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SIMPLY CONNECTED CLOSED SMOOTH 5-MANIFOLDS WITH EFFECTIVE SMOOTH $U(2)$-ACTIONS

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0. Introduction. In [1] Asoh classified connected closed smooth manifolds of dimension less than 5, which admit a non-trivial smooth $SU(2)$-action up to $SU(2)$-equivariant diffeomorphisms. In [5] Nakanishi investigated an equivariant classification of smooth $SO(3)$-actions on closed connected orientable smooth 5-manifolds such that the orbit space is an orientable surface. In [4] Hudson classified simply connected closed 5-manifolds with $SO(3)$-actions admitting at least one singular orbit up to $SO(3)$-equivariant diffeomorphisms. The purpose of this note is to prove the following theorem.

THEOREM. *Suppose that a compact simply connected smooth 5-manifold M^5 without boundary admits an effective smooth $U(2)$-action $\phi: U(2)\times M^5 \to M^5$. Then M^5 is $U(2)$-equivariantly diffeomorphic to one of the following $U(2)$-manifolds.*

(I) *S^5 on which $U(2)$ acts by*

$$\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} \zeta \\ z \\ w \end{pmatrix}\right) \mapsto \begin{pmatrix} \zeta(ad-bc)^k \\ (az+cw)(ad-bc)^{-m} \\ (bz+dw)(ad-bc)^{-m} \end{pmatrix}$$

where $(k, 2m-1)=1$.

(II) $M_k^5 = U(2) \underset{O(2)}{\times} D^2 \cup_{f_k} S^3 \times D^2$,

where the $O(2)$-action on 2-disk D^2 is the natural one, $U(2)$ acts on $S^3\times D^2$ by

$$\left(A, \left(\begin{pmatrix} z \\ w \end{pmatrix}, \zeta\right)\right) \mapsto \left(A\begin{pmatrix} z \\ w \end{pmatrix}(\det A)^{-2k}, \quad (\det A)^2\zeta\right).$$

$U(2)/O(1)$ is $U(2)$-equivariantly diffeomorphic to $\partial(U(2)\underset{O(2)}{\times} D^2)$ by $[A]\mapsto [A, 1]$ and the attaching map f_k is given by

$$f_k([A]) = \left(A\begin{pmatrix} 1 \\ 0 \end{pmatrix}(\det A)^{-2k}, \quad (\det A)^2\right).$$

$$\text{(III)}\quad M\left\{\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} A & C \\ B & D \end{pmatrix}\right\} = U(2) \underset{T^2}{\times} L(b, d)\ ,$$

where a, b, c, d, A, B, C, D *are integers satisfying the conditions*: $ad - bc = \pm 1$ *and* $(A + B, b) = 1 = (A - B, C - D)$ *and where* $L(b, d)$ *is a* 3-*dimensional lens space for* $b \neq 0$, *while we put*

$$L(0, \pm 1) = \left\{\begin{pmatrix} z \\ w \end{pmatrix} \in C^2;\ (|z| - R)^2 + |w|^2 = r^2\right\}$$

for fixed r, R *with* $0 < r < R$. *The* T^2-*action on* $L(b, d)$ *is as follows. Put*

$$\begin{pmatrix} X & U \\ Y & V \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

If $b \neq 0$, *then*

$$\left(\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}, \begin{bmatrix} z \\ w \end{bmatrix}\right) \mapsto \begin{bmatrix} zx^Y y^{-X} \\ w(x^B y^{-A})^\delta \end{bmatrix},$$

where $x^b = \xi$, $y^b = \eta$ *and* $\delta = ad - bc$. *If* $b = 0$, *then*

$$\left(\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}\right) \mapsto \begin{pmatrix} z\xi^{-Y}\eta^X \\ w\xi^{-V}\eta^U \end{pmatrix},$$

where $(|z| - R)^2 + |w|^2 = r^2$.

REMARK. (i) In (II), M_k^5 is diffeomorphic to the Wu-manifold $SU(3)/SO(3)$ by [2, Theorem 2.3] and in particular M_0^5 is $U(2)$-equivariantly diffeomorphic to $SU(3)/SO(3)$ admitting the following $U(2)$-action:

$$(A, [X]) \mapsto \left[\begin{pmatrix} (\det A)^{-1} & 0 \\ 0 & A \end{pmatrix} X\right],$$

where $X \in SU(3)$ and $[X] \in SU(3)/SO(3)$.

(ii) In (II), denote ϕ by ϕ_k in case $M^5 = M_k^5$. If $j \neq k$, then ϕ_j is not weakly equivariant to ϕ_k.

The remainder of this note is divided into three sections. In Section 1, we state necessary lemmas and show that the principal orbits of the $U(2)$-action are of condimension one or two. In Section 2, we show that if the codimension of the principal orbits is one, then M^5 is $U(2)$-equivariantly diffeomorphic to either (I) with $k \neq 0$, (II) or (III) with $AD - BC \neq 0$. In Section 3, we show that if the codimension of the principal orbits is two, then M^5 is $U(2)$-equivariantly diffeomorphic to either (I) with $k = 0$ or (III) with $AD - BC = 0$.

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1. Preliminaries. In the first place, we define symbols and notations.

$G_k^3 = \{A \in U(2);\ (\det A)^k = 1\} \quad (k \neq 0)$.

$T^2 = \left\{\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix} \in U(2);\ |\xi| = 1 = |\eta|\right\}$.

$N^2 = T^2 \cup \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T^2$.

$G_{m,n}^1 = \left\{\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix} \in U(2);\ \xi^n = \eta^m\right\} \quad (m \neq 0 \quad \text{or} \quad n \neq 0)$.

$C_{p,q,k} = \left\{\begin{pmatrix} \omega^p & 0 \\ 0 & \omega^q \end{pmatrix} \in U(2);\ \omega^k = 1\right\} \quad (k \neq 0 \quad \text{and} \quad (p, q, k) = 1)$.

$\Delta_k = C_{1,-1,2k+1} \cup \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} C_{1,-1,2k+1} \quad (k \geqq 1)$.

$S^{2m-1} = \{\boldsymbol{a} \in \boldsymbol{C}^m;\ \|\boldsymbol{a}\| = 1\}$ $(m \geqq 1)$, where $\boldsymbol{C}^m$ is the complex vector space of m-dimensional complex column vectors.

$\boldsymbol{Z}_k = \boldsymbol{Z}/k\boldsymbol{Z}$ or $\{\omega \in S^1;\ \omega^k = 1\}$ $(k \neq 0)$, where $\boldsymbol{Z}$ is the additive group of all rational integers.

[ ] denotes an equivalence class of a certain equivalence relation.

For integers k, q with $(k, q)=1$ and $k\neq 0$, a free $\boldsymbol{Z}_k$-action $\phi_{k,q}\colon S^3 \times \boldsymbol{Z}_k \to S^3$ is defined by

$$\phi_{k,q}\left(\begin{pmatrix} z \\ w \end{pmatrix},\ \omega\right) = \begin{pmatrix} z\,\omega \\ w\,\omega^q \end{pmatrix} .$$

We denote the orbit space of the action $\phi_{k,q}$ by $L(k, q)$. It is called a 3-dimensional lens space. For convenience, we put

$$L(0, q) = \left\{\begin{pmatrix} z \\ w \end{pmatrix} \in \boldsymbol{C}^2;\ (|z| - R)^2 + |w|^2 = r^2\right\} ,$$

where $q = \pm 1$ and $0 < r < R$. Then $L(0, q)$ is diffeomorphic to $S^1 \times S^2$.

Let G be a compact Lie group and let K be a closed subgroup. Let ρ be a smooth K-action on a smooth manifold X. Then K acts on $G \times X$ by $(h, (g, x)) \mapsto (gh^{-1}, \rho(h, x))$. We denote the orbit space of this K-action by $G \times_K X$ or $(G \times X)/K$. Define a canonical G-action on $G \times_K X$ by $(g',$

$[g, x]) \mapsto [g'g, x]$. This smooth G-manifold $G \underset{K}{\times} X$ is called a twisted product. By (H) we denote the type of the principal isotropy subgroup of ϕ, that is, every principal isotropy subgroup of ϕ is conjugate to H. The $U(2)$-action ϕ is effective if and only if each principal isotropy subgroup does not contain any proper normal subgroup of $U(2)$. The proper normal subgroups of $U(2)$ are G_k^3 $(k \neq 0)$ or subgroups of $G_{1,1}^1$, where $G_{1,1}^1$ is the center of $U(2)$. Thus ϕ is effective if and only if $H \cap G_{1,1}^1 = \{E_2\}$, where E_2 is the unit matrix of $U(2)$.

LEMMA 1.1. *Suppose that a closed subgroup H of $U(2)$ satisfies $H \cap G_{1,1}^1 = \{E_2\}$. Then H is conjugate to one of the following subgroups:*

$$C_{p,q,k} \quad ((p-q, k) = 1) , \qquad \Delta_k , \qquad G_{r,r+1}^1 ,$$

where k, p, q, r are some integers.

PROOF. Since the closed subgroup of $U(2)$ whose dimension is greater than one contains a non-trivial subgroup of $G_{1,1}^1$ except $\{E_2\}$, H must be a finite subgroup or a 1-dimensional closed subgroup of $U(2)$. It is easy to see that if H is a 1-dimensional closed subgroup, then H is conjugate to $G_{r,r+1}^1$ for some r. We also see easily that if H is a finite cyclic group of $U(2)$, then H is conjugate to $C_{p,q,k}$ for some p, q, k with $(p-q, k) = 1$.

Thus we have only to prove that non-cyclic H is conjugate to Δ_k for some k. Now let H be a non-cyclic finite subgroup of $U(2)$. Moreover suppose that $H \cap G_{1,1}^1 = \{E_2\}$. We define two homomorphisms $\omega: S^1 \times SU(2) \to U(2)$, $\pi: S^1 \times SU(2) \to SU(2)$ by $\omega(\alpha, A) = \alpha A$, $\pi(\alpha, A) = A$. The homomorphism ω is surjective and its kernel $\omega^{-1}(E_2)$ is equal to $\{(1, E_2), (-1, -E_2)\}$. For a subgroup G of $U(2)$ we put $\tilde{G} = \omega^{-1}(G)$. The following facts are clear or well known:

(a) $\pi \mid \tilde{H}$ is injective, i.e., $\tilde{H} \cong \pi(\tilde{H})$,

(b) $g \in \tilde{H}$ and $\pi(g) = -E_2$ implies $g = (-1, -E_2)$.

(c) Up to conjugacy, a non-cyclic finite subgroup of $SU(2)$ is isomorphic to one of the following:

$$D_{4n}^* = \{x, y \mid x^2 = (xy)^2 = y^n, x^4 = 1\} \quad \text{(binary dihedral group, } n \geqq 2) ,$$
$$T^* = \{x, y \mid x^2 = (xy)^3 = y^3, x^4 = 1\} \quad \text{(binary tetrahedral group)} ,$$
$$O^* = \{x, y \mid x^2 = (xy)^3 = y^4, x^4 = 1\} \quad \text{(binary octahedral group)} ,$$
$$I^* = \{x, y \mid x^2 = (xy)^3 = y^5, x^4 = 1\} \quad \text{(binary icosahedral group)} .$$

By (a), (b), (c), the subgroup H is isomorphic to one of D_{2n} (dihedral group), T (tetrahedral group), O (octahedral group), I (icosahedral group).

If H is isomorphic to D_{2n}, then $\tilde{H}$ is isomorphic to D_{4n}^* by π. Suppose

$x, y \in \pi(\tilde{H})$ satisfy $x^2 = (xy)^2 = y^n$, $x^4 = E_2$ and let $(u, x), (v, y) \in \tilde{H}$ $(\subset S^1 \times SU(2))$. Then by (b), $u^2 = u^2v^2 = v^n = -1$ since $x^2 = (xy)^2 = y^n = -E_2$. Hence we see that $u = \pm\sqrt{-1}$, $v = -1$ and $n \equiv 1 \pmod 2$. Up to conjugacy in $SU(2)$, we can put

$$y = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, \quad \lambda = \exp(\pi\sqrt{-1}/n) .$$

Let

$$X = \omega(u, x) \quad Y = \omega(v, y) = \begin{pmatrix} -\lambda & 0 \\ 0 & -\bar{\lambda} \end{pmatrix} .$$

Then $X^2 = E_2 \neq X$ and $X \notin G^1_{1,1}$.
Hence we may put

$$X = \begin{pmatrix} a & \bar{z} \\ z & -a \end{pmatrix} \qquad (a^2 + |z|^2 = 1 , \quad a \in R) .$$

Since $YXY = X$, we see that $a = 0$ and $|z| = 1$. Thus

$$X = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \bar{z} \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} Y \begin{pmatrix} 1 & 0 \\ 0 & \bar{z} \end{pmatrix} .$$

Hence H is conjugate to a subgroup of $U(2)$ which is generated by matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \quad (\alpha = \exp(2\pi\sqrt{-1}/(2k+1)) \quad \text{for some} \quad k \geqq 1) .$$

Therefore if H is isomorphic to the dihedral group D_{2n} $(n \geqq 2)$, then H is conjugate to Δ_k.

Next suppose that H is isomorphic to T (tetrahedral group). Then $\pi(\tilde{H})$ is isomorphic to T^*. Suppose $x, y \in \pi(\tilde{H})$ satisfy $x^2 = (xy)^3 = y^3$, $x^4 = E_2$ and let $(u, x), (v, y) \in \tilde{H}$ $(\subset S^1 \times SU(2))$. Then since $x^2 = (xy)^3 = y^3 = -E_2$, we have $u^2 = (uv)^3 = v^3 = -1$ by (b). This is a contradiction. Thus H is not isomorphic to the tetrahedral group.

In the same manner, we can show that H is isomorphic neither to the octahedral group O nor to the icosahedral group I. Hence if H is a non-cyclic finite subgroup, then H is conjugate to Δ_k for some $k \geqq 1$.

q.e.d.

By Lemma 1.1, we have the following corollary.

COROLLARY 1.2. *The codimension of the principal orbits of ϕ is one or two.*

We omit the proof of the following lemma.

LEMMA 1.3. (i) *A 3-dimensional closed subgroup of* $U(2)$ *is* G_k^3 *for some* $k \neq 0$ *and is normal in* $U(2)$.

(ii) *A 2-dimensional closed subgroup of* $U(2)$ *is conjugate to* T^2 *or* N^2.

(iii) *The identity component of a 1-dimensional closed subgroup of* $U(2)$ *is conjugate to* $G_{p,q}^1$ *for some* p, q *with* $(p, q) = 1$.

Define a smooth map $p_{j,k}: U(2) \to L(k, 1)$ for j, k with $j^2 + k^2 \neq 0$ by

$$p_{j,k}\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right) = \left[\begin{pmatrix} a \\ b \end{pmatrix}(ad - bc)^{-j/k}\right] \quad (k \neq 0) ,$$

$$p_{j,0}\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right) = \begin{pmatrix} (R + r(|a|^2 - |b|^2))(ad - bc)^j \\ r(2\bar{a}b) \end{pmatrix} .$$

Then $p_{j,k}$ induces the following $U(2)$-diffeomorphism.

LEMMA 1.4. $U(2)/G_{j,k-j}^1 \cong L(k, 1)$ *for* j, k *with* $j^2 + k^2 \neq 0$.

The following lemmas are proved easily.

LEMMA 1.5. *Assume that the integers* m, n, k, p, q *satisfy the conditions* $k \neq 0$, $(p, q, k) = 1$ *and* $(m, n) = 1$. *Then* $G_{m,n}^1 \cdot C_{p,q,k} = G_{mj,nj}^1$, *where* $j = k \Big/ \left(k, \begin{vmatrix} m & p \\ n & q \end{vmatrix}\right)$, *and there exists a complex representation* ρ: $G_{mj,nj}^1 \to S^1$ *whose kernel is* $C_{p,q,k}$. *Moreover, the representation* ρ *is unique up to complex-conjugation.*

LEMMA 1.6. *Let* K *be a* 1*-dimensional closed subgroup of* $U(2)$ *and let* H *be a finite cyclic subgroup of* $K \cap T^2$. *Suppose that the order of* H *is greater than one,* $H \cap G_{1,1}^1 = \{E_2\}$ *and* K/H *is connected. Then* K *is a subgroup of* T^2, *or there exists* $g \in N^2$ *such that* $H = gO(1)g^{-1}$ *and* $K = gO(2)g^{-1}$.

LEMMA 1.7. *For* $a, b, c, d \in \boldsymbol{Z}$ *with* $ad - bc = \pm 1$, *define a diffeomorphism* $f: S^1 \times S^1 \to S^1 \times S^1$ *by* $f(z, w) = (z^a w^b, z^c w^d)$. *The closed smooth* 3*-manifold* $S^1 \times D^2 \cup_f S^1 \times D^2$ *formed by attaching* $S^1 \times D^2$ *to* $S^1 \times D^2$ *on their boundaries under* f *is diffeomorphic to the* 3*-dimensional lens space* $L(b, d)$.

PROOF. In the first place we treat the case $b \neq 0$. Put

$$U_1 = \left\{\begin{bmatrix} Z \\ W \end{bmatrix} \in L(b, d);\ |Z| \leqq |W|\right\} , \quad U_2 = \left\{\begin{bmatrix} Z \\ W \end{bmatrix} \in L(b, d);\ |Z| \geqq |W|\right\} .$$

Define $\phi_s: S^1 \times D^2 \to U_s$ $(s = 1, 2)$ by

$$\phi_1(z, w) = \left[(1 + |w|^2)^{-1/2}\begin{pmatrix}\bar{w}\zeta^{-a}\\ \zeta^{-\delta}\end{pmatrix}\right], \quad \phi_2(z, w) = \left[(1 + |w|^2)^{-1/2}\begin{pmatrix}\zeta^{-1}\\ w\zeta^{-d}\end{pmatrix}\right],$$

where $\zeta^b = z$ and $\delta = ad - bc$. Then since $\phi_1 = \phi_2 \circ f$ on $S^1 \times S^1$, $\phi_1 \cup_f \phi_2$ induces a diffeomorphism of $S^1 \times D^2 \cup_f S^1 \times D^2$ onto $L(b, d)$. Next we treat the case $b = 0$. Put

$$U_1 = \left\{\begin{pmatrix}Z\\ W\end{pmatrix} \in \boldsymbol{C}^2;\ (|Z| - R)^2 + |W|^2 = r^2 \quad\text{and}\quad |Z| \geqq R\right\},$$

$$U_2 = \left\{\begin{pmatrix}Z\\ W\end{pmatrix} \in \boldsymbol{C}^2;\ (|Z| - R)^2 + |W|^2 = r^2 \quad\text{and}\quad |Z| \leqq R\right\}.$$

Define $\phi_s\colon S^1 \times D^2 \to U_s$ $(s = 1, 2)$ by

$$\phi_1(z, w) = \begin{pmatrix}(R + r(1 - |w|^2)^{1/2})z^a\\ rz^c w^d\end{pmatrix}, \quad \phi_2(z, w) = \begin{pmatrix}(R - r(1 - |w|^2)^{1/2})z\\ rw\end{pmatrix},$$

where for the convenience of notations, even if $|w| < 1$, we regard w^{-1} as $\bar{w}$ since $d = \pm 1$. Then since $\phi_1 = \phi_2 \circ f$ on $S^1 \times S^2$, $\phi_1 \cup_f \phi_2$ induces a diffeomorphism of $S^1 \times D^2 \cup_f S^1 \times D^2$ onto $L(0, d)$. q.e.d.

We omit the proof of the following lemma.

LEMMA 1.8. *For $a, b, c, d, A, B, C, D \in \boldsymbol{Z}$ with $ad - bc = \pm 1$, define the same T^2-action on $L(b, d)$ that is defined in Part* (III) *of the main theorem. Then*

$$M\left\{\begin{pmatrix}a & c\\ b & d\end{pmatrix}, \begin{pmatrix}A & C\\ B & D\end{pmatrix}\right\} = U(2) \underset{T^2}{\times} L(b, d)$$

is simply connected if and only if $(A + B, b) = 1$. The canonical $U(2)$-action on

$$M\left\{\begin{pmatrix}a & c\\ b & d\end{pmatrix}, \begin{pmatrix}A & C\\ B & D\end{pmatrix}\right\} = U(2) \underset{T^2}{\times} L(b, d)$$

is effective if and only if $(A - B, C - D) = 1$.

Let G be a compact Lie group. Let X_1, X_2 be compact connected manifolds on which G acts smoothly. Assume that ∂X_1 is equivariantly diffeomorphic to ∂X_2 as G-manifolds. Denote by $M(f) = X_1 \cup_f X_2$ the compact connected G-manifold formed from X_1 and X_2 by the identification of points of ∂X_1 and ∂X_2 under a G-diffeomorphism $f\colon \partial X_1 \to \partial X_2$. The following lemma is described in [8, p. 161].

LEMMA 1.9. *Let $f, f'\colon \partial X_1 \to \partial X_2$ be G-invariant diffeomorphisms.*

Then $M(f)$ is equivariantly diffeomorphic to $M(f')$ as G-manifolds, if f is G-diffeotopic to f'.

In particular, if ∂X_1 and ∂X_2 are both equivariantly diffeomorphic to G/H as G-manifolds for a closed subgroup H of G, then every G-invariant diffeomorphism $\partial X_1 = G/H \to G/H = \partial X_2$ is the right translation by an element of $W_G(H) = N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G. We thus have the following corollary.

COROLLARY 1.10. *If $\alpha, \beta \in W_G(H)$ belong to the same component of $W_G(H)$, then $M(\alpha) = X_1 \cup_\alpha X_2$ is G-equivariantly diffeomorphic to $M(\beta) = X_1 \cup_\beta X_2$.*

Let K_1, K_2 be two closed subgroups of G such that $H \subset K_1 \cap K_2$ and let ρ_s $(s = 1, 2)$ be a k_s-dimensional orthogonal representation of K_s such that K_s/H is diffeomorphic to $O(k_s)/O(k_s - 1)$ by ρ_s. Let L be a closed subgroup of G. Suppose that $K_1 \cup K_2 \subset L$. Then the following lemma is proved easily.

LEMMA 1.11. *For each $\alpha \in W_L(H) \subset W_G(H)$,*

$$G \underset{L}{\times} (L \underset{K_1}{\times} D^{k_1} \cup_\alpha L \underset{K_2}{\times} D^{k_2})$$

is G-equivariantly diffeomorphic to $G \underset{K_1}{\times} D^{k_1} \cup_\alpha G \underset{K_2}{\times} D^{k_2}$.

2. The proof when ϕ admits principal orbits of codimension 1. Then the principal isotropy subgroup of ϕ is finite. We denote the type of the principal isotropy subgroup by (H). Using some results due to Uchida [8, Sections 1 and 5] concerning manifolds which admit a Lie group action with codimension one orbits, we easily see the following facts:

Each principal orbit of ϕ is $U(2)$-equivariantly diffeomorphic to $U(2)/H$ and there are only two singular orbits $U(2)(x_1) \cong U(2)/K_1$, $U(2)(x_2) \cong U(2)/K_2$ where K_1, K_2 are some closed subgroups of $U(2)$ such that $H \subset K_1 \cap K_2$. In fact, there are two slice representations ρ_1, ρ_2 of K_1, K_2 respectively and there is an element $\alpha \in W_{U(2)}(H)$ such that M^5 is $U(2)$-equivariantly diffeomorphic to

$$M(\alpha, \rho_1, \rho_2) = U(2) \underset{K_1}{\times} D^{k_1} \cup_\alpha U(2) \underset{K_2}{\times} D^{k_2} .$$

(2.1) $K_s \neq U(2)$. In fact, $U(2)$ does not act transitively on $\partial D^5 = S^4$.

(2.2) K_s is not 2-dimensional. In fact, neither T^2 nor N^2 acts transitively on $\partial D^3 = S^2$.

(2.3) K_s is not finite, since $U(2)/K_s$ is a singular orbit.

(2.4) Both K_1 and K_2 are not 3-dimensional. For otherwise, $U(2)/K_1$

and $U(2)/K_2$ would be $U(2)$-equivariantly diffeomorphic to S^1. Since in the manifold M^5 the codimension of the orbit $U(2)/K_2$ is greater than 2, $U(2)/K_1$ is simply connected by [8, Lemma 2.2.3]. This is a contradiction.

(2.5) Suppose that K_1 is 3-dimensional and K_2 is 1-dimensional. Then $K_1 = G_k^3$ for some k with $k \neq 0$. Since $\partial D^4 = S^3$ is not homeomorphic to G_k^3/Δ_j for any $j \geqq 1$, K is not conjugate to Δ_j for any $j \geqq 1$. First we investigate the slice representation $\rho: G_k^3 \to O(4)$ of the isotropy subgroup G_k^3. Since the identity component of G_k^3 is $SU(2)$ and $K_1 = G_k^3$ acts transitively on $\partial D^4 = S^3$ by ρ, we have $G_k^3/H = S^3$. Hence $SU(2) \cap H = \{E_2\}$. Thus the restriction $\rho\,|\,SU(2)$ of ρ to $SU(2)$ is a real representation induced by the natural $SU(2)$-action on $\boldsymbol{C}^2$. From this fact it follows that for some p with $|p| < |k|$ we have

$$\rho(A)\cdot\begin{pmatrix} z \\ w \end{pmatrix} = A\begin{pmatrix} z \\ w \end{pmatrix}(\det A)^{-p}\,, \quad \text{where} \quad A \in G_k^3 \quad \text{and} \quad \begin{pmatrix} z \\ w \end{pmatrix} \in \boldsymbol{C}^2 = \boldsymbol{R}^4\,.$$

Hence $H = C_{p,1-p,k}$ with $(2p-1, k) = 1$. Since $X_1 = U(2) \underset{K_1}{\times} D^4$, $X_2 = U(2) \underset{K_2}{\times} D^2$, $X_1 \cap X_2 = U(2)/H$ and $X_1 \cup X_2 = M^5$, $H_1(X_1) = H_1(U(2)/G_k^3) = H_1(S^1) = \boldsymbol{Z}$, $H_1(X_2) = H_1(U(2)/K_2)$, $H_1(X_1 \cap X_2) = H_1(U(2)/C_{p,1-p,k}) = \boldsymbol{Z}$ and $H_1(X_1 \cup X_2) = H_1(M^5) = 0$. By Mayer-Vietoris homology sequence of X_1 and X_2, $H_1(X_1 \cap X_2) \to H_1(X_1) \oplus H_1(X_2) \to H_1(X_1 \cup X_2)$ is an exact sequence. Hence $H_1(X_2) = H_1(U(2)/K_2) = 0$. Now we study K_2. Since $H_1(U(2)/O(2)) \neq 0$, K_2 is not conjugate to $O(2)$. If $H = C_{p,1-p,k} = \{E_2\}$, then we may regard K_2 as a closed subgroup of T^2. If $k \geqq 2$, then by Lemma 1.6, $K_2 \subset T^2$. Hence $K_2 = G^1_{m,n} \cdot C_{p,1-p,k}$ for some m, n with $(m, n) = 1$. By Lemma 1.5, $K_2 = G^1_{mj,nj}$ where $j = k/(k, m-(m+n)p)$. Since $U(2)/K_2 = L(j(m+n), 1)$ by Lemma 1.4, $H_1(X_2) = H_1(X_2) = H_1(L(j(m+n), 1)) = \boldsymbol{Z}_{j(m+n)} = 0$. Hence $|j(m+n)| = 1$. Thus $j = \pm 1$ and $m + n = \pm 1$. Since in general, $G^1_{\mu,\nu} = G^1_{-\mu,-\nu}$ for $(\mu, \nu) = 1$, it is no loss of generality to suppose that $j = 1$ and $m + n = 1$. Then $m \equiv p \pmod{k}$. Since $(2p-1, k) = 1$, we have $(2m-1, k) = 1$. By Lemma 1.5, there is a unique slice representation $\sigma: G^1_{m,1-m} \to O(2)$ whose kernel is $H = C_{p,1-p,k} = C_{m,1-m,k}$. By Lemma 1.10, there exists at most one $U(2)$-equivariant diffeomorphism class of such M^5 for k, m with $(2m-1, k) = 1$, since $W_{U(2)}(H) = T^2$ (or $U(2)$) is connected. We see easily that in this case M^5 is $U(2)$-equivariantly diffeomorphic to S^5 of (I) with $k \neq 0$ in the main theorem.

(2.6) Suppose that both K_1 and K_2 are 1-dimensional. Then the principal isotropy subgroup H is conjugate to $C_{p,q,k}$ for some p, q, k with $(p-q, k) = 1$ or to Δ_k for some $k \geqq 1$. Now we consider the Mayer-Vietoris homology sequence of X_1 and X_2. Since $H_1(X_1 \cap X_2) = H_1(U(2)/H)$, $H_1(X_s) = H_1(U(2) \underset{K_s}{\times} D^2) = H_1(U(2)/K_s)$ $(s = 1, 2)$ and $H_1(X_1 \cup X_2) = H_1(M^5)$, the sequence $H_1(U(2)/H) \xrightarrow{\mu} H_1(U(2)/K_1) \oplus H_1(U(2)/K_2) \xrightarrow{\nu} H_1(M^5)$ is exact. Since M^5 is simply connected, we have $H_1(M^5) = 0$. Therefore μ is surjective. The principal isotropy subgroup H is not conjugate to Δ_k for any $k \geqq 1$. In fact, assume that H is conjugate to Δ_k for some $k \geqq 1$. Then $H_1(U(2)/H) = \boldsymbol{Z}$ and K_2 must be conjugate to one of the following subgroups:

$$L_1 = G^1_{1,-1} \cup \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} G^1_{1,-1}\,, \quad L_2 = G^1_{2,-2} \cup \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} G^1_{2,-2}\,,$$

$$L_3 = G^1_{2k+1,2k+1} \cup \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} G^1_{2k+1,2k+1}\,.$$

$H_1(U(2)/L_1) = \boldsymbol{Z}$, $H_1(U(2)/L_2) = \boldsymbol{Z} \oplus \boldsymbol{Z}_2$ and $H_1(U(2)/L_3) = \boldsymbol{Z}_4$. Thus μ: $H_1(U(2)/H) \to H_1(U(2)/K_1) \oplus H_1(U(2)/K_2)$ is not surjective. This is a contradiction. Hence H is a finite cyclic subgroup of $U(2)$. We may regard H as a subgroup of $T^2 \cap K_1 \cap K_2$.

(i) Assume that either K_1 or K_2 is conjugate to $O(2)$. We may put $K_1 = O(2)$. Then $H_1(X_1 \cap X_2) = H_1(U(2)/H) = \boldsymbol{Z}$, since $H = O(1)$, where $O(1) = C_{0,1,2}$ by Lemma 1.6. Since the homomorphism μ: $H_1(U(2)/H) \to H_1(U(2)/K_1) \oplus H_1(U(2)/K_2)$ is surjective, we have $H_1(U(2)/K_2) = 0$. Hence K_2 is not conjugate to $O(2)$. By Lemma 1.6, we can regard K_2 as a closed subgroup of T^2. Let the identity component of K_2 be $G^1_{a,b}$ with $(a, b) = 1$. Then $K_2 = G^1_{a,b} \cdot H$. By Lemma 1.5, $G^1_{a,b} \cdot C_{0,1,2} = G^1_{a,b}$ or $G^1_{2a,2b}$. Hence by Lemma 1.4, $U(2)/K_2 = L(a+b, 1)$ or $L(2(a+b), 1)$. Since in general $H_1(L(k, q)) = \boldsymbol{Z}_k$ and $H_1(U(2)/K_2) = 0$, we have $K_2 = G^1_{a,b}$ with $a + b = \pm 1$. Without loss of generality, we may assume that $b = 1 - a$ and a is even. Hence if $K_1 = O(2)$, then $H = O(1)$ and $K_2 = G^1_{2k,1-2k}$ for some integer k. Now the $O(2)$-action on the 2-disk D^2 whose principal isotropy subgroup is conjugate to $O(1)$ is necessarily the $O(2)$-action induced by the canonical 2-dimensional real representation and by Lemma 1.5, the $G^1_{2k,1-2k}$-action on the 2-disk D^2 whose principal isotropy subgroup is conjugate to $O(1)$ is necessarily the $G^1_{2k,1-2k}$-action induced by the following 1-dimensional complex representation

$$\left(\begin{pmatrix} \tau^{2k} & 0 \\ 0 & \tau^{1-2k} \end{pmatrix}, \zeta\right) \mapsto \tau^2\zeta$$

where $\tau \in \boldsymbol{C}$ with $|\tau| = 1$ and $\zeta \in \boldsymbol{C}$. For $\alpha \in W_{U(2)}(O(1))$. Put

$$M^5_k(\alpha) = U(2) \underset{O(2)}{\times} D^2 \cup_\alpha (U(2) \times D^2)/G^1_{2k,1-2k}\,.$$

Since $W_{U(2)}(O(1)) = T^2/O(1)$, where $O(1) = C_{0,1,2}$, we have $M^5_k(\alpha) = M^5_k([E_2])$ by

Lemma 1.10. Now we shall show that $M_k^5([E_2])$ is simply connected. Let $i_s\colon X_1 \cap X_2 \to X_s$ $(s=1, 2)$ be a natural inclusion, where $X_1 = U(2) \times_{O(2)} D^2$ and $X_2 = (U(2) \times D^2)/G^1_{2k,1-2k}$. Since the induced homomorphism $i_{1*}\colon \pi_1(X_1 \cap X_2) = \pi_1(U(2)/O(1)) \to \pi_1(X_1) = \pi_1(U(2)/O(2))$ is surjective and $\pi_1(X_2) = \pi_1(L(1, 1)) = 0$, we see that $M_k^5([E_2]) = X_1 \cup_{[E_2]} X_2$ is simply connected by van Kampen's theorem. Next we study the $U(2)$-manifold X_2. We have the following commutative diagram:

$$\begin{array}{ccc} [A] & \longmapsto & [A, 1] \\ U(2)/O(1) & \longrightarrow & (U(2) \times D^2)/G^1_{2k,1-2k} \\ \Big\downarrow f_k & & \Big\downarrow F_k \\ S^3 \times S^1 & \subset & S^3 \times D^2 , \end{array}$$

where

$$F_k([A, \zeta]) = \left(A\begin{pmatrix}1\\0\end{pmatrix}(\det A)^{-2k},\ (\det A)^2\zeta\right),$$

$$f_k([A]) = \left(A\begin{pmatrix}1\\0\end{pmatrix}(\det A)^{-2k},\ (\det A)^2\right).$$

Define a $U(2)$-action on $S^3 \times D^2$ by

$$\left(A, \left(\begin{pmatrix}z\\w\end{pmatrix}, \zeta\right)\right) \mapsto \left(A\begin{pmatrix}z\\w\end{pmatrix}(\det A)^{-2k},\ (\det A)^2\zeta\right).$$

Then F_k is a $U(2)$-equivariant diffeomorphism. Therefore if one of the two singular isotropy subgroups of the $U(2)$-action ϕ is conjugate to $O(2)$, then the manifold M^5 is $U(2)$-equivariantly diffeomorphic to

$$M_k^5 = U(2) \times_{O(2)} D^2 \cup_{f_k} S^3 \times D^2 .$$

By Mayer-Vietoris homology sequence, we have $H_2(M_k^5) = \boldsymbol{Z}_2$. Hence M_k^5 is diffeomorphic to the Wu-manifold $SU(3)/SO(3)$ by [2, Theorem 2.3]. Now put

$$S\Lambda(3) = \{L \in SU(3);\ {}^tL = L\}$$

and let $U(2)$ act on $S\Lambda(3)$ by

$$(A, X) \mapsto \begin{pmatrix}\delta^{-1} & 0\\ 0 & A\end{pmatrix} X \begin{pmatrix}\delta^{-1} & 0\\ 0 & {}^tA\end{pmatrix},$$

where $A \in U(2)$, $X \in S\Lambda(3)$ and $\delta = \det A$. Then $SU(3)/SO(3)$ admitting the $U(2)$-action in Remark in Section 0 is $U(2)$-equivariantly diffeomorphic to $S\Lambda(3)$ with the above $U(2)$-action by the map sending $[U]$ to $U{}^tU$, where $U \in SU(3)$ and $[U] \in SU(3)/SO(3)$. Denote the isotropy subgroup at $X \in SA(3)$ by $U(2)_X$. For $X \in SA(3)$, put

$$X = \begin{pmatrix} \lambda & \mu & \nu \\ \mu & \alpha & \gamma \\ \nu & \gamma & \beta \end{pmatrix}.$$

If $|\lambda| = 1$, then $U(2)_X$ is conjugate to $O(2)$. If $0 < |\lambda| < 1$, then $U(2)_X$ is conjugate to $C_{0,1,2} = O(1)$. If $\lambda = 0$, then $U(2)_X$ is conjugate to $G^1_{0,1} = U(1)$. Hence $M^5_0 = SA(3) = SU(3)/SO(3)$.

Denote ϕ by ϕ_k in case $M^5 = M^5_k$. Next we show that if $j \neq k$, then ϕ_j is not weakly equivariant to ϕ_k. Suppose that ϕ_j is weakly equivariant to ϕ_k. Then there exists an automorphism α of $U(2)$ and there exists a diffeomorphism $f\colon M^5_j \to M^5_k$ such that the following diagram is commutative:

$$\begin{array}{ccc} U(2) \times M^5_j & \xrightarrow{\phi_j} & M^5_j \\ \downarrow{\scriptstyle \alpha \times f} & & \downarrow{\scriptstyle f} \\ U(2) \times M^5_k & \xrightarrow{\phi_k} & M^5_k \end{array}.$$

The automorphism α maps the center of $U(2)$ into itself i.e., induces an automorphism of $G^1_{1,1}$. Therefore we have the following commutative diagram:

$$\begin{array}{ccc} S^1 \times M_j & \xrightarrow{\psi_j} & M_j \\ \downarrow{\scriptstyle \beta \times f} & & \downarrow{\scriptstyle f} \\ S^1 \times M_k & \xrightarrow{\psi_k} & M_k \end{array},$$

where ψ_j, ψ_k are the S^1-actions induced by the restriction of the $U(2)$-actions ϕ_j, ϕ_k to $G^1_{1,1}$ respectively and β is the automorphism of S^1 induced by α. The isotropy types of ψ_j are $(\boldsymbol{Z}_1)$, $(\boldsymbol{Z}_2)$, $(\boldsymbol{Z}_{4j-1})$ and the isotropy types of ψ_k are $(\boldsymbol{Z}_1)$, $(\boldsymbol{Z}_2)$, $(\boldsymbol{Z}_{4k-1})$. Hence $|4j - 1| = |4k - 1|$. Thus $j = k$. q.e.d.

(ii) Suppose that neither K_1 nor K_2 is conjugate to $O(2)$. By Lemma 1.6, we may assume that $K_1, K_2 \subset T^2$. Let ρ_s $(s = 1, 2)$ be a 2-dimensional real representation such that the induced K_s-action on D^2 is transitive on ∂D^2 and the kernel of ρ_s is equal to H. For $\alpha \in W_{U(2)}(H)$, let M^5 be $U(2)$-equivariantly diffeomorphic to

$$M^5_\alpha = M(\alpha, \rho_1, \rho_2) = U(2) \underset{K_1}{\times} D^2 \cup_\alpha U(2) \underset{K_2}{\times} D^2 .$$

Now if $p + q \not\equiv 0 \pmod{k}$, then the normalizer of $H = C_{p,q,k}$ in $U(2)$ is T^2.

If $p+q\equiv 0 \pmod k$ then the normalizer of $H=C_{p,q,k}$ in $U(2)$ is N^2. Hence $W_{U(2)}(H)$ is connected or has two components. If α belongs to the identity component of $W_{U(2)}(H)$, then by Corollary 1.10 and Lemma 1.11, M_α^5 is $U(2)$-equivariantly diffeomorphic to

$$U(2) \underset{T^2}{\times} (T^2 \underset{K_1}{\times} D^2 \cup_{[E_2]} T^2 \underset{K_2}{\times} D^2)\ .$$

If $W_{U(2)}(H)=N^2/H$ and α belongs to the component of $[\lambda]$, then $H=C_{1,-1,2j+1}=C_{j,j+1,2j+1}$ for some j. By the same corollary and lemma we have

$$M_\alpha^5 \cong U(2) \underset{N^2}{\times} (N^2 \underset{K_1}{\times} D^2 \cup_{[\lambda]} N^2 \underset{K_2}{\times} D^2)\ ,$$

where $\lambda=\begin{pmatrix}0&1\\1&0\end{pmatrix}\in N^2$.

Now we shall prove the following lemma.

LEMMA 2.1. *$N^2 \underset{K_1}{\times} D^2 \cup_{[\lambda]} N^2 \underset{K_2}{\times} D^2$ is N^2-equivariantly diffeomorphic to*

$$N^2 \underset{T^2}{\times} (T^2 \underset{K_1}{\times} D^2 \cup_{[E_2]} T^2 \underset{K_\lambda}{\times} D^2)\ ,$$

where $K_\lambda=\lambda K_2\lambda^{-1}$ and K_λ acts on D^2 by $\rho_\lambda(h)=\rho_2(\lambda^{-1}h\lambda)$ $(h\in K_\lambda)$. Moreover $H=\lambda H\lambda^{-1}\subset K_\lambda\subset T^2$.

PROOF. Identify N^2/H with $\partial(N^2 \underset{K}{\times} D^2)$ by the map $[A]\mapsto[A,1]$, where $K=K_1$, K_2 or K_λ. Define an N^2-diffeomorphism $\chi\colon N^2\times D^2\to N^2 \underset{K_\lambda}{\times} D^2$ by $([A,\zeta])=[A\lambda^{-1},\zeta]$. Then $(N^2 \underset{K_2}{\times} D^2, N^2/H)$ is N^2-equivariantly diffeomorphic to $(N^2 \underset{K_\lambda}{\times} D^2, N^2/H)$ by χ, where $(\chi\,|\,N^2/H)([A])=[A\lambda^{-1}]$. Hence χ induces an N^2-diffeomorphism of $N^2 \underset{K_1}{\times} D^2\cup_{[\lambda]} N^2\times D^2$ onto $N^2 \underset{K_1}{\times} D^2 \cup_{[E_2]} N^2 \underset{K_\lambda}{\times} D^2$. Thus $N^2 \underset{K_1}{\times} D^2 \cup_{[\lambda]} N^2 \underset{K_2}{\times} D^2$ is N^2-equivariantly diffeomorphic to $N^2 \underset{T^2}{\times} (T^2 \underset{K_1}{\times} D^2 \cup_{[E_2]} T^2 \underset{K_\lambda}{\times} D^2)$ by Lemma 1.11. q.e.d.

By this lemma, $U(2) \underset{N^2}{\times} (N^2 \underset{K_1}{\times} D^2 \cup_{[\lambda]} N^2 \underset{K_2}{\times} D^2)$ is $U(2)$-equivariantly diffeomorphic to $U(2) \underset{N^2}{\times} (N^2 \underset{T^2}{\times} (T^2 \underset{K_1}{\times} D^2 \cup_{[E_2]} T^2 \underset{K_\lambda}{\times} D^2)) = U(2) \underset{T^2}{\times} (T^2 \underset{K_1}{\times} D^2 \cup_{[E_2]} T^2 \underset{K_\lambda}{\times} D^2)$. Thus if neither K_1 nor K_2 is conjugate to $O(2)$, then

$$M^5 \cong M(\alpha,\rho_1,\rho_2) \cong U(2) \underset{T^2}{\times} (T^2 \underset{K_1}{\times} D^2 \cup_{[E_2]} T^2 \underset{K_2}{\times} D^2)\ .$$

Now we investigate $L=T^2 \underset{K_1}{\times} D^2 \cup_{[E_2]} T^2 \underset{K_2}{\times} D^2$. Since K_1, K_2 are the 1-dimensional closed subgroups of T^2, we have $K_1=G^1_{A,B}$, $K_2=G^1_{X,Y}$ for some $A,B,X,Y\in\boldsymbol{Z}$ with $A^2+B^2\neq 0$, $X^2+Y^2\neq 0$. Then we can put

$$\rho_1\left(\begin{pmatrix}\xi&0\\0&\eta\end{pmatrix}\right)=\xi^{-D}\eta^{C}\ ,\qquad \rho_2\left(\begin{pmatrix}\xi&0\\0&\eta\end{pmatrix}\right)=\xi^{-V}\eta^{U}\ ,$$

where C, D and U, V must satisfy $AD-BC\neq 0$ and $XV-YU\neq 0$, respectively, since the K_s-action on ∂D^2 induced by ρ_s is transitive. Define the T^2-action $\tilde{\rho}_s$ on $S^1\times D^2$ by

$$\tilde{\rho}_1\left(\begin{pmatrix}\xi & 0\\ 0 & \eta\end{pmatrix}, (z, w)\right) = (\xi^{-B}\eta^{A}z,\ \xi^{-D}\eta^{C}w)$$

$$\tilde{\rho}_2\left(\begin{pmatrix}\xi & 0\\ 0 & \eta\end{pmatrix}, (z, w)\right) = (\xi^{-Y}\eta^{X}z,\ \xi^{-V}\eta^{U}w)\ .$$

Moreover, define the map $\hat{\rho}_s$ of $T^2\underset{K_s}{\times} D^2$ onto $S^1\times D^2$ by

$$\hat{\rho}_1\left(\left[\begin{pmatrix}\xi & 0\\ 0 & \eta\end{pmatrix}, \zeta\right]\right) = (\xi^{-B}\eta^{A},\ \xi^{-D}\eta^{C}\zeta)$$

$$\hat{\rho}_2\left(\left[\begin{pmatrix}\xi & 0\\ 0 & \eta\end{pmatrix}, \zeta\right]\right) = (\xi^{-Y}\eta^{X},\ \xi^{-V}\eta^{U}\zeta)\ .$$

Then $(T^2\underset{K_s}{\times} D^2, T^2/H)$ is T^2-equivariantly diffeomorphic to $(S^1\times D^2, S^1\times S^1)$ by $\hat{\rho}_s$, where

$$(\hat{\rho}_1\,|\,T^2/H)\left(\left[\begin{pmatrix}\xi & 0\\ 0 & \eta\end{pmatrix}\right]\right) = \hat{\rho}_1\left(\left[\begin{pmatrix}\xi & 0\\ 0 & \eta\end{pmatrix}, 1\right]\right) = (\xi^{-B}\eta^{A},\ \xi^{-D}\eta^{C})$$

$$(\hat{\rho}_2\,|\,T^2/H)\left(\left[\begin{pmatrix}\xi & 0\\ 0 & \eta\end{pmatrix}\right]\right) = \hat{\rho}_2\left(\left[\begin{pmatrix}\xi & 0\\ 0 & \eta\end{pmatrix}, 1\right]\right) = (\xi^{-Y}\eta^{X},\ \xi^{-V}\eta^{U})\ .$$

Furthermore, we have the following commutative diagram

$$\begin{array}{ccccccc}
T^2\underset{K_1}{\times} D^2 & \longleftarrow & T^2/H & \xrightarrow[\text{identity}]{[E_2]} & T^2/H & \longrightarrow & T^2\underset{K_2}{\times} D^2\\
\downarrow{\scriptstyle \hat{\rho}_1} & & \downarrow{\scriptstyle \hat{\rho}_1|T^2/H} & & \downarrow{\scriptstyle \hat{\rho}_2|T^2/H} & & \downarrow{\scriptstyle \hat{\rho}_2}\\
S^1\times D^2 & \hookleftarrow & S^1\times S^1 & \xrightarrow{f} & S^1\times S^1 & \hookrightarrow & S^1\times D^2
\end{array}$$

where $f=(\hat{\rho}_2|T^2/H)\cdot(\hat{\rho}_1|T^2/H)^{-1}$ is a T^2-equivariant diffeomorphism. The map f is an automorphism of the topological group $S^1\times S^1$. Hence for some $a, b, c, d\in \boldsymbol{Z}$ with $ad-bc=\pm 1$, we have $f(z, w)=(z^a w^b, z^c w^d)$. On the other hand, $f(\xi^{-B}\eta^{A}, \xi^{-D}\eta^{C})=\hat{\rho}_2\hat{\rho}_1^{-1}(\xi^{-B}\eta^{A}, \xi^{-D}\eta^{C})=(\xi^{-Y}\eta^{X}, \xi^{-V}\eta^{U})$ for each $\xi, \eta\in \boldsymbol{C}$ with $|\xi|=1$ and $|\eta|=1$. Hence $\xi^{-Y}\eta^{X}=\xi^{-Ba-Db}\eta^{Aa+Cb}$, $\xi^{-V}\eta^{U}=\xi^{-Bc-Dd}\eta^{Ac+Cd}$ for arbitrary $\xi, \eta\in\boldsymbol{C}$ with $|\xi|=1$, $|\eta|=1$. Therefore

$$\begin{pmatrix}X & U\\ Y & V\end{pmatrix}=\begin{pmatrix}A & C\\ B & D\end{pmatrix}\begin{pmatrix}a & c\\ b & d\end{pmatrix}.$$

Moreover $L=T^2\underset{K_1}{\times} D^2\cup_{[E_2]} T^2\underset{K_2}{\times} D^2$ is T^2-equivariantly diffeomorphic to $S^1\times D^2\cup_f S^1\times D^2$. By Lemma 1.7, as a T^2-manifold we have $S^1\times D^2$

$\cup_f S^1 \times D^2 \cong L(b, d)$, where the T^2-action on $L(b, d)$ is the same action that is defined in Part (III) of the main theorem described in the introduction. Hence $T^2 \underset{K_1}{\times} D^2 \cup_{[E_2]} T^2 \underset{K_2}{\times} D^2 \cong L(b, d)$ as a T^2-manifold. Thus as a $U(2)$-manifold $M^5 = M(\alpha, \rho_1, \rho_2) \cong U(2) \underset{T^2}{\times} L(b, d)$ for some $a, b, c, d, A, B, C, D \in \boldsymbol{Z}$ with $ad - bc = \pm 1$ and $AD - BC \neq 0$. Moreover by Lemma 1.8, we have $(A + B, b) = 1$ and $(A - B, C - D) = 1$, since $M^5 \cong M^5_\alpha = M(\alpha, \rho_1, \rho_2) \cong U(2) \underset{T^2}{\times} L(b, d)$ is simply connected and the $U(2)$-action on M^5 is effective. Then $H = G^1_{A,B} \cap G^1_{C,D} = G^1_{X,Y} \cap G^1_{U,V} = C_{r,r+1,AD-BC}$, where $\begin{pmatrix} r \\ r+1 \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ for some $\alpha, \beta \in \boldsymbol{Z}$. Therefore in this case M^5 is $U(2)$-equivariantly diffeomorphic to the $U(2)$-manifold

$$M\left\{\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} A & C \\ B & D \end{pmatrix}\right\} = U(2) \underset{T^2}{\times} L(b, d)$$

of Part (III) with $AD - BC \neq 0$ in the main theorem.

3. The proof when ϕ admits principal orbits of codimension 2. Then the principal isotropy subgroup of ϕ is 1-dimensional. We denote the type of the principal isotropy subgroup by (H). Then we may regard H as $G^1_{r,r+1}$ for some $r \in \boldsymbol{Z}$ by Lemma 1.1.

(3.1) Suppose that $U(2)$ appears as an isotropy subgroup. We investigate 5-dimensional real representations of $U(2)$. Let V be the 5-dimensional real vector space of all symmetric 3×3 real matrices with trace 0. Let τ be the 5-dimensional real representation of $SO(3)$ on V defined by $\tau(A, X) = AXA^{-1}$ for $A \in SO(3)$, $X \in V$. We denote by λ_1 the canonical 2-dimensional complex representation of $U(2)$ or $SU(2)$. Denote the determinant representation of $U(2)$ by λ_2. Let ρ be the natural homomorphism of $U(2)$ onto $SO(3) \cong U(2)/G^1_{1,1}$. There are only the following three possibilities of irreducible real representations of $SU(2)$ with dimension less than six: $\rho_0\colon SU(2) \to SU(2)/\{\pm E_2\} \cong SO(3)$, $r(\lambda_1)\colon SU(2) \to SO(4)$, $\sigma_0\colon SU(2) \to SO(5)$, where ρ_0 is the restriction of the above ρ to $SU(2)$, $r(\lambda_1)$ is the underlying real representation of the complex representation λ_1 and $\sigma_0 = \tau \circ \rho_0$ (composition of τ and ρ_0 as maps). These representations can be uniquely extended, respectively, to the following representations: $\rho\colon U(2) \to SO(3)$, $r(\lambda_1\lambda_2^m)\colon U(2) \to SO(4)$, $\sigma\colon U(2) \to SO(5)$, where $r(\lambda_1\lambda_2^m)$ is the underlying real representation of the complex representation $\lambda_1\lambda_2^m(A) = A(\det A)^m$ for $A \in U(2)$ and $\sigma = \tau \circ \rho$. Thus the following are all the 5-dimensional real representations of $U(2)$: $\rho + r(\lambda_2^n)\colon U(2) \to SO(3) \times SO(2) \subset O(5)$, $r(\lambda_1\lambda_2^m) + 1\colon U(2) \to SO(4) \times SO(1) \subset O(5)$, $\sigma\colon U(2) \to SO(5) \subset O(5)$, where $r(\lambda_2^n)$ is the underlying real representation of the complex representation $\lambda_2^n(A) = (\det A)^n$ for $A \in U(2)$ and 1 is the trivial 1-dimensional real representation. Hence the 5-dimensional real representation of $U(2)$ which induces an effective action with principal orbits of codimension 2 is $r(\lambda_1\lambda_2^m) + 1$ for $m = 0, -1$. Therefore if $U(2)$ appears as an isotropy subgroup, then such action is of two isotropy types $(G^1_{m,m+1}, U(2))$ for $m = 0, -1$. We denote the set of fixed points of this action by $F(U(2), M^5)$ or F. It follows from [3, IV 8.6. Theorem] that the orbit space M^* of this action is a 2-disk D^2 and $F(U(2), M^5) = \partial D^2 = S^1$. Denote by U the $U(2)$-invariant closed tubular neighborhood of $F = S^1$ in M^5 and let X be the closure of $M^5 - U$ in M^5. Then X is also $U(2)$-invariant. Since $U(2)/G^1_{m,m+1} \cong S^3$ for $m = 0, -1$ and $W_{U(2)}(G^1_{m,m+1}) = S^1$, we have

$$X \cong S^3 \underset{S^1}{\times} F(G^1_{m,m+1}, X)$$

by [7, Lemma 4.2], where $F(G^1_{m,m+1}, X) = \{x \in X;\ G^1_{m,m+1} \subset U(2)_x\}$ and $W_{U(2)}(G^1_{m,m+1}) = S^1$ acts freely on $F(G^1_{m,m+1}, X)$. Moreover we have the S^3-bundle $X \to X/U(2)$ with a $U(2)$-action. Now the orbit space $X/U(2)$ is the 2-dimensional disk D^2. Thus X is $U(2)$-equivariantly diffeomorphic to $S^3 \times D^2$. Moreover, ∂X is $U(2)$-equivariantly diffeomorphic to $S^3 \times S^1$, hence so is ∂U. On the other hand, $U \to F = S^1$ is a D^4-bundle with a $U(2)$-action. Thus U is $U(2)$-equivariantly diffeomorphic to $D^4 \times S^1$. Consequently, there exists a $U(2)$-equivariant diffeomorphism $f\colon S^3 \times S^1 \to S^3 \times S^1$, so that M^5 is $U(2)$-equivariantly diffeomorphic to the manifold $M(f) = D^4 \times S^1 \cup_f S^3 \times D^2$. Now for such f there exist a smooth map $\alpha\colon S^1 \to S^1$ and a diffeomorphism $\beta\colon S^1 \to S^1$ such that $f(q, \zeta) = (q\alpha(\zeta), \beta(\zeta))$ for $(q, \zeta) \in S^3 \times S^1$. Extend f to the $U(2)$-equivariant diffeomorphism $F\colon D^4 \times S^1 \to D^4 \times S^1$ defined by $F(tq, \zeta) = (tq\alpha(\zeta), \beta(\zeta))$ $(0 \leqq t \leqq 1)$. Then F induces a $U(2)$-equivariant diffeomorphism $S^5 = D^4 \times S^1 \cup_{\mathrm{id}} S^3 \times D^2 \to M(f) = D^4 \times S^1 \cup_f S^3 \times D^2$, where id is the identity map of $S^3 \times S^1$. Consequently, M^5 is $U(2)$-equivariantly diffeomorphic to S^5 of (I) with $k = 0$ in the main theorem.

(3.2) G^3_k does not appear as an isotropy subgroup of ϕ. Indeed, the identity component of a 1-dimensional closed subgroup of G^3_k is $G^1_{1,-1}$.

(3.3) N^2 does not appear as an isotropy subgroup of ϕ. Indeed, suppose that N^2 is an isotropy subgroup and $\rho\colon N^2 \to O(3)$ is the slice representation of ρ. Then the identity component of its principal isotropy subgroup is $G^1_{1,-1}$. This is a contradiction.

(3.4) Suppose that T^2 appears as an isotropy subgroup of ϕ. For $r \in \boldsymbol{Z}$, let $\zeta_r\colon T^2 \to S^1$ be a complex representation defined by

$$\zeta_r\left(\begin{pmatrix}\xi & 0\\ 0 & \eta\end{pmatrix}\right) = \xi^{r+1}\eta^{-r}$$

and let $r(\zeta_r)$ be the real representation induced by ζ_r. The slice representation of T^2 whose principal isotropy subgroup is $G^1_{r,r+1}$ is necessarily $\rho_r = r(\zeta_r) + 1\colon T^2 \to SO(2) \times SO(1) \subset O(3)$, where 1 is the 1-dimensional trivial representation. The isotropy type of the T^2-action induced by the slice representation ζ_r is $(G^1_{r,r+1}, T^2)$. Thus if T^2 appears as an isotropy subgroup of ϕ, then by (3.1), (3.2), (3.3) and this fact, ϕ is of two isotropy types $(G^1_{r,r+1}, T^2)$ for some $r \in \boldsymbol{Z}$.

Denote by $M_{(T^2)}$ the set of all points whose isotropy groups are conjugate to T^2. Since the isotropy type (T^2) is maximal, $M_{(T^2)}$ is a $U(2)$-invariant closed submanifold of M^5. By [3, IV 8.6 Theorem], the orbit spaces $M^5/U(2)$ and $M_{(T^2)}/U(2)$ are homeomorphic to D^2 and $\partial D^2 = S^1$, respectively.

Denote by $F(T^2, M_{(T^2)})$ or F the set of all points of $M_{(T^2)}$ whose isotropy subgroup contains T^2. We identify $U(2)/T^2$ with S^2 as $U(2)$-spaces. By [7, Lemma 4.2] we have $M_{(T^2)} = U(2) \times_{N^2} F = (U(2)/T^2) \times_{W(T^2)} F$, where $W(T^2) = W_{U(2)}(T^2) = N^2/T^2$ and $W(T^2)$ acts freely on F. We may identify $W(T^2)$ with $S^0 = \{\pm 1\}$. S^0 acts on S^2 by $(\pm 1, \boldsymbol{a}) \mapsto \pm \boldsymbol{a}$, where $\boldsymbol{a} \in S^2$ and $\pm 1 \in S^0$. Thus $M_{(T^2)} = S^2 \times_{S^0} F$ as a $U(2)$-manifold. Moreover, $F/S^0 = M_{(T^2)}/U(2) = S^1$. Since S^0 acts freely on F, we see that $F \to S^1$ is a principal S^0-bundle over S^1. Hence $F = S^1$ or $S^1 \times S^0$.

Denote the normal bundle of $M_{(T^2)}$ in M^5 by ν. First we show that ν has a $U(2)$-invariant complex structure, so that ν is an orientable real plane bundle with a $U(2)$-action. Next we show that $F = S^1 \times S^0$ by means of the Gysin sequence. Consider the following commutative diagram:

$$\begin{array}{ccc} \mu = j^*\nu & \longrightarrow & \nu \\ \downarrow & & \downarrow \\ F & \xrightarrow{j} & M_{(T^2)} = U(2) \times_{N^2} F \end{array}$$

where j is the inclusion map and $\mu = j^*\nu$ is the induced bundle. Then μ is a real plane bundle with N^2-action and $\nu = U(2) \times_{N^2} \mu$. Thus if μ has an N^2-invariant complex structure, then it naturally induces a $U(2)$-invariant complex structure on ν. Now we introduce a canonical complex structure on μ.

Since the T^2-action which is the restriction of the N^2-action on μ leaves F fixed, each element of T^2 induces an automorphism of every fiber of μ. In particular, consider the $G^1_{1,1}$-action which is the restriction of such a T^2-action. Since the above N^2-action is induced by the $U(2)$-action ϕ and the isotropy type of ϕ is $(G^1_{r,r+1}, T^2)$, such a $G^1_{1,1}$-action is free on the associated sphere bundle $S(\mu)$ of μ. Thus we can define the complex structure on μ by means of the action of $\sqrt{-1}E_2 \in G^1_{1,1}$. Then since $G^1_{1,1}$ is the center of $U(2)$, such a complex structure is compatible with the N^2-action on μ, i.e., N^2-invariant. Hence ν has a $U(2)$-invariant complex structure and the normal bundle ν is an orientable plane bundle. In order to prove that $F = S^1 \times S^0$ let us assume $F = S^1$ and derive a contradiction. Then in the principal bundle $F = S^1 \xrightarrow{p} S^1$ the projection p is the map $p(z) = z^2$ for $z \in F = S^1$. Consider the bundle $M_{(T^2)} = S^2 \underset{S^0}{\times} F \to S^2/S^0 = P_2$ (real projective plane). This is the sphere bundle associated to the complex line bundle $\xi = S^2 \underset{S^0}{\times} C \to P_2$, where the S^0-action $S^0 \times C \to C$ is defined by $(\pm 1, z) \mapsto \pm z$. Since the bundle ξ can be regarded as a real orientable plane bundle, we can apply the Gysin sequence of the sphere bundle $M_{(T^2)} \to P_2 = S^2/S_0$. Thus the following sequence is exact:

$$0 = H_3(P_2) \to H_1(P_2) \to H_2(M_{(T^2)}) \to H_2(P_2) = 0 .$$

Hence $H_2(M_{(T^2)}) \cong H_1(P_2) \cong Z_2$. Now denote by U a $U(2)$-invariant closed tubular neighborhood of $M_{(T^2)}$ in M^5 and let E be the closure of $M^5 - U$ in M^5. Then E is also $U(2)$-invariant. Moreover, we have the bundle $E \to E/U(2)$ whose typical fiber is $U(2)/G^1_{r,r+1}$. The orbit space $E/U(2)$ is diffeomorphic to the 2-disk D^2. By Lemma 1.4, $U(2)/G^1_{r,r+1} = L(2r+1, 1)$. Thus $E = L(2r+1, 1) \times D^2$. Hence $\partial U = \partial E = L(2r+1, 1) \times S^1$. On the other hand, the bundle $\partial U \to M_{(T^2)}$ can be regarded as the sphere bundle associated with the normal bundle ν of $M_{(T^2)}$ in M^5. Since it has been already proved that ν is orientable, we can apply the Gysin sequence of the above sphere bundle and get the exact sequence

$$0 = H_4(M_{(T^2)}) \to H_2(M_{(T^2)}) \to H_3(\partial U) .$$

Here $H_2(M_{(T^2)}) \cong Z_2$, $H_3(\partial U) = H_3(L(2r+1, 1) \times S^1) \cong Z$. This is a contradiction. Therefore $F \neq S^1$. Hence $F = S^0 \times S^1$. Since $M_{(T^2)} = (U(2)/T^2) \underset{W(T^2)}{\times} F = S^2 \underset{S^0}{\times} F$ as a $U(2)$-space, $M_{(T^2)} = (U(2)/T^2) \times S^1 = S^2 \times S^1$ as a $U(2)$-space.

In the following commutative diagram of bundles

$$\begin{array}{ccc} \mu & \longrightarrow & \nu \\ \downarrow & & \downarrow \\ F & \xrightarrow{j} & M_{(T^2)} \end{array}$$

we have $\mu = j^*\nu$ and $\nu = U(2) \times_{N^2} \mu$. Since an orientable plane bundle over S^1 is trivial, $\mu = F \times \boldsymbol{C} = (S^0 \times S^1) \times \boldsymbol{C}$ as a bundle with an N^2-action. Thus $\nu = U(2) \times_{N^2} (F \times \boldsymbol{C})$ as a bundle with a $U(2)$-action. The T^2-action, which is the restriction of such a $U(2)$-action, induces a T^2-action on the fiber $\boldsymbol{C}$. Since the $U(2)$-action on the associated D^2-bundle of ν coincides with the $U(2)$-action on the $U(2)$-invariant closed tubular neighborhood U by ϕ, the principal isotropy type of such T^2-action on $\boldsymbol{C}$ is $(G^1_{r,r+1})$. Now we consider the plane bundle $\pi: U(2) \times_{T^2} (S^1 \times \boldsymbol{C}) \to M_{(T^2)} = (U(2)/T^2) \times S^1 = S^2 \times S^1$ with a $U(2)$-action, where T^2 acts on $S^1 \times \boldsymbol{C}$ by

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, (\tau, \zeta)\right) \mapsto (\tau, \alpha^{r+1}\beta^{-r}\zeta) \quad \text{and} \quad \pi([A, (\tau, \zeta)]) = ([A], \tau) .$$

Define a map $h: U(2) \times_{T^2} (S^1 \times \boldsymbol{C}) \to U(2) \times_{N^2} (S^0 \times S^1 \times \boldsymbol{C})$ by $h([A, (\tau, \zeta)]) = [A, (1, \tau, \zeta)]$. Then h is a $U(2)$-equivariant isomorphism of vector bundles with $U(2)$-actions. We consider the plane bundle $\pi: L(2r1+, 1) \times_{S^1} (S^1 \times \boldsymbol{C}) \to S^2 \times S^1 = M_{(T^2)}$, where S^1 acts on $L(2r + 1, 1)$ and $S^1 \times \boldsymbol{C}$ by

$$\left(\tau, \begin{bmatrix} z \\ w \end{bmatrix}\right) \mapsto \left[\begin{pmatrix} z \\ w \end{pmatrix}\tau^{1/(2r+1)}\right] \quad \text{and} \quad (\tau, (\xi, \eta)) \mapsto (\xi, \tau\eta)$$

respectively. $U(2)$ acts on $L(2r + 1, 1)$ by

$$\left(A, \begin{bmatrix} z \\ w \end{bmatrix}\right) \mapsto A \cdot \begin{bmatrix} z \\ w \end{bmatrix} = \left[A\begin{pmatrix} z \\ w \end{pmatrix}(\det A)^{-r/(2r+1)}\right]$$

and the above projection π is defined by

$$\pi\left(\left[\begin{bmatrix} z \\ w \end{bmatrix}, (\xi, \eta)\right]\right) = \left(\begin{pmatrix} |z|^2 - |w|^2 \\ 2\,\mathrm{Re}\,(\bar{z}w) \\ 2\,\mathrm{Im}\,(\bar{z}w) \end{pmatrix}, \xi\right) .$$

$U(2) \times_{T^2} (S^1 \times \boldsymbol{C})$ is $U(2)$-equivariantly isomorphic to $L(2r + 1, 1) \times_{S^1} (S^1 \times \boldsymbol{C})$ by the map

$$[A, (\xi, \eta)] \mapsto \left[A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, (\xi, \eta)\right] .$$

Thus we may regard $L(2r + 1, 1) \times_{S^1} (S^1 \times \boldsymbol{C})$ as the normal bundle ν. Hence $U = L(2r + 1, 1) \times_{S^1} (S^1 \times D^2)$. On the other hand, $E = L(2r + 1, 1) \times D^2 = L(2r + 1, 1) \times_{S^1} (D^2 \times S^1)$, where S^1 acts on $D^2 \times S^1$ by $(\tau, (t\xi, \eta)) \mapsto (t\xi, \tau\eta)$ $(0 \leqq t \leqq 1, |\xi| = |\eta| = 1)$. Now as $U(2)$-manifolds

$$\partial(L(2r+1, 1) \underset{S^1}{\times} (S^1 \times D^2)) = \partial(L(2r+1, 1) \underset{S^1}{\times} (D^2 \times S^1))$$
$$= L(2r+1, 1) \times S^1 .$$

Denote by $M(f) = L(2r+1, 1) \underset{S^1}{\times} (S^1 \times D^2) \cup_f L(2r+1, 1) \underset{S^1}{\times} (D^2 \times S^1)$ the manifold which we obtain from $L(2r+1, 1) \underset{S^1}{\times} (S^1 \times D^2)$ and $L(2r+1, 1) \underset{S^1}{\times} (D^2 \times S^1)$ by identifying their boundaries under a $U(2)$-equivariant diffeomorphism $f: L(2r+1, 1) \times S^1 \to L(2r+1, 1) \times S^1$. For any $U(2)$-equivariant diffeomorphism $f: L(2r+1, 1) \times S^1 \to L(2r+1, 1) \times S^1$, $M(f)$ is $U(2)$-equivariantly diffeomorphic to $M(\mathrm{id})$ where id is the identity map of $L(2r+1, 1) \times S^1$. In fact, for every $U(2)$-equivariant diffeomorphism f of $L(2r+1, 1) \times S^1$, there exist a smooth map $\alpha: S^1 \to L(2r+1, 1)$ and a diffeomorphism $\beta: S^1 \to S^1$ such that

$$f\left(A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \zeta\right) = (A \cdot \alpha(\zeta), \beta(\zeta)) ,$$

where $A \in U(2)$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in L(2r+1, 1)$ and $\zeta \in S^1$. By means of f we define a $U(2)$-equivariant diffeomorphism $F: L(2r+1, 1) \underset{S^1}{\times} (S^1 \times D^2) \to L(2r+1, 1) \underset{S^1}{\times} (S^1 \times D^2)$ by

$$F\left(\left[A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, (\xi, t\eta)\right]\right) = [A \cdot \alpha(\xi), (\beta(\xi), t\eta)] ,$$

where $A \in U(2)$, $0 \leqq t \leqq 1$ and $|\xi| = 1 = |\eta|$. F induces a $U(2)$-equivariant diffeomorphism of $M(\mathrm{id})$ onto $M(f)$.

Therefore for any $U(2)$-equivariant diffeomorphism f of $L(2r+1, 1) \times S^1$, $M(f)$ is $U(2)$-equivariantly diffeomorphic to $L(2r+1, 1) \underset{S^1}{\times} S^3$. Consequently, $M^5 = U \cup E$ is $U(2)$-equivariantly diffeomorphic to $L(2r+1, 1) \underset{S^1}{\times} S^3$. Now suppose that $a, b, c, d, A, B, C, D \in \boldsymbol{Z}$ satisfy the condition of (III) and $AD - BC = 0 = (A - B)(X - Y)$. Then $b = \pm 1$. If $X - Y = 0$ (resp. $A - B = 0$), then $X = Y = 0$ (resp. $A = B = 0$) and for some $r \in \boldsymbol{Z}$ we have

$$\begin{pmatrix} A \\ B \end{pmatrix} = \pm \begin{pmatrix} r \\ r+1 \end{pmatrix} \quad \left(\text{resp. } \begin{pmatrix} X \\ Y \end{pmatrix} = \pm \begin{pmatrix} r \\ r+1 \end{pmatrix}\right) .$$

Hence $L(b, d)$ is T^2-equivariantly diffeomorphic to S^3 admitting the following T^2-action

$$\left(\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}\right) \mapsto \begin{pmatrix} z \\ w\xi^{r+1}\eta^{-r} \end{pmatrix} .$$

Consequently under this situation

$$M\left\{\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} A & C \\ B & D \end{pmatrix}\right\} = U(2) \underset{T^2}{\times} S^3 .$$

On the other hand, $U(2) \underset{T^2}{\times} S^3$ is $U(2)$-equivariantly diffeomorphic to $L(2r+1, 1) \underset{S^1}{\times} S^3$ by

$$[A, q] \mapsto \left[A\begin{pmatrix} 1 \\ 0 \end{pmatrix}(\det A)^{-r/(2r+1)}, q\right],$$

where $A \in U(2)$ and $q \in S^3$. Therefore M^5 is $U(2)$-equivariantly diffeomorphic to the $U(2)$-manifold

$$M\left\{\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} A & C \\ B & D \end{pmatrix}\right) = U(2) \underset{T^2}{\times} L(b, d)$$

of (III) with $AD - BC = 0 = (A - B)(X - Y)$ for some $a, b, c, d, A, B, C, D \in \boldsymbol{Z}$.

(3.5) Suppose that each isotropy subgroup of ϕ is 1-dimensional, that is, for some r the identity component of each isotropy subgroup is conjugate to $G^1_{r,r+1}$. Then it follows from [7, Lemma 4.2] that

$$\begin{aligned} M^5 &\cong (U(2) \times F(G^1_{r,r+1}, M^5))/N_{U(2)}(G^1_{r,r+1}) \cong U(2) \underset{T^2}{\times} F(G^1_{r,r+1}, M^5) \\ &\cong ((U(2)/G^1_{r,r+1}) \times F(G^1_{r,r+1}, M^5))/(T^2/G^1_{r,r+1}) \cong L(2r+1, 1) \underset{S^2}{\times} F(G^1_{r,r+1}, M^5) , \end{aligned}$$

where $F(G^1_{r,r+1}, M^5)$ is the closed 3-dimensional submanifold of all points of M^5 whose isotropy subgroups contain $G^1_{r,r+1}$, S^1 acts on $L(2r+1, 1)$ by

$$\left(\begin{pmatrix} z \\ w \end{pmatrix}, \tau\right) \mapsto \left[\begin{pmatrix} z \\ w \end{pmatrix}\tau^{1/(2r+1)}\right],$$

on $F(G^1_{r,r+1}, M^5)$ almost freely (i.e., each isotropy subgroup is discrete) and $U(2)$ acts on $L(2r+1, 1)$ by

$$\left(A, \begin{bmatrix} z \\ w \end{bmatrix}\right) \mapsto \left[A\begin{pmatrix} z \\ w \end{pmatrix}(\det A)^{-r/(2r+1)}\right].$$

Now we investigate $F^3 = F(G^1_{r,r+1}, M^5)$. The above S^1-action on F^3 is without fixed points and effective since each principal isotropy subgroup of ϕ is conjugate to $G^1_{r,r+1}$. The orbit space $M^5/U(2)$ is homeomorphic to the orbit space F^3/S^1. Since M^5 is simply connected, by [3, II 6.3. Corollary] $M^5/U(2) \cong F^3/S^1$ is a simply connected compact topological 2-manifold. Hence it is D^2 or S^2. It follows from [3, IV 3.12. Theorem and IV 8.3. Proposition] that $F^3/S^1 \cong M^5/U(2) \cong S^2$. Therefore by [6, Theorems 2 and 4], F^3 is S^1-equivariantly diffeomorphic to a 3-dimensional lens space admitting an effective S^1-action with at most two exceptional orbits. Let $\boldsymbol{Z}_{m_1}, \boldsymbol{Z}_{m_2}$ $(m_1 \neq 0,\ m_2 \neq 0)$ be the two exceptional isotropy subgroups, where $\boldsymbol{Z}_{m_s} = \{\omega \in \boldsymbol{C};\ \omega^{m_s} = 1\}$ $(s = 1, 2)$. For each exceptional orbit $S^1/\boldsymbol{Z}_{m_s}$, $s = 1, 2$, there exists an invariant closed tubular neighborhood U_s such that $F^3 = U_1 \cup U_2$, $U_1 \cap U_2 = \partial U_1 = \partial U_2$. Moreover U_s is a compact connected smooth manifold on which S^1 acts smoothly and is S^1-equivariantly diffeomorphic to a twisted product $S^1 \underset{\boldsymbol{Z}_{m_s}}{\times} D^2$, where $\boldsymbol{Z}_{m_s}$ acts on 2-disk D^2 by $\sigma_s(\omega, w) = \omega^{n_s} w$ $((m_s, n_s) = 1)$. Define an S^1-action $\tilde{\sigma}_s$ on $S^1 \times D^2$ by

$$\tilde{\sigma}_s(\tau, (z, w)) = (\tau^{m_s} z, \tau^{n_s} w)\ .$$

Moreover, define the map $\hat{\sigma}_s$ of $S^1 \underset{\boldsymbol{Z}_{m_s}}{\times} D^2$ onto $S^1 \times D^2$ by

$$\hat{\sigma}_s([\xi, \eta]) = (\xi^{m_s}, \xi^{n_s}\eta)\ .$$

Then $\hat{\sigma}_s$ is an S^1-equivariant diffeomorphism. Hence $(U_s, \partial U_s)$ is S^1-equivariantly diffeomorphic to $(S^1 \times D^2, S^1 \times S^1)$. Moreover the manifold F^3 is S^1-equivariantly diffeomorphic to $S^1 \times D^2 \cup_f S^1 \times D^2$ where $f\colon S^1 \times S^1 \to S^1 \times S^1$ is an S^1-equivariant diffeomorphism such that the following diagram is commutative:

$$\begin{tikzcd}
S^1 \times (S^1 \times S^1) \arrow[r, "\tilde{\sigma}_1"] \arrow[d, "1_{S^1} \times f"] & S^1 \times S^1 \arrow[d, "f"] \\
S^1 \times (S^1 \times S^1) \arrow[r, "\tilde{\sigma}_2"] & S^1 \times S^1\ .
\end{tikzcd}$$

Now we must study the map f. Define another S^1-action ρ on $S^1 \times S^1$ by $\rho(\tau, (z, w)) = (\tau z, w)$. Then every S^1-equivariant diffeomorphism of $S^1 \times S^1$ admitting the S^1-action ρ onto itself is S^1-diffeotopic to the map $(z, w) \mapsto (zw^k, w^\delta)$ for some k and $\delta = \pm 1$. Define a diffeomorphism $\bar{\sigma}_s$: $S^1 \times S^1 \to S^1 \times S^1$ by $\bar{\sigma}_s(z, w) = (z^{m_s} w^{p_s}, z^{n_s} w^{q_s})$, where $m_s q_s - n_s p_s = 1$. Then $\tilde{\sigma}_s \circ (1_{S^1} \times \bar{\sigma}_s) = \bar{\sigma}_s \circ \rho$ and $f_0 = \bar{\sigma}_2 \circ f \circ \bar{\sigma}_1^{-1}$ is an S^1-equivariant diffeomorphism of $S^1 \times S^1$ admitting the S^1-action ρ onto itself. Hence for some k, f_0 is S^1-diffeotopic to the S^1-equivariant diffeomorphism $g_0(z, w) = (zw^k, w^\delta)$, where $\delta = \pm 1$. Therefore f is S^1-diffeotopic to the S^1-equivariant diffeomorphism g with $\tilde{\sigma}_2 \circ (1_{S^1} \times g) = g \circ \tilde{\sigma}_1$ defined by

$$g(z, w) = (z^a w^b, z^c w^d)\ ,$$

where $\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} m_1 & n_1 \\ p_1 & q_1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ k & \delta \end{pmatrix} \begin{pmatrix} m_2 & n_2 \\ p_2 & q_2 \end{pmatrix}$ and the following diagram is commutative

$$\begin{array}{ccccc} S^1 \times S^1 & \xrightarrow{f_0 \cong g_0} & S^1 \times S^1 & & \\ \downarrow{\scriptstyle \bar\sigma_1} & & \downarrow{\scriptstyle \bar\sigma_2} & & \\ S^1 \times D^2 \supset S^1 \times S^1 & \xrightarrow{f \cong g} & S^1 \times S^1 \subset S^1 \times D^2 \,. \end{array}$$

By Lemma 1.9, $S^1 \times D^2 \cup_f S^1 \times D^2$ is S^1-equivariantly diffeomorphic to $S^1 \times D^2 \cup_g S^1 \times D^2$. Therefore F^3 is the 3-dimensional lens space $L(b, d)$ admitting the following S^1-action:

$$\left(\zeta, \begin{bmatrix} Z \\ W \end{bmatrix}\right) \mapsto \begin{bmatrix} Zz^{-m_2} \\ Wz^{-m_1\delta} \end{bmatrix} \qquad (b \neq 0,\ z^b = \zeta,\ ad - bc = \delta)$$

$$\left(\zeta, \begin{pmatrix} z \\ w \end{pmatrix}\right) \mapsto \begin{pmatrix} z\zeta^{m_2} \\ w\zeta^{n_2} \end{pmatrix} \qquad (b = 0)\ .$$

Put

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} = -\begin{pmatrix} r \\ r+1 \end{pmatrix}(m_1, n_1)\ , \qquad \begin{pmatrix} X & U \\ Y & V \end{pmatrix} = -\begin{pmatrix} r \\ r+1 \end{pmatrix}(m_2, n_2)\ .$$

Then

$$\begin{pmatrix} X & U \\ Y & V \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \qquad AD - BC = 0$$

and the above S^1-action on $L(b, d)$ induces the T^2-action on $L(b, d)$ in Part (III) of the main theorem described in the introduction. Therefore if each isotropy subgroup of ϕ is of dimension 1, then M^5 is $U(2)$-equivariantly diffeomorphic to

$$M\left\{\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} A & C \\ B & D \end{pmatrix}\right\} = U(2) \underset{T^2}{\times} L(b, d)$$

for some $a, b, c, d, A, B, C, D \in \boldsymbol{Z}$ with $ad - bc = \pm 1$, $AD - BC = 0$, $A - B \neq 0$ and $(A - B)a + (C - D)b \neq 0$. Moreover by Lemma 1.8, $(A + B, b) = 1$ and $(A - B, C - D) = 1$, since $M^5 = U(2) \underset{T^2}{\times} L(b, d)$ is simply connected and the $U(2)$-action on M^5 is effective. Then $H = G^1_{r,r+1}$ where

$$r = -\begin{vmatrix} A & C \\ p_1 & q_1 \end{vmatrix}, \qquad \begin{vmatrix} A - B & C - D \\ p_1 & q_1 \end{vmatrix} = 1$$

for some $p_1, q_1 \in \boldsymbol{Z}$. Moreover there are at most two non-principal orbits and they are exceptional orbits. Consequently in this case, the $U(2)$-manifold M^5 is $U(2)$-equivariantly diffeomorphic to the $U(2)$-manifold of (III) with $AD - BC = 0 \neq (A - B)(X - Y)$ for some $a, b, c, d, A, B, C, D \in \boldsymbol{Z}$.

Here we complete the proof of the main theorem.

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