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# UNIQUENESS OF THE NORMAL CONNECTIONS AND CONGRUENCE OF ISOMETRIC IMMERSIONS

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Let $f$ be an isometric immersion of a Riemannian manifold $M$ into a Riemannian manifold $\tilde{M}$ of constant curvature and let $N_f$ be the normal bundle. The normal connection is a metric linear connection in the bundle $N_f$ which satisfies the Codazzi equation for the second fundamental form $\alpha$. The first aim of the present paper is to prove the following result: in the case where the first normal space $N_1(x)$ coincides with the normal space $N(x)$ at each point $x$ of $M$, a metric linear connection in the bundle $N_f$ which satisfies the equation of Codazzi type coincides with the normal connection. This fact can be derived as a special case of the general treatment of connections in the bundles of normal spaces of higher orders as given by O. Kowalski [6], but simplicity of the result under our assumption is remarkable. Indeed, we shall define the torsion tensor of an arbitrary linear connection in the bundle $N_f$ in such a way that the normal connection can be characterized, still under the assumption that $N_1(x) = N(x)$ for every point $x$, as a unique metric linear connection in $N_f$ whose torsion tensor is 0. This then is an analogue of the uniqueness theorem of the Riemannian connection as a linear metric connection with zero torsion in the tangent bundle of a Riemannian manifold.

The second aim of the paper is to apply the result above to obtain congruence theorems for isometric immersions which satisfies $N_1(x) = N(x)$ for all points $x$ or whose second fundamental forms are parallel. Isometric immersions into a Euclidean space with parallel second fundamental forms have been essentially determined by D. Ferus [2], [3], [4], and our result has a close bearing on part of the proof of the main result in [4].

**1. Uniqueness of the normal connection.** Let $f$ be an isometric immersion of a Riemannian manifold $M$ into a Riemannian manifold $M$ of constant curvature. For each point $x$ of $M$, let $N(x)$ be the normal space and let $N_f$ be the normal bundle. The second fundamental form

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$\alpha$ defines for each point $x$ of $M$ a bilinear symmetric mapping of $T_x(M) \times T_x(M)$ into $N(x)$. Thus $\alpha$ is a section of the bundle Hom $(T(M) \otimes T(M), N_f)$, where $T(M)$ is the tangent bundle of $M$. It is known that $\alpha$ satisfies the equation of Codazzi:

$$(1) \qquad (\nabla_X^* \alpha)(Y, Z) = (\nabla_Y^* \alpha)(X, Z)$$

where $X, Y, Z$ are tangent vectors to $M$ and $(\nabla_X^* \alpha)(Y, Z)$ is defined by

$$(2) \qquad (\nabla_X^* \alpha)(Y, Z) = \nabla_X^{\perp} \alpha(Y, Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z) .$$

Here $\nabla^{\perp}$ denotes the normal connection in the bundle $N_f$. (For the standard terminology on isometric immersions, see [5].)

More generally, let $\hat{\nabla}$ be an arbitrary linear connection in the vector bundle $N_f$. We define the *torsion tensor* of $\hat{\nabla}$ as the section $\hat{T}$ of

$$\mathrm{Hom}\,(T(M) \otimes T(M) \otimes T(M),\ N_f)$$

defined by

$$(3) \qquad \hat{T}(X, Y)Z = (\hat{\nabla}_X^* \alpha)(Y, Z) - (\hat{\nabla}_Y^* \alpha)(X, Z) ,$$

where $\hat{\nabla}_X^* \alpha$ is defined by

$$(4) \qquad (\hat{\nabla}_X^* \alpha)(Y, Z) = \hat{\nabla}_X \alpha(Y, Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z) ,$$

in other words, using $\hat{\nabla}$ instead of $\nabla^{\perp}$ in (2). Thus the torsion tensor of $\hat{\nabla}$ is identically 0 if and only if $\hat{\nabla}$ satisfies the equation of Codazzi's type:

$$(5) \qquad (\hat{\nabla}_X^* \alpha)(Y, Z) = (\hat{\nabla}_Y^* \alpha)(X, Z) .$$

There is another way of defining the torsion tensor $\hat{T}$. We interpret $\alpha$ as a section $\rho$ of the bundle

$$\mathrm{Hom}\,(T(M),\ \mathrm{Hom}\,(T(M),\ N_f))$$

by setting

$$\rho(X)Y = \alpha(X, Y) .$$

Given a linear connection $\hat{\nabla}$ in the bundle $N_f$, we have a linear connection in the vector bundle Hom $(T(M), N_f)$ over $M$ defined by

$$(\hat{\nabla}_X \tau)(Y) = \hat{\nabla}_X(\tau(Y)) - \tau(\nabla_X Y) ,$$

where $\tau$ is a section of Hom $(T(M), N_f)$ and $X$, $Y$ are vector fields on $M$. Using this connection we have

$$(6) \qquad \hat{T}(X, Y) = \hat{\nabla}_X(\rho(Y)) - \hat{\nabla}_Y(\rho(X)) - \rho([X, Y]) .$$

To prove (6) it is sufficient to note

$$\begin{aligned}\hat{\nabla}_X(\rho(Y))Z &= \hat{\nabla}_X(\rho(Y)Z) - \rho(Y)(\nabla_X Z)\\ &= \hat{\nabla}_X(\alpha(Y, Z)) - \alpha(\nabla_X Z, Y)\,,\\ \hat{\nabla}_Y(\rho(X))Z &= \hat{\nabla}_Y(\alpha(X, Z)) - \alpha(\nabla_Y Z, X)\end{aligned}$$

and

$$\rho([X, Y])Z = \alpha([X, Y], Z) = \alpha(\nabla_X Y, Z) - \alpha(\nabla_Y X, Z)\,.$$

Now (6) gives another expression for the torsion tensor of $\hat{\nabla}$ which is quite similar to the expression for the torsion tensor of a linear connection on a manifold:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]\,.$$

We shall now prove

THEOREM 1. *Let $f$ be an isometric immersion of a Riemannian manifold $M$ into a Riemannian manifold $\tilde{M}$ of constant curvature. Assume that the first normal space $N_1(x)$ coincides with the normal space $N(x)$ at each point $x$. Then a metric linear connection $\hat{\nabla}$ with zero torsion tensor in the normal bundle $N_f$ coincides with the normal connection $\nabla^{\perp}$.*

PROOF. For any section $\xi$ of $N_f$ and for any vector field $X$ on $M$, we set

$$K(X)\xi = \nabla^{\perp}_X\xi - \hat{\nabla}_X\xi\,.$$

Then $K$ is a section of $\mathrm{Hom}\,(T(M), \mathrm{Hom}\,(N_f, N_f))$. For any $X \in T_x(M)$, $K(X)$ is a skew-symmetric endomorphism of the normal space $N(x)$, because both $\nabla^{\perp}_X$ and $\hat{\nabla}_X$ are metric connections in $N_f$. Since both connections satisfy the equation of Codazzi's type, we obtain from (1) and (5)

$$K(X)\alpha(Y, Z) = K(Y)\alpha(X, Z) \tag{7}$$

for any $X, Y, Z \in T_x(M)$. Using skew-symmetry of the endomorphism of the form $K(X)$, $X \in T_x(M)$, we obtain

$$\begin{aligned}\langle K(Z)\alpha(X_1, X_2), \alpha(Y_1, Y_2)\rangle + \langle \alpha(X_1, X_2), K(Z)\alpha(Y_1, Y_2)\rangle &= 0\\ \langle K(X_1)\alpha(X_2, Y_1), \alpha(Z, Y_2)\rangle + \langle \alpha(X_2, Y_1), K(X_1)\alpha(Z, Y_2)\rangle &= 0\\ \langle K(X_2)\alpha(Z, X_1), \alpha(Y_1, Y_2)\rangle + \langle \alpha(Z, X_1), K(X_2)\alpha(Y_1, Y_2)\rangle &= 0\\ -\langle K(Y_1)\alpha(Y_2, Z), \alpha(X_1, X_2)\rangle - \langle \alpha(Y_2, Z), K(Y_1)\alpha(X_1, X_2)\rangle &= 0\\ -\langle K(Y_2)\alpha(Z, X_1), \alpha(X_2, Y_1)\rangle - \langle \alpha(Z, X_1), K(Y_2)\alpha(X_2, Y_1)\rangle &= 0\,.\end{aligned}$$

Adding up these five equations and making use of (7) we obtain

$$2\langle K(Z)\alpha(X_1, X_2), \alpha(Y_1, Y_2)\rangle = 0\,.$$

Since $N_1(x)$ is spanned by vectors of the form $\alpha(X, Y)$, $X, Y \in T_x(M)$, our assumption $N_1(x) = N(x)$ implies that

$$\langle K(Z)\xi, \eta\rangle = 0 \quad \text{for all} \quad \xi, \eta \in N(x) .$$

This means $K(X) = 0$, that is, $\nabla^{\perp}_X = \hat{\nabla}_X$ for every $X \in T_x(M)$.

**2. Congruence of isometric immersions.** In this section, let $M$ be one of the standard models of Riemannian manifolds of constant curvature, namely, the Euclidean space $E^m$, the sphere $S^m$, the real projective space $P^m(R)$ of constant positive curvature, and the hyperbolic space $H^m$.

THEOREM 2. *Let $f$ and $\hat{f}$ be two isometric immersions of a connected Riemannian manifold $M$ into $\tilde{M}$. If there exists a bundle isomorphism $\Phi$ of the normal bundle $N_f$ for $f$ onto the normal bundle $N_{\hat{f}}$ for $\hat{f}$ which preserves the metrics and the second fundamental forms and if $N_1(x) = N(x)$ for all $x \in M$ for the immersion $f$, then $f$ and $\hat{f}$ are congruent (by an isometry of $\tilde{M}$).*

PROOF. Define a linear connection $\hat{\nabla}$ in the bundle $N_f$ as follows. For any section $\xi$ of $N_f$ and for any vector field $X$ on $M$, we set

$$\hat{\nabla}_X\xi = \Phi^{-1}(\hat{\nabla}^{\perp}_X\Phi(\xi)) ,$$

where $\hat{\nabla}^{\perp}$ denotes the normal connection in $N_f$ for $f$. Since $\Phi$ preserves metrics, $\hat{\nabla}$ is a metric connection. Since $\Phi$ preserves the second fundamental forms, i.e. $\Phi\alpha(X, Y) = \hat{\alpha}(X, Y)$, the connection $\hat{\nabla}$ in $N_f$ satisfies the equation of Codazzi's type for $\alpha$. By Theorem 1 we conclude that $\hat{\nabla}$ coincides with $\nabla^{\perp}$. This means that $\Phi$ preserves the normal connections. By a well-known result in [7], $f$ and $\hat{f}$ differ by an isometry of $\tilde{M}$.

We now consider isometric immersions with parallel second fundamental forms.

THEOREM 3. *Let $f$ and $\hat{f}$ be two isometric immersions of a Riemannian manifold $M$ into $\tilde{M}$ such that their second fundamental forms $\alpha$ and $\hat{\alpha}$ are parallel. If there exists a bundle isomorphism $\Phi$ of $N_f$ onto $N_{\hat{f}}$ which preserves the metrics and the second fundamental forms, then $f$ and $\hat{f}$ are congruent.*

PROOF. Since $\nabla^*\alpha = 0$, the first normal spaces $N_1(x)$ for $f$ are parallel relative to the normal connection in $N_f$. By a theorem of Erbacher [1], there is a complete totally geodesic submanifold $M_1$ of $\tilde{M}$ such that $f(M) \subset M_1$. (Note that the theorem is valid for $P^m(R)$ as well. Also note that $M_1$ is again one of the model spaces.) The normal bundle for the immersion $f$ of $M$ into $M_1$ is denoted by $N_f^1$, its fibers being $N_1(x)$. Similarly, there is a complete totally geodesic submanifold $\hat{M}_1$ of $\tilde{M}$ such that $\hat{f}(M) \subset \hat{M}_1$, for which the normal bundle $N_{\hat{f}}^1$ has fibers $\hat{N}_1(x)$, namely, the first normal spaces for $\hat{f}$.

The bundle isomorphism $\Phi$ of $N_f$ onto $N_{\hat{f}}$ induces an isomorphism of $N_f^1$ onto $N_{\hat{f}}^1$ which preserves the metrics and the second fundamental forms. Thus by Theorem 2, we conclude that there is an isometry $\phi$ of $M_1$ onto $\hat{M}_1$ such that $\phi \circ f = \hat{f}$. We may now extend $\phi$ to an isometry of $M$ onto itself.

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