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CLASS NUMBERS OF IMAGINARY ABELIAN NUMBER FIELDS, I

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In this paper we give some applications of Brauer-Siegel theorem. Especially we show that there exist only a finite number of imaginary abelian number fields whose class numbers are not greater than any given integer. In the following k always denotes a normal algebraic number field of degree n. Let d, h and R be absolute value of its discriminant, class number and regulator, respectively. Then Brauer-Siegel theorem (or its proof by Brauer)[1] shows:

For any given positive number ε, there exists a positive number δ (depending only on ε) such that

$$\left|\frac{\log hR}{\log\sqrt{d}}-1\right|<\varepsilon$$

if $n/\log d<\delta$. We apply Brauer-Siegel theorem in this form, though the usual form suffices for abelian number fields.

1. We assume that k is imaginary from now on. Let k_0 be the maximal real subfield of k. Let d_0, h_0 and R_0 be absolute value of discriminant of k_0, and similarly. Then $h_1=h/h_0$ is a positive integer which is called the first factor of the class number of k. Let d_{k/k_0} be the relative discriminant, and let d_1 be its absolute norm. Then

$$d=d_1\cdot d_0^2$$

holds.

If k_0 is also normal, the unit group of k has a subgroup of finite index generated by the units of k_0 and the roots of unity in k. If we put this index q, it holds

1) For Brauer-Siegel theorem, see [2] or [5, Chapter IX].

$$qR = 2^{n/2-1}R_0\,.$$

THEOREM 1. *Let k be an imaginary normal extension of the rational number field Q of degree n. Let d be the absolute value of its discriminant. We assume that the maximal real subfield k_0 of k is also normal over Q. Then the first factor h_1 of the class number is arbitrarily large, if $n/\log d$ is sufficiently small.*

PROOF. Above remark shows

$$hR \leqq h_1 h_0 \cdot 2^{n/2} R_0\,.$$

It is known that there exists a positive constant C such that

$$\frac{\log h_0 R_0}{\log \sqrt{d_0}} \leqq C$$

for any k_0 [2, formula (6)] (The left hand side is 0/0 for $k_0 = Q$, but no trouble is caused in the following). If $\log d_0/\log d < 1/2C$, it holds

$$\frac{\log hR}{\log\sqrt{d}} \leqq \frac{\log h_1 h_0 2^{n/2} R_0}{\log \sqrt{d}}$$

$$= \frac{\log h_1 + n \log 2/2 + \log h_0 R_0}{\log \sqrt{d}}$$

$$\leqq \frac{\log h_1}{\log \sqrt{d}} + \frac{n \log 2}{\log d} + \frac{\log h_0 R_0}{2C \log \sqrt{d_0}}$$

$$\leqq \frac{\log h_1}{\log \sqrt{d}} + \frac{n \log 2}{\log d} + \frac{1}{2}\,.$$

The second inequality does not hold if $k_0 = Q$ or $\log h_0 R_0 < 0$ in the above. But the last inequality holds also in these cases.
If $n/\log d$ is sufficiently small, Brauer-Siegel theorem shows that

$$\frac{1}{3} \leqq \frac{\log h_1}{\log \sqrt{d}}.$$

If $\log d_0/\log d \geqq 1/2C$, it holds

$$\frac{\log hR}{\log \sqrt{d}} \leqq \frac{\log h_1}{\log \sqrt{d}} + \frac{n \log 2}{\log d} + \frac{\log h_0 R_0}{2 \log \sqrt{d_0}}.$$

If $n/\log d$ is sufficiently small, $n/\log d_0$ is small enough. Then it also holds

$$\frac{1}{3} \leqq \frac{\log h_1}{\log \sqrt{d}}.$$

Theferore it holds

$$\frac{1}{3} \log \sqrt{d} \leqq \log h_1,$$

if $n/\log d$ is sufficiently small. Then $\log \sqrt{d}$ is large, and $\log h_1$ becomes large arbitrarily.

REMARK 1. Let k be a (not necessarily normal) totally imaginary algebraic number field which has a totally real subfield k_0 such that $[k : k_0] = 2$. If k runs over such fields with bounded degrees, h_1 goes to infinity as shown by Brauer-Siegel theorem for non-normal case and the above argument.

REMARK 2. Let $\{k_1, k_2, \cdots\}$ be a sequence of fields satisfying the conditions of Theorem 1 such that $n/\log d$ goes to zero. Then the above proof shows that $\liminf \log h_1/\log \sqrt{d} \geqq \frac{1}{2}$.

2. In this section we prove our main theorem.

THEOREM 2. *For any integer N, there exist only a finite number of imaginary abelian number fields whose first factors h_1 of class numbers are not greater than N.*[2)]

By Theorem 1 we only need to prove the following proposition.

PROPOSITION 1. *$n/\log d$ is sufficiently small for almost all abelian number fields (not necessarily imaginary).*

PROOF. Let k denote an abelian number field. Let $p_1, p_2, \cdots, p_r$ be the prime numbers which are ramified in k. We prove the proposition in two steps.

2) This theorem can also be proved by applying Landau's estimate for $L(1, \chi)$ and by Siegel's theorem for imaginary quadratic fields.

(1) If $p_1p_2\cdots p_r$ is sufficiently large, $n/\log d$ is small enough. Let e_i be the ramification index of p_i. Then any prime divisor $\mathfrak{p}_i$ of p_i appears in the different of k with exponent $\geqq e_i-1$. Therefore

$$d \geqq \prod_{p_i} \prod_{\mathfrak{p}_i | p_i} N_{k/Q}\mathfrak{p}_i^{e_i-1} = \prod_{p_i} p_i^{n(1-1/e_i)}$$

holds. Hence it holds

$$\log d \geqq n \sum_i (1-1/e_i) \log p_i \geqq \frac{n}{2} \sum_i \log p_i .$$

This proves our assertion.

(2) We fix prime numbers $p_1=2,\ p_2,\cdots,p_r$. Let k be an abelian number field in which at most above primes are ramified. If n is fixed, there exist only a finite number of such fields of degrees $\leqq n$. So it suffices to prove that $n/\log d$ is arbitrarily small if n is sufficiently large. Such a field is contained in some field of the $p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$-th roots of unity. So its Galois group over Q is isomorphic to a subgroup of a direct sum of cyclic groups of orders $2, p_2-1,\cdots,p_r-1, p_1^{a_1-2}, p_2^{a_2-1},\cdots,p_r^{a_r-1}$. If n is sufficiently large, the Galois group has a cyclic factor group of order $p_i^{2f_i}$ for some i and some large integer f_i. Then k contains a cyclic subfield F' of degree $p_i^{2f_i}$. As the p_i-parts of the ramification indices of p_j for $j\neq i$ are bounded, we can assume that they are smaller than $p_i^{f_i}$. Then only a prime p_i can be ramified in the subfield F of F' of degree $p_i^{f_i}$. Therefore F is contained in the field of the $p_i^{a_i}$-th roots of unity. Taking its degree into account, F is contained in the field E of the $p_1^{f_1+2}$-th roots of unity if $i=1$, and it is contained in the field E of the $p_i^{f_i+1}$-th roots of unity if $i\neq 1$. Then it is known [5, IV. Theorem 3] that

$$d_E = \begin{cases} p_i^{p_i^{f_i+1}(f_i+1)} & \text{if } i=1 \\ p_i^{p_i^{f_i}[(f_i+1)(p_i-1)-1]} & \text{if } i\neq 1 . \end{cases}$$

It holds

$$d_E = \begin{cases} N_{F/Q}d_{E/F}\cdot d_F^{p_i} & \text{if } i=1 \\ N_{F/Q}d_{E/F}\cdot d_F^{p_i-1} & \text{if } i\neq 1 . \end{cases}$$

If $i\neq 1$, p_i is tamely ramified in E/F, so $N_{F/Q}d_{E/F}=p_i^{p_i-2}$. In the case $i=1$, let ζ denote a primitive 2^{f_1+2}-th root of unity. Let ζ^σ be its conjugate over F. As $\zeta^{\sigma^2}=\zeta$, ζ^σ is equal to one of $-\zeta, \zeta^{-1}$and $-\zeta^{-1}$. If $\zeta^\sigma=-\zeta$, F contains ζ^2 which is a primitive 2^{f_1+1}-th root of unity. Then F is not cyclic for $f_1 \geqq 2$. This is a contradiction. Hence ζ^σ is equal to ζ^{-1} or $-\zeta^{-1}$. Then the relative different of E/F is generated by $\zeta-\zeta^{-1}$ or $\zeta+\zeta^{-1}$. As $1-\zeta$, $1+\zeta$, $1-\sqrt{-1}\,\zeta$ and $1+\sqrt{-1}\,\zeta$ are all generators of the prime divisor of 2, $N_{F/Q}d_{E/F}=2^2$ holds. Therefore it holds

$$p_i \log d_F + p_i \log p_i \geqq \frac{1}{2} f_i p_i^{f_i+1} \log p_i \,.$$

Hence

$$\frac{[F:Q]}{\log\ d_F} = \frac{p_i^{f_i}}{\log\ d_F} \leqq \frac{3}{f_i\ \log\ p_i}.$$

holds for large f_i. As k contains F, and as

$$\log\ d = \log\ N_F d_{k/F} + [k:F]\ \log\ d_F\,,$$

it holds

$$\frac{n}{\log\ d} \leqq \frac{[k:F]\,[F:Q]}{[k:F]\log\ d_F} \leqq \frac{3}{f\ \log\ p_i}.$$

If we take f_i sufficiently large, $n/\log\ d$ is small enough. This proves the proposition and also proves Theorem 2.

REMARK. Proposition 1 can also be proved for relative abelian extensions. Step (2) in the proof is given by Hasse's conductor-discriminant formula.

3. We give some remarks relating to preceding sections. First we show the index q in section 1 is not greater than 2. This has been known for abelian case by [3, § 20].

PROPOSITION 2. *Let k be a (not necessarily normal) totally imaginary algebraic number field of finite degree. We assume that k has a totally real subfield k_0 such that $[k:k_0]=2$. Let E and E_0 be unit groups of k and k_0 respectively. Let W be a group of the roots of unity in k. Then it holds*

$$(E:WE_0) \leqq 2\,.$$

PROOF. As E and E_0 have the same rank, E/WE_0 is a finite abelian group. Let $\varepsilon \in E$ be such that $\varepsilon^l \in WE_0$ for odd prime l. We put

$$\varepsilon^l = \zeta\varepsilon_0\,, \qquad \zeta \in W, \varepsilon_0 \in E_0\,.$$

Then it holds

$$(\varepsilon\varepsilon^\sigma)^l = (\zeta\zeta^\sigma)\cdot\varepsilon_0{}^2\,,$$

where ε^σ is the conjugate of ε over k_0. We note that $\zeta\zeta^\sigma$ is equal to 1 or -1 as k_0 is real. If we put $\varepsilon_1 = \pm\varepsilon\varepsilon^\sigma$ according as the sign of $\zeta\zeta^\sigma$, it holds $\varepsilon_0 = \varepsilon_2{}^l$ for

$$\varepsilon_2 = \varepsilon_1{}^{(l+1)/2}/\varepsilon_0\,.$$

Then $\varepsilon/\varepsilon_2$ is a root of unity, so holds $\varepsilon \in WE_0$. Now let $\eta \in E$ be such that $\eta^4 \in WE_0$. We put

$$\eta^4 = \zeta_1\eta_0\,, \qquad \zeta_1 \in W, \eta_0 \in E_0\,.$$

If $\zeta_1\zeta_1{}^\sigma$ is equal to -1, -1 is a square in k_0, and a contradiction. Hence if we put $\eta_1 = \eta\eta^\sigma, \eta_1{}^2 = \pm\eta_0$ holds and η^2/η_1 is a root of unity. Then $\eta^2 \in WE_0$ holds, and this proves that E/WE_0 is of type $(2, 2, \cdots, 2)$. Let ε and η be elements of E not contained in WE_0. We put

$$\varepsilon^2 = \zeta^i\varepsilon_0 \text{ and } \eta^2 = \zeta^j\eta_0\,,$$

where ε_0 and η_0 are elements of E_0 and ζ is a generator of W. Substituting elements of same classes modulo WE_0 for ε and η, we can assume i and j are equal to 0 or 1. If $\sqrt{-1}$ is not contained in k, $-\zeta$ is a square in k. So we can assume $i=j=0$. Then ε^2 and η^2 are in E_0, and ε/η is in E_0 by Kummer theory. Hence $(E:WE_0) \leqq 2$ holds in this case. If $\sqrt{-1}$ is in k, and if i or j is equal to 0, $\varepsilon/\sqrt{-1}$ or $\eta/\sqrt{-1}$ is in E_0. Then ε or η is contained in WE_0, which contradicts to the hypothesis. If $\sqrt{-1}$ is contained in k, and if $i=j=1$, ε/η is contained in WE_0. This shows $(E:WE_0) \leqq 2$ holds also in this case.

Above proposition shows that

$$2^{n/2-2}R_0 \leqq R \leqq 2^{n/2-1}R_0$$

holds in the situation of the proposition. Now let k be a field of the l^f-th roots of unity, where l is a prime number. Then it is known that $q=1$. It holds

$$d = l^{l^{f-1}[f(l-1)-1]},$$

and for the norm of relative discriminant it holds

$$d_1 = N_k(\zeta - \zeta^{-1}) = \begin{cases} l \text{ if } l \neq 2 \\ 2^2 \text{ if } l = 2, \end{cases}$$

where ζ is a primitive l^f-th root of unity. So $\log d_1/\log d_0$ goes to 0 when l goes to infinity or f goes to infinity. If l goes to infinity for fixed f or f goes to infinity for fixed l, it holds

$$\frac{\log hR}{\log \sqrt{d}} = \frac{\log h_1}{\log \sqrt{d}} + \frac{(n/2-1) \log 2}{\log \sqrt{d}} + \frac{\log h_0R_0}{\log \sqrt{d}}$$

$$\rightarrow \frac{\log h_1}{\log \sqrt{d}} + \frac{\log h_0R_0}{2 \log \sqrt{d_0}} \rightarrow \frac{\log h_1}{\log \sqrt{d}} + \frac{1}{2}.$$

Hence it holds

$$\frac{\log h_1}{\log \sqrt{d}} \rightarrow \frac{1}{2}.$$

Let $f=1$, then $d=l^{l-2}$ and

$$\log h_1 \sim \frac{1}{4} \log d = \frac{l-2}{4} \log l \sim \frac{l}{4} \log l$$

holds when l goes to infinity. This has been obtained by Ankeny-Chowla [1] and Siegel [6]. Now we fix a prime l and let f go to infinity. Then

$$\log h_1 \sim \frac{1}{2} \log \sqrt{d} = \frac{1}{4} l^{f-1} [f(l-1)-1] \log l$$

holds. Iwasawa [4] has shown that the l-part h_l of h_1 is of the form

$$h_l = l^{e_f} \qquad \text{with } e_f = \mu l^f + \lambda f + c,$$

where μ, λ and c are constants. Then it holds

$$\log h_1 - \log h_l \rightarrow \infty, \text{ as } f \rightarrow \infty.$$

Therefore non-l-part of h_1 goes to infinity as f goes to infinity.

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