(u^j{}_k)\,.$$

COROLLARY 2.2. *Let U be a self-conjugate solution of* (17) *and* (19) *such that $U(x)$ is nonsingular at every $x \in G \cup \Gamma_2$. Then for every $W \in \Omega$ for which $W=0$ on Γ_1 we have*

(18′)
$$\int_G [\bigtriangledown W^T \widetilde{A}\,\bigtriangledown W - W^T PW]\,dx \geqslant \int_{\Gamma_2} W^T g(x)\,W ds\,,$$

with equality holding if, and only if, $W \equiv UK$.

When $m=n=1$ (18′) again reduces to an inequality of Beesack [2]. Moreover, this inequality also contains an elementary proof of the extremal property of the first eigenfunction associated with the operator defined by

$$\bigtriangledown \cdot (\widetilde{A}\bigtriangledown U) + PU = \lambda U$$

and the boundary condition (19), providing of course such a U exists. It follows that the comparison theorems of Kreith [5] and Swanson [12, Theorem 1] as

well as bounds on eigenvalues may be deduced from this inequality directly. As an illustration of this we shall prove an extension of Kreith's comparison theorem [5] for equation (17). To this end let $\widetilde{A}_1$ and $\widetilde{A}_2$ be two tensor functions and P_1 and P_2 be two matrix functions satisfying the same assumptions as $\widetilde{A}$ and P. As before let Γ_1 and Γ_2 be two disjoint sets whose union is the boundary ∂G. Further we assume two matrix functions g_1 and g_2 to be defined on Γ_2.

THEOREM 3. *Let U be a self-conjugate solution of*

$$(20) \qquad \bigtriangledown \cdot (\widetilde{A}_1 \bigtriangledown U) + P_1 U = 0, \quad x \in G$$

such that

$$\begin{cases} U = 0 & on \; \Gamma_1 \\ U_{\sigma_1} = g_1 U & on \; \Gamma_2. \end{cases}$$

Let W be a solution of

$$(21) \qquad \bigtriangledown \cdot (\widetilde{A}_2 \bigtriangledown W) + P_2 W = 0, \quad x \in G$$

such that

$$\begin{cases} W = 0 & on \; \Gamma_1 \\ W_{\sigma_2} = g_2 W & on \; \Gamma_2. \end{cases}$$

If

$$(22) \quad \int_G [\bigtriangledown W^T(\widetilde{A}_2 - \widetilde{A}_1) \bigtriangledown W + W^T(P_1 - P_2)W]\,dx \geqslant \int_{\Gamma_2} W^T(g_2 - g_1)\,W ds,$$

then either $\det U(x_0) = 0$ for some $x_0 \in G \cup \Gamma_2$ or $W \equiv UK$.

PROOF. Suppose the contrary and let $\det U(x) \neq 0$ for all $x \in G \cup \Gamma_2$. Then by Cor. 2.2,

$$(23) \qquad \int_G [\bigtriangledown W^T \widetilde{A}_1 \bigtriangledown W - W^T P_1 W]\,dx \geqslant \int_{\Gamma_2} W^T g_1 W ds,$$

where equality holds if, and only if, $W \equiv UK$. On the other hand, if we multiply (21) by W^T and integrate by parts, we get

$$\int_G [\bigtriangledown W^T \widetilde{A}_2 \bigtriangledown W - W^T P_2 W]\,dx = \int_{\Gamma_2} W^T g_2 W ds.$$

Combining this with (23) we arrive at

$$\int_G [\nabla W^T (\widetilde{A}_1 - \widetilde{A}_2) \nabla W + W^T (P_2 - P_1) W] \, dx \geqslant \int_{\Gamma_2} W^T (g_1 - g_2) \, W \, ds$$

which contradicts (22) unless equality holds. However, by Cor. 2.2, this situation occurs if, and only if, $W \equiv UK$, and the result follows.

In view of Theorem 2 we can state the following

COROLLARY 3.1. *Suppose V is a self-conjugate sub-solution of* (20), *i.e., a solution of*

(24) $$\nabla \cdot (\widetilde{A}_1 \nabla V) + P_1 V \leqslant 0$$

for which (11) *and the same boundary conditions hold. Further we suppose that strict inequality holds in* (24) *for at least one interior point of G. If W is the same as above then* $\det V(x_0) = 0$ *for some* $x_0 \in G$, *and $W \equiv VK$ only if V is in fact a solution of* (20).

We remark that by using the method of Swanson [13] one can also extend these results to the case of unbounded domains G. For a comparison theorem derived by means of a generalized Picone identity, c.f. [3], [6] and [8].

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