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WEAK SOLUTIONS OF NAVIER-STOKES EQUATIONS

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Introduction. Consider the initial-value problem for the Navier-Stokes equation in a domain Ω of $\boldsymbol{R}^n$:

$$
\text{(N-S)} \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u\cdot\nabla u + \nabla p = f\,; \quad \nabla\cdot u = 0\,, \quad x\in\Omega\,, \quad 0<t<T\,. \\ u|_\Gamma = 0\,; \quad u|_{t=0} = a \end{cases}
$$

(Γ: the boundary of Ω) where $u = u(x, t)$ is the unknown velocity vector $(u^1, u^2, \cdots, u^n)$; $p = p(x, t)$ is the unknown pressure; $a = a(x)$ is the initial velocity vector field; $f = f(x, t)$ is a given external force. Here we use the notation:

$$
u\cdot\nabla v = \sum_{i=1}^n u^i \frac{\partial v}{\partial x_i}\,; \quad \nabla\cdot u = \sum_{i=1}^n \frac{\partial u^i}{\partial x_i}
$$

for vector functions u, v.

In his famous paper [8], E. Hopf showed the existence of the so-called Hopf's weak solution to the problem (N-S). The first purpose of the present paper is to show the existence of a weak solution, belonging to some class of functions introduced by J. L. Lions [14], which seems to have a somewhat stronger property than the Hopf's weak solution.

In the general case the uniqueness of a weak solution has been not known. Lions-Prodi [15] gave the uniqueness theorem when $n = 2$. C. Foias [15] introduced function spaces $L^{r,r'}$ (for the definition see the chapter 1 of this paper), and showed that if $\Omega = \boldsymbol{R}^n$, and if there is a weak solution u in $L^{r,r'}$ with $r > n$, and with $n/r + 2/r' < 1$, then this u is the only weak solution of (N-S). J. Serrin [23] gave a similar theorem under the assumptions that Ω is a general domain of $\boldsymbol{R}^n$ ($n = 2, 3, 4$), and that a pair of exponents r, r' satisfies $r > n$ and $n/r + 2/r' \leqq 1$. The second purpose is to generalize the Foias-Serrin uniqueness theorem in two directions. First we shall remove the artificial restriction on the dimension n imposed in the theorem of Serrin. Secondly, we shall show that if there is a weak solution u in $L^{n,\infty}$ which is right continuous for t as an L^n-valued function, then u is the only weak solution. Recently von Wahl [26] obtained similar results (the uniqueness in the class $C([0, T); L^n))$ under the assumptions that the initial velocity and the external force are regular to some extent, and that Ω is a bounded domain, His method is however different from ours.

In the celebrated paper [13], J. Leray considered the case $\Omega = \boldsymbol{R}^3$, and constructed a weak solution. At the very end of the paper cited above he posed the problem whether or not the energy of the flow $(1/2)\int_{R^3} |u(x, t)|^2 dx$ tends to zero as $t \to \infty$. Our third purpose is to give an affirmative answer to this; the more general situations will be considered. T. Kato has obtained similar results on the decay of strong solutions with small initial value by a different method from ours.

1. Results.

1.1. Before stating our results we introduce some function spaces, and give our definition of weak solutions of (N-S). $C_{0,\sigma}^\infty$ is the set of all C^∞ (vector) functions $\phi = (\phi^1, \phi^2, \cdots, \phi^n)$ with support in Ω, such that $\nabla \cdot \phi = 0$. L_σ^2 is the closure of $C_{0,\sigma}^\infty$ with respect to the L^2-norm $\|\cdot\|$; $(\cdot, \cdot)$ denotes the L^2-inner product. L^p stands for the usual (vector-valued) L^p-space over Ω, $1 \leqq p \leqq \infty$. $H_{0,\sigma}^1$ denotes the closure of $C_{0,\sigma}^\infty$ with respect to the norm

$$\|\phi\|_{H^1} = \|\phi\| + \|\nabla\phi\|$$

where $\nabla\phi = \partial_x\phi = (\partial\phi^i/\partial x_j;\ i, j = 1, 2, \cdots, n)$. Y is the set of all ϕ in $H_{0,\sigma}^1 \cap L^n$. Equipped with the norm

$$\|\phi\|_Y = \|\phi\|_{H^1} + \|\phi\|_{L^n},$$

Y is a Banach space.

When X is a Banach space, its norm is a denoted by $\|\cdot\|_X$; $C^k([t_1, t_2]; X)$, $L^p((t_1, t_2); X)$ are then usual Banach spaces, where t_1, and t_2 are real numbers such that $t_1 < t_2$. $H^1((t_1, t_2); X)$ is the closure of $C^1([t_1, t_2]; X)$ with respect to the norm

$$\int_{t_1}^{t_2} (\|w(t)\|_X + \|w_t(t)\|_X)dt$$

$(w_t = \partial w/\partial t)$. In this paper we shall denote by M various constants.

We can now introduce the assumptions on the initial function a and the external force f, and state the definition of weak solutions of (N-S).

ASSUMPTION 1. The initial function $a = a(x)$ is in L_σ^2.

ASSUMPTION 2. The function $f = f(\cdot, t)$ is in L^2 for almost all t in $(0, T)$, and $Pf(t)$ is an L_σ^2-valued integrable function on $(0, T)$. (P: the projection on L_σ^2 (in L^2)).

Throughout the present paper, we make the above assumptions. Our definition of a weak solution of (N-S) is as follows.

DEFINITION. Let a and f be as above. A measurable function u on $\Omega \times (0, T)$ is called a weak solution of the initial-valued problem (N-S) if

(i) $u \in L^2((0, T'); H^1_{0,\sigma})$ for any T' with $0 < T' < T$;

(ii) $u \in L^\infty((0, T); L^2_\sigma)$;

(iii)

$$\int_0^T \{-(u, \Phi_t) + (\nabla u, \nabla \Phi) + (u \cdot \nabla u, \Phi)\} dt = \int_0^T (f, \Phi) dt + (a, \Phi(0)) \tag{1.1}$$

for all Φ in $H^1((0, T); Y)$ such that for some $T_0 < T$, $\Phi(\cdot, t) = 0$ on (T_0, T), $(\Phi(0) = \Phi(\cdot, 0))$.

The above definition is essentially due to J. Lions [14]. There are many other definitions of weak solutions. Concerning the relation between the Hopf's weak solution and the weak solution in our sense, we have

PROPOSITION 1. *Any weak solution in the above sense is a Hopf's weak solution. The converse is true when $C^\infty_{0,\sigma}$ is dense in Y. $C^\infty_{0,\sigma}$ is dense in Y if one of the following conditions is satisfied*:

(a) $2 \leqq n \leqq 4$;

(b) *Ω is a star-shaped bounded domain*;

(c) $\Omega = \boldsymbol{R}^n$.

(For the proof, see the appendix).

Concerning the (weak) continuity (in t) of weak solutions, we have the result of G. Prodi [20] (see also J. Serrin [23]).

PROPOSITION 2. (Prodi) *Suppose that u is a weak solution of* (N-S). *After suitable modification of its value of $u(t)$ on a set of measure zero of the time interval* $[0, T]$, *we have that $u(\cdot, t)$ is continuous for t in the weak topology of L^2_σ, and that for any $0 \leqq s \leqq t < T$,*

$$\int_s^t \{-(u, \Phi_t) + (\nabla u, \nabla \Phi) + (u \cdot \nabla u, \Phi)\} dt = \int_s^t (f, \Phi) dt - (u(t), \Phi(t)) + (u(s), \Phi(s)) \tag{1.2}$$

for every Φ in $H^1((s, t); Y)$. Here and in what follows we simply write $u(t)$, $\Phi(t)$ for $u(\cdot, t)$, $\Phi(\cdot, t)$.

In what follows we shall mean by a weak solution a weak solution redefined as above.

1.2. Our result on the existence of weak solutions now reads:

THEOREM 1. *Let the assumptions* 1 *and* 2 *hold. Then there is a weak solution u of the problem* (N-S). *Moreover,*

$$\|u(t)\|^2 + 2\int_0^t \|\nabla u\|^2 dt \leqq 2\int_0^t (f, u)dt + \|a\|^2 ; \quad (0 \leqq t < T) \tag{1.3}$$

$$\lim_{t\to 0} \|u(t) - a\| = 0 . \tag{1.4}$$

REMARKS. 1. The existence of a Hopf's weak solution is well-known. For the existence of our weak solution, see J. Lions [14].

2. For the existence of strong solutions, see Kiselev-Ladyzhenskaya [10], Fujita-Kato [5], Giga-Miyakawa [6].

3. If Ω is a bounded domain, then the energy inequality (of strong form)

$$\|u(t)\|^2 + 2\int_s^t \|\nabla u\|^2 dt \leqq 2\int_s^t (f, u)dt + \|u(s)\|^2$$

holds for almost all $s \geqq 0$, including $s = 0$, and all $t > s$. However, it is not known whether or not the above energy inequality of the strong form does hold for a general domain Ω. Thus in the general case it is not known whether or not there is a weak solution of (N-S) with $f = 0$, such that $\|u(t)\|$ monotonously decreases with t.

1.3. We next proceed to our uniqueness results. To this end we first define a function space $L^{r,r'}$. If $w = w(x, t)$ is defined and measurable in a cylindrical domain $\Omega \times (t_1, t_2)$ of space-time, we set

$$\|w(t)\|_{L^r} = \left(\int_\Omega |w(x, t)|^r dx\right)^{1/r}$$

and

$$|w|_{r,r'} = \begin{cases} \left(\int_{t_1}^{t_2} \|w(t)\|_{L^r}^{r'} dt\right)^{1/r'} & (\text{if } 1 \leqq r' < \infty) \\ \sup\limits_{t_1 \leqq t \leqq t_2} \|w(t)\|_{L^r} & (\text{if } r' = \infty) . \end{cases}$$

Here r and r' are considered to be independent exponents with $1 \leqq r, r' \leqq \infty$.

DEFINITION. We say that $w = w(x, t)$ is contained in the class $L^{r,r'}(\Omega \times (t_1, t_2))$ if w is defined and measurable in $\Omega \times (t_1, t_2)$, and $|w|_{r,r'} < \infty$.

REMARK. It is easy to see that

$$L^{r,r'}(\Omega \times (t_1, t_2)) = L^{r'}((t_1, t_2); L^r) . \tag{1.5}$$

(see H. Rikimaru [21]).

Our uniqueness theorems read:

THEOREM 2. *Let the assumptions* 1 *and* 2 *hold. Let* u, v *be weak solutions of the problem* (N-S). *Suppose also that*

$$\|v\|^2 + 2\int_0^t \|\nabla v\|^2 dt \leqq 2\int_0^t (f, v)dt + \|a\|^2, \quad 0 < t < T, \tag{1.6}$$

and that $u \in L^{r,r'}(\Omega \times (0, T))$ *for a pair of exponents* r, r' *satisfying*

$$\frac{n}{r} + \frac{2}{r'} \leqq 1 \tag{1.7}$$

and also $r > n$. *Then* $u = v$ *on* $[0, T)$.

THEOREM 3. *Let the assumptions* 1 *and* 2 *hold. Let* u, v *be weak solutions of the problem* (N-S). *Suppose that* v *satisfies the inequality* (1.6) *and that* $u \in L^\infty((0, T); L^n)$. *If there is an* s $(0 \leqq s < T)$ *with* $u = v$ *on* $[0, s]$, *and if* u *is right continuous for* t *at* $t = s$ *in the norm of* L^n, *then there is a* $\delta > 0$ *such that* $u = v$ *on* $[0, s + \delta)$.

COROLLARY. *Let the assumptions* 1 *and* 2 *hold. Let* u, v *be weak solutions of* (N-S). *Suppose that* v *satisfies the inequality* (1.6) *and* $u \in L^\infty((0, T); L^n)$. *If* u *is right continuous for all* t *in* $[0, T)$ *in the norm of* L^n, *then* $u = v$ *on* $[0, T)$.

REMARKS. 1. If $n = 2$, then it can be shown that any weak solution u in $L^\infty((0, T); L^2)$ is continuous for all t in $(0, T)$. The uniqueness theorem for $n = 2$ due to Prodi-Lions [15] can be obtained.

2. C. Foias [4] first introduce function spaces $L^{r,r'}$ and showed that the uniqueness theorem (similar to Theorem 2 above) holds if $\Omega = \boldsymbol{R}^n$; $r > n$ and $n/r + 2/r' < 1$. On the other hand, J. Serrin [23] gave the uniqueness theorem under the assumptions that Ω is a general domain; $2 \leqq n \leqq 4$; $r > n$; $n/r + 2/r' \leqq 1$. Thus Theorem 2 may be considered as a generalization of the Foias-Serrin uniqueness theorem.

3. Recently von Wahl [26](1) gave the uniqueness theorem similar to Theorem 3 above. Under the assumptions that $a = a(x)$, and $f = f(x, t)$ are regular to some extent, and that Ω is a bounded domain, he showed that the uniqueness theorem holds in the class $C([0, T); L^n)$, by using the a-priori estimates due to Solonnikov [24]; the method is different from ours.

(1) After the completion of the present paper, Professor von Wahl kindly informed the author of their recent work [27], in which they independently showed similar results (Theorems 2 and 3 above) by using the Yosida approximation; the author would like to express his sincere thanks to Professor von Wahl for it.

1.4. We are next concerned with the problem whether or not $\|u(t)\| \to 0$ as $t \to \infty$. We first define the operator A_0 in L^2_σ. Let A_0 be the operator in L^2_σ defined by: $A_0\phi = -\Delta\phi$; $D(A_0) = C^\infty_{0,\sigma}$. ($D(S)$; domain of S). Then the A_0 thus defined is clearly symmetric and positive in L^2_σ. Moreover we have $(A_0\phi, \phi) = \|\nabla\phi\|^2$. Hence A_0 admits the self-adjoint extension A (called the Friedrichs extension of A_0) in L^2_σ. It is then easy to see that A is positive and satisfies:

$$\|A^{1/2}\phi\| = \|\nabla\phi\| . \tag{1.8}$$

From the above identity it follows that the zero is not an eigenvalue of A. Thus A is a strictly positive self-adjoint operator in L^2_σ. Now we make the following assumption on A.

ASSUMPTION 3. For some non-negative α,

$$(I + A)^{-\alpha}\phi \in L^n \quad \text{for all} \quad \phi \quad \text{in} \quad L^2_\sigma .$$

In many cases the above assumption is satisfied:

PROPOSITION 3. *The above assumption is satisfied with* $\alpha = (n-2)/4$ *if one of the following conditions is satisfied.*

(i) $2 \leqq n \leqq 4$;

(ii) $\Omega = \boldsymbol{R}^n$, $n \geqq 2$.

PROOF. Define the operator B in $L^2(\boldsymbol{R}^n)$ by: $B\phi = -\Delta\phi$, $D(B) = H^2(\boldsymbol{R}^n)$ (Sobolev space). By the Sobolev inequality

$$\|\phi\|_{L^n(\boldsymbol{R}^n)} \leqq M\|(I + B)^\alpha\phi\| , \quad \phi \in D(B^\alpha) \tag{1.9}$$

($\alpha = (n-2)/4$). If $\Omega = \boldsymbol{R}^n$, then $(I+A)^{-\alpha} = (I+B)^{-\alpha}P$. ($P$: the projection on L^2_σ). By (1.9), $(I + B)^{-\alpha}$ is a bounded operator from $L^2(\boldsymbol{R}^n)$ to $L^n(\boldsymbol{R}^n)$. Hence $(I + A)^{-\alpha}$ is a bounded operator from $L^2_\sigma(\boldsymbol{R}^n)$ to $L^n(\boldsymbol{R}^n)$. We next suppose that $2 \leqq n \leqq 4$. Let E be the extension operator from $L^2_\sigma(\Omega)$ to $L^2(\boldsymbol{R}^n)$: $E\phi(x) = \phi(x)$ (if $x \in \Omega$); $= 0$ (if $x \notin \Omega$). Since $(I + B)^{1/2}E(I + A)^{-1/2}$ is a bounded operator by (1.8), it follows from the interpolation theorem that $(I + B)^\beta E(I + A)^{-\beta}$ is a bounded operator for $0 \leqq \beta \leqq 1/2$. Hence by (1.9) we see that $(I + A)^{-\alpha}$ is a bounded operator from L^2_σ to L^n.

Our result on the decay of solutions reads:

THEOREM 4. *Let* $T = \infty$. *Let the assumptions* 1, 2 *and* 3 *hold. Let* u *be a weak solution of* (N-S) *with* $\int_0^\infty \|\nabla u\|^2 dt < \infty$. *Then* $\|(I + A)^{-\alpha}u(t)\|$ *tends to zero as* $t \to \infty$.

The following two corollaries are immediate consequences of Theorem 4.

COROLLARY 1. *Under the assumptions of Theorem* 4,

$$\lim_{t\to\infty}\int_t^{t+1}\|u(s)\|^2 ds = 0 . \tag{1.10}$$

COROLLARY 2. *Let the assumptions of Theorem* 4 *hold. If* $\|u(t)\|$ *tends to some constant, say* c, *as* $t\to\infty$, *then we have* $c = 0$.

J. Leray [13] considered the case $\Omega = \boldsymbol{R}^3$ and $f = 0$, and constructed a weak solution u that becomes smooth (in x and t) for large t, say $t > T$; moreover, $\|u(t)\|$ monotonously decreases with $t > T$. He posed a problem whether or not $\|u(t)\|\to 0$ as $t\to\infty$. Corollary 2, together with Proposition 3, gives an affirmative answer to it. More generally, if Ω is a domain of $\boldsymbol{R}^3$, $f = 0$, and if u is a generalized solution (in the sense of Ladyzhenskaya [12]), then $\|u(t)\|$ monotonously decreases with t, and hence tends to zero as $t\to\infty$, by Corollary 2.

REMARKS. 1. We can construct a weak solution that $\|u(t)\|$ tends to zero as $t\to\infty$. (see K. Masuda [18]). T. Kato constructed a strong solution with $\|u(t)\|\to 0$ as $t\to\infty$ by a method different from ours. (His result was motivation for the present work.)

2. For the decay of $\|u(t)\|_{L^\infty}$ and $\|\nabla u(t)\|$ see Masuda [17, 18], J. G. Heywood [7], P. Maremonti [16].

3. Theorem 4 and the outline of its proof have been reported in Masuda [19].

2. Preliminaries.

2.1. We first recall elementary properties of the mollifier $J_h[w]$ of $w, h > 0$. Let ρ be a C^∞ function in $\boldsymbol{R}^1$ with support in $|t|\leqq 1$, such that $\rho(t) = \rho(-t)$, $\rho(t)\geqq 0$, and $\int_{-\infty}^{\infty}\rho(t)dt = 1$. We set $\rho_h(t) = h^{-1}\rho(t/h)$. Let s, t be fixed numbers such that $0\leqq s < t < +\infty$. Let X be a Banach space. For w in $L^p((s, t); X)$, $1\leqq p < \infty$, we define the mollifier $J_h[w]$ of w by

$$J_h[w](\tau) = \int_s^t \rho_h(\tau-\sigma)w(\sigma)d\sigma , \tag{2.1}$$

Then the following lemma is well-known, and is easy to prove.

LEMMA 2.1. *We have*

(i) *For each fixed* h, J_h *is a bounded operator from* $L^p((s, t); X)$ *into* $C^1([s, t]; X)$.

(ii) *For each fixed* w *in* $L^p((s, t); X)$, $J_h[w]\to w$ *as* $h\to 0$ *in* $L^p((s, t); X)$;

(iii) *If* $w \in C([s,t]; X)$, *then* $J_h[w](t) \to (1/2)w(t)$ *and* $J_h[w](s) \to (1/2)w(s)$ *as* $h \to 0$ *in the norm of* X.

LEMMA 2.2. *Let* X_0 *be a dense subset of a Banach space* X. *Then any function* $\Phi \in H^1((s,t); X)$ *can be approximated by a sequence* $\{\Phi_N\}$, *in the topology of* $H^1((s,t); X)$, *such that each* Φ_N *has the form*

$$\Phi_N(\tau) = \sum_{\text{finite}} \lambda_j(\tau)\phi_j \tag{2.2}$$

where λ_j *is some* C^∞ *function on* $\mathbf{R}^1$ *and* ϕ_j *is some element of* X_0. *Similarly, any function in* $L^2((s,t); X)$ *can be approximated by a sequence of functions of the form* (2.2) *in the topology of* $L^2((s,t); X)$.

PROOF. Since $C^1([s,t]; X)$ is dense in $H^1((s,t); X)$, we may assume that Φ is in $C^1([s,t]; X)$. Since X_0 is dense in X by hypothesis, for any positive integer N, there is a $\phi_{N,j}$ in X_0 with $\|\phi_{N,j} - \Phi(t_j)\| < 1/N^2$, $j = 0, 1, \cdots, N$. ($t_j = s + j\Delta_N$; $\Delta_N = (t-s)/N$). Set

$$\tilde{\Phi}_N(\tau) = \phi_{N,j} + \Delta_N^{-1}(\tau - t_j)(\phi_{N,j+1} - \phi_{N,j}) , \tag{2.3}$$

if $t_j \leqq \tau \leqq t_{j+1}$. It is easy to see that $\tilde{\Phi}_N \in H^1((s,t); X)$. Moreover $\tilde{\Phi}_N$ tends to Φ as $N \to \infty$ in $H^1((s,t); X)$. Indeed, we have

$$\tilde{\Phi}'_N(\tau) - \Phi'(\tau) = \Delta_N^{-1}\left[\phi_{N,j+1} - \Phi(t_{j+1}) - (\phi_{N,j} - \Phi(t_j)) + \int_{t_j}^{t_j+1} (\Phi'(\sigma) - \Phi'(\tau))d\sigma\right]$$

if $t_j \leqq \tau \leqq t_{j+1}$. Therefore

$$\|\tilde{\Phi}'_N(\tau) - \Phi'(\tau)\| \leqq 2\Delta_N + \sup_{|\sigma-\sigma'|<1/N} \|\Phi'(\sigma) - \Phi'(\sigma')\|$$

from which it follows that the integral

$$\int_s^t \|\tilde{\Phi}'_N(\tau) - \Phi'(\tau)\|^2 d\tau$$

tends to zero as $N \to \infty$. Thus we can see that $\tilde{\Phi}_N \to \Phi$ in $H^1((s,t); X)$; note that clearly $\tilde{\Phi}_N \to \Phi$ in $C([s,t]; X)$. Extend $\tilde{\Phi}_N$ to function on $\mathbf{R}^1$: $\tilde{\Phi}_N(\tau) = \phi_{N,0}$ (if $\tau \leqq t_0$); $= \phi_{N,N}$ (if $\tau \geqq t_N$). Then we mollify $\tilde{\Phi}_N$:

$$\Phi_N(\tau) \equiv \int_{-\infty}^{\infty} \rho_{1/N}(\tau - \sigma)\tilde{\Phi}_N(\sigma)d\sigma .$$

The Φ_N thus defined is a desired function of the form (2.2). The latter statement can be proved similarly.

2.2. In this subsection we shall give some estimates for $(w_1 \cdot \nabla w_2, w_3)$.

LEMMA 2.3. *Let* ϕ_1, ϕ_2 *be in* $H^1_{0,\sigma}$, $\phi_3 \in L^r$, *and* $\phi_4 \in Y$, *where* $n \leqq r \leqq \infty$. *Then*

(i) $\|\phi_1\phi_3\| \leqq M\|\nabla\phi_1\|^{n/r}\|\phi_1\|^{1-n/r}\|\phi_3\|_{L^r}$;

(ii) $|(\phi_1 \cdot \nabla\phi_2, \phi_3)| \leqq M\|\nabla\phi_1\|^{n/r}\|\phi_1\|^{1-n/r}\|\nabla\phi_2\|\,\|\phi_3\|_{L^r}$;
(iii) $|(\phi_3 \cdot \nabla\phi_2, \phi_1)| \leqq M\|\nabla\phi_1\|^{n/r}\|\phi_1\|^{1-n/r}\|\nabla\phi_2\|\,\|\phi_3\|_{L^r}$;
(iv) $(\phi_4 \cdot \nabla\phi_1, \phi_2) = -(\phi_4 \cdot \nabla\phi_2, \phi_1)$.

PROOF. By the Hölder inequality,

$$\|\phi_1\phi_3\| \leqq M\|\phi_1\|_{L^{n'}}^{n/r}\|\phi_1\|^{1-n/r}\|\phi_3\|_{L^r}\ , \quad \left(\frac{1}{n'} = \frac{1}{2} - \frac{1}{n}\right).$$

Hence the statement (i) follows from the Sobolev inequality:

$$\|\phi\|_{L^{n'}} \leqq M\|\nabla\phi\|\ .$$

The statements (ii), (iii) follow from (i). Let $\{\phi_{i,j}\}_{j=1}^{\infty}$ be a sequence in $C_{0,\sigma}^{\infty}$ such that $\phi_{i,j} \to \phi_i$ as $j \to \infty$ in $H_{0,\sigma}^1$, $i = 1, 2, 4$. Then by (ii) and (iii),

$$\begin{aligned}(\phi_4 \cdot \nabla\phi_2, \phi_1) &= \lim_{j\to\infty}\lim_{k\to\infty}\lim_{l\to\infty}(\phi_{4,l} \cdot \nabla\phi_{2,j}, \phi_{1,k}) \\ &= -\lim_{j\to\infty}\lim_{k\to\infty}\lim_{l\to\infty}(\phi_{2,j}, \phi_{4,l} \cdot \nabla\phi_{1,k}) \\ &= -(\phi_2, \phi_4 \cdot \nabla\phi_1)\ ,\end{aligned}$$

showing (iv).

LEMMA 2.4. *Let* $w_1 \in L^2((s, t); H_{0,\sigma}^1) \cap L^{\infty}((s, t); L^2)$, $w_2 \in L^2((s, t); H_{0,\sigma}^1)$, $w_3 \in L^{r'}((s, t); L^r)$, *and* $w_4 \in L^2((s, t); Y)$ *where* $n/r + 2/r' = 1$, $r \geqq n$. *If* $1 \leqq r' < \infty$, *we set*

$$g(s, t) = \left(\int_s^t \|w_1\|^2\|w_3\|_{L^r}^{r'}\,dt\right)^{1/r'}$$

and if $r' = \infty$, *we set*

$$g(s, t) = \operatorname*{ess\,sup}_{\tau}\|w_3(\tau)\|_{L^n} \quad (s \leqq \tau \leqq t)\ .$$

Then:

$$\text{(i)} \qquad \int_s^t |(w_1 \cdot \nabla w_2, w_3)|dt + \int_s^t |(w_3 \cdot \nabla w_2, w_1)|dt \leqq Mg(s, t)\left(\int_s^t \|\nabla w_1\|^{qn/r}\|\nabla w_2\|^q dt\right)^{1/q}\ ,$$

$(q = 2r/(n + r))$ *M being a constant independent of* w_1, w_2, w_3, *and* s, t.

$$\text{(ii)} \qquad \int_s^t (w_4 \cdot \nabla w_1, w_2)dt = -\int_s^t (w_4 \cdot \nabla w_2, w_1)dt\ .$$

PROOF. The proofs of (i), (ii) follow from Lemma 2.3.

Let ζ be a monotone increasing C^{∞} function in $\boldsymbol{R}^1$ such that $0 \leqq \zeta \leqq 1$, $|\partial_s\zeta(s)| \leqq 1$ (for all s in $\boldsymbol{R}^1$), and $\zeta(s) = 1$ (if $|s| \leqq 1$); $=0$ (if $|s| \geqq 4$). Set

$\zeta_k(x)=\zeta(|x|/k)$, $(x\in \boldsymbol{R}^m)$ $k=1, 2, \cdots$. Then a sequence $\{\zeta_k\}_{k=1}^{\infty}$ will be called a sequence of m-dimensional cut-off functions. Then:

LEMMA 2.5. *For any $\varepsilon>0$ and w_3 in $C([0, T']; L^n)$, there is a constant M, an integer N, and functions $\psi_j(x)$ $(i=1, \cdots, N)$ in L^2 such that the inequality*

$$(2.4)\qquad \int_s^t |(w_1\cdot\nabla w_2, w_3)|dt \leqq \varepsilon\int_s^t (\|\nabla w_1\|^2+\|\nabla w_2\|^2+\|w_1\|\,\|\nabla w_2\|)dt + M\sum_{i=1}^{N}\int_s^t |(w_1, \psi_i)|^2 dt$$

holds for all w_1, w_2 in $L^2((s, t); H^1_{0,\sigma})$, and $0\leqq s<t\leqq T'$.

PROOF. We fix w_1, w_2; and define the linear functional on $C([s, t]; L^n)$.

$$I[w]=\int_s^t (w_1\cdot\nabla w_2, w)dt\ .$$

Then we decompose $I[w_3]$ in the form:

$$(2.5)\qquad I[w_3]=I[w_{3,1}]+I[w_{3,2}]+I[w_{3,3}]$$

where

$$w_{3,1}(x, t)=(1-\zeta_p(x))w_3(x, t)\ ;$$
$$w_{3,2}(x, t)=\zeta_p(x)(1-\eta_q(|w_3(x, t)|))w_3(x, t)\ ;$$
$$w_{3,3}(x, t)=\zeta_p(x)\eta_q(|w_3(x, t)|)w_3(x, t)\ .$$

Here $\{\zeta_p\}$, $\{\eta_q\}$ be sequences of n-dimensional, 1-dimensional cut-off functions, respectively. We shall estimate each term on the RHS of (2.5). By Lemma 2.4 (i), (ii),

$$(2.6)\qquad |I[w_{3,i}]|\leqq M\int_s^t \|\nabla w_1\|\,\|\nabla w_2\|dt \sup_{0\leqq\tau\leqq T'} \|w_{3,i}(\tau)\|_{L^n}\ ,\quad i=1, 2\ .$$

We shall show that $\sup_{0\leqq\tau\leqq T'}\|w_{3,i}(\tau)\|_{L^n}$ is sufficiently small for large p, and q. From hypothesis it easily follows that $\|w_{3,1}(\tau)\|_{L^n}$ is continuous for τ. Moreover the family of continuous functions $\|w_{3,1}(\tau)\|_{L^n}$ on $[0, T']$ is monotone decreasing in p, and converges to zero for each fixed τ by Lebesgue convergence theorem. Hence it follows from the Dini theorem that $\|w_{3,1}(\tau)\|_{L^n}$ converges to zero as $p\to\infty$, uniformly on $[0, T']$. Hence we can take p so large that

$$(2.7)\qquad |I[w_{3,1}]|\leqq \frac{\varepsilon}{4}\int_s^t \|\nabla w_1\|\,\|\nabla w_2\|dt\ ,\quad 0\leqq s<t\leqq T'$$

(see (2.6)): we fix such a p. Using the elementary inequality

$$|\eta_q(|\xi|)\xi-\eta_q(|\xi'|)\xi'|\leqq \sup_s(\eta(s)+|s\eta'(s)|)|\xi-\xi'|$$

for two vectors ξ, ξ', we can see that $\|w_{3,2}(\tau)\|_{L^n}$ is continuous for τ. Also the family of continuous functions $\|w_{3,2}(\tau)\|_{L^n}$ is monotone decrersing in q and tends to zero as $q \to \infty$ for each fixed τ (and a fixed p). Hence by the Dini theorem, it converges to zero as $q \to \infty$, uniformly on $[0, T']$. Hence we can take q so large that

$$(2.8) \qquad |I[w_{3,2}]| \leqq \frac{\varepsilon}{4}\int_s^t \|\nabla w_1\|\,\|\nabla w_2\| dt\,, \quad 0 \leqq s < t \leqq T'$$

(see (2.6)): we fix such a q.

We finally proceed to the estimate of $I[w_{3,3}]$. We have

$$(2.9) \qquad |I[w_{3,3}]| \leqq \int_s^t \|\zeta_p w_1\|\,\|\nabla w_2\| dt \sup_{s\leqq\tau\leqq t} \|\eta_q(|w_3(\tau)|)w_3(\tau)\|_{L^\infty}\,.$$

A trivial calculation gives:

$$(2.10) \qquad \|\eta_q(|w_3(\tau)|)w_3\|_{L^\infty} \leqq 4q\,, \quad 0 \leqq \tau \leqq T'\,.$$

On the other hand since $H_0^1(\Omega_p)$ is compactly imbedded in $L^2(\Omega_p)$ ($\Omega_p = \{x \in \Omega; |x| \leqq 4p\}$), and since $\zeta_p w_1 \in H_0^1(\Omega_p)$, it follows from the Friedrichs inequality (Courant-Hilbert [2; p. 489]) that for any $\varepsilon' > 0$ there is an integer N and functions ω_i in $L^2(\Omega_p)$ ($i = 1, \cdots, N$) with

$$\|\zeta_p w_1(\tau)\| \leqq \varepsilon'\|\nabla(\zeta_p w_1(\tau))\| + M\sum_{i=1}^N |(\zeta_p w_1(\tau), \omega_i)_{L^2(\Omega_p)}| \quad \text{a.e. in } (s, t)\,.$$

($(\cdot, \cdot)_{L^2(\Omega_p)}$ denotes the L^2-inner product over Ω_p.) Hence since $|\partial_x\zeta_p(x)| \leqq 1$, we have

$$(2.11) \qquad \|\zeta_p w_1(\tau)\| \leqq \varepsilon'\|\nabla w_1(\tau)\| + \varepsilon'\|w_1(\tau)\| + M\sum_{i=1}^N |(w_1(\tau), \psi_i)|$$

where $\psi_i(x) = \zeta_p(x)\omega_i(x)$ ($x \in \Omega_p$); $= 0$ ($x \in \Omega\setminus\Omega_p$). Thus, by the Schwarz inequality, (2.9), (2.10), (2.11), we have

$$(2.12) \qquad |I[w_{3,3}]| \leqq 4q\varepsilon'\int_s^t (\|\nabla w_1\|^2 + \|\nabla w_2\|^2 + \|w_1\|\,\|\nabla w_2\|)dt + M\sum_{i=1}^N \int_s^t |(w_1, \psi_i)|^2 dt\,.$$

Taking ε' so small that $4q\varepsilon' < 1$, and collecting all the estimates (2.7), (2.9), (2.12), we obtain the desired estimate (2.4).

We wish to relax the assumption, made in Lemma 2.5, that w_3 is continuous on $[0, T']$ in the norm of L^n. To this end we prepare a lemma:

LEMMA 2.6. *Let f be a non-negative and integrable function on $[s, T']$, and $\{g_k\}_{k=1}^\infty$ be a sequence of non-negative functions in $L^\infty(s, T')$. Suppose that $\int_s^t f(\tau)d\tau > 0$ for any t in (s, T'). Suppose also that for each* *fixed* t $g_k(t)$ *decreases monotonously to zero as* $k \to \infty$, *and for each fixed* k $g_k(t)$ *is right continuous for* t *at* $t = s$. *Then for any* $\varepsilon > 0$, *there is an* N *such that*

$$\int_s^t f(\tau) g_k(\tau) d\tau \leqq \varepsilon \int_s^t f(\tau) d\tau$$

for all t *in* (s, T') *and* $k > N$.

PROOF. Put

$$z_k(t) = \int_s^t f g_k dt \Big/ \int_s^t f dt \,, \quad t > s \,, \quad k = 1, \cdots .$$

If we define $z_k(s) = g_k(s)$, then z_k is continuous for t in $[s, T']$. Indeed, it is clearly continuous for t in $(s, T']$. It is also easy to see that

$$|z_k(t) - g_k(s)| \leqq \sup_{s < \tau < t} |g_k(\tau) - g_k(s)| \,,$$

from which it follows that z_k is continuous on $[s, T']$. On the other hand for each fixed t $z_k(t)$ decreases monotonously to zero as $k \to \infty$. Hence by the Dini theorem $z_k(t)$ converges to zero as $k \to \infty$, uniformly on $[s, T']$. This proves Lemma 2.6.

LEMMA 2.7. *Let* $w \in L^2((s, T'); H^1_{0,\sigma})$, *and* $u \in L^\infty((s, T'); L^n)$. *Suppose that* $\int_s^t \|w\|^2 dt > 0$ *for any* t *in* (s, T'). *Suppose also that* u *is right continuous for* t *at* $t = s$ *in the norm of* L^n. *Then for any* $\varepsilon > 0$

$$\int_s^t |(w \cdot \nabla w, u)| dt \leqq \varepsilon \int_s^t \|\nabla w\|^2 dt + M \int_s^t \|w\|^2 dt \,, \quad s \leqq t \leqq T' \,, \tag{2.13}$$

M being a constant independent of t.

PROOF. If we set

$$u_1 = (1 - \zeta_p(x)) u(x, t) \,; \qquad u_2 = \zeta_p(x)(1 - \eta_q(|u(x, t)|)) u(x, t) \,;$$
$$u_3 = \zeta_p(x) \eta_q(|u(x, t)|) u(x, t) \,,$$

then in the same way as in the proof of Lemma 2.5 we can get, by $u = u_1 + u_2 + u_3$,

$$\int_s^t |(w \cdot \nabla w, u)| dt \leqq M \int_s^t \|\nabla w\|^2 (\|u_1\|_{L^n} + \|u_2\|_{L^n}) dt$$
$$+ 4qM \int_s^t \|\zeta_p w\| \, \|\nabla w\| dt \,.$$

From hypothesis it follows that for each fixed p $\|u_1(t)\|_{L^n}$ is right continuous for t at $t = s$, and for each fixed t $\|u_1(t)\|_{L^n}$ decreases monotonously to zero as $p \to \infty$, and that $\int_s^t \|\nabla w\|^2 dt > 0$ for t in (s, T'). Thus by Lemma 2.6, there is a p_0 such that

$$\int_s^t \|\nabla w\|^2 \|u_1\|_{L^n} dt \leqq \varepsilon \int_s^t \|\nabla w\|^2 dt\,, \quad p \geqq p_0\,, \quad s \leqq t \leqq T'\,.$$

Similarly we can see that for each fixed p there is a q_o with

$$\int_s^t \|\nabla w\|^2 \|u_2\|_{L^n} dt \leqq \varepsilon \int_s^t \|\nabla w\|^2 dt\,, \quad q \geqq q_o\,, \quad s \leqq t \leqq T'\,.$$

By the Hölder inequality, for each fixed p and q,

$$\int_s^t \|\zeta_p w\| \, \|\nabla w\| dt \leqq \varepsilon \int_s^t \|\nabla w\|^2 dt + M\varepsilon^{-1} \int_s^t \|w\|^2 dt\,, \quad s \leqq t \leqq T'\,.$$

Collecting all the estimates above, we can get the desired estimate (2.13).

3. Existence of weak solutions; Proof of Theorem 1. Following Hopf [8], we first construct approximate solutions of the problem (N-S) by the well-known Galerkin method, in the Banach space $Y = H^1_{0,\sigma} \cap L^n$. To this end we need the following.

LEMMA 3.1. *The Banach space Y is separable.*

PROOF. Define the extension E: $Y \to H^1(R^n) \cap L^n(\boldsymbol{R}^n)$ by $(Eu)(x) = u(x)$ (if $x \in \Omega$); $= 0$ (if $x \in \boldsymbol{R}^n \backslash \Omega$). By the identification $u \leftrightarrow Eu$, Y can be regarded as a closed subspace of $H^1(R^n) \cap L^n(\boldsymbol{R}^n)$. By virtue of Lions [12; p. 6], $H^1(\boldsymbol{R}^n) \cap L^n(\boldsymbol{R}^n)$ is separable. Hence, Y is separable.

Now by Lemma 3.1 just proved, there exists a sequence $\{\phi_k\}_{k=1}^\infty$ of linearly independent vectors which is total in Y. Since $C_{0,\sigma}^\infty \subset Y \subset L^2_\sigma$, and since $C_{0,\sigma}^\infty$ is dense in L^2_σ, it follows that $\{\phi_k\}_{k=1}^\infty$ is also total in L^2_σ; we may assume, without loss of generality, that it is a complete orthonormal system in L^2_σ. Using $\{\phi_k\}$, we construct approximate solution $u_m = u_m(x, t)$ of the problem (N-S) which has the form

$$u_m(x, t) = \sum_{l=1}^m c_{ml}(t) \phi_l(x)\,. \tag{3.1}$$

Here the coefficient $c_{ml} = c_{ml}(t)$ $(l = 1, 2, \cdots, m)$ is a solution of a system of ordinary differential equation

$$dc_{ml}/dt + \sum_{i=1}^m a_{il} c_{mi} + \sum_{i,p=1}^m a_{ipl} c_{mi} c_{mp} = f_l \quad (l = 1, 2, \cdots, m) \tag{3.2}$$

with the initial condition

$$c_{ml}(0) = c_{0,l} \quad (l = 1, 2, \cdots, m) \tag{3.3}$$

where

$$a_{il} = (\nabla\phi_i, \nabla\phi_l)\,; \quad a_{ipl} = (\phi_i \cdot \nabla\phi_p, \nabla\phi_l)\,; \quad f_l = (f, \phi_l)\,; \quad c_{0,l} = (a, \phi_l)\,.$$

We note that a_{ipl} is finite by Lemma 2.3. If $\lambda_l \in H^1((0, T))$ $(1 \leqq l \leqq m)$, then noting the relation

$$(3.4) \qquad (u_m(t), \phi_l) = c_{ml} ,$$

we multiply the both side of (3.2) by $\lambda_l(t)$ and integrate it in t over the interval (s, t); and there results:

$$(3.5) \qquad \int_s^t \{-(u_m, \Phi_t) + (\nabla u_m, \nabla \Phi) + (u_m \cdot \nabla u_m, \Phi)\} dt = \int_s^t (f, \Phi) dt - (u_m(t), \Phi(t)) + (u_m(s), \Phi(s)) ,$$

where $\Phi = \lambda_l(t)\phi_l(x)$. Putting $\lambda_l(t) = c_{ml}(t)$ in the above identity, and taking the summation with respect to l, we find

$$(3.6) \qquad \| u_m(t) \|^2 + 2\int_0^t \| \nabla u_m \|^2 dt = 2\int_0^t (u_m, f) dt + \| a_m \|^2$$

where $a_m = u_m(0)$, since we have $(u_m \cdot \nabla u_m, u_m) = 0$ by Lemma 2.3. Since $\| a_m \| \leqq \| a \|$, it follows from the assumption 2 that

$$(3.7) \qquad \| u_m(t) \|^2 + \int_0^t \| \nabla u_m \|^2 dt \leqq M_1 , \quad 0 \leqq t < T ,$$

M_1 being a constant independent of m, t. (see Ladyzhenskaya [12; Chapter 6, Section 3]). As is well-known the above a priori estimate (3.7) guarantees the global existence of solution of (3.2), (3.3). Moreover, we have:

LEMMA 3.2. *For each fixed* j, *the family* $\{(u_m(t), \phi_j)\}_{j=1}^{\infty}$ *forms a uniformly bounded and equicontinuous family of continuous functions on* $[0, T]$.

PROOF. The uniform boundedness is an immediate consequence of (3.7). A simple calculation yields

$$(u_m(t), \phi_j) - (u_m(s), \phi_j) = \int_s^t ((\partial/\partial\tau) u_m(\tau), \phi_j) dt$$
$$= -\int_s^t (\nabla u_m, \nabla \phi_j) d\tau - \int_s^t (u_m \cdot \nabla u_m, \phi_j) d\tau + \int_s^t (f, \phi_j) d\tau$$
$$(\equiv I_1 + I_2 + I_3) .$$

We shall estimate I_j, $j = 1, 2, 3$. By the Schwarz inequality and (3.7),

$$(3.8) \qquad |I_1| \leqq M(t-s)^{1/2}$$

and

$$(3.9) \qquad |I_3| \leqq M \int_s^t \| Pf \| dt$$

M being a constant independent of m, s, t. Applying to I_2 Lemma 2.5 with $w_1 = w_2 = u_m$ and $w_3 = \phi_j$, we see that for any $\varepsilon' > 0$, there holds

$$|I_2| \leqq \varepsilon' \int_s^t \|\nabla u_m\|^2 dt + M \int_s^t \|u_m\|^2 dt ;$$

and hence, by (3.7)

$$|I_2| \leqq M_1 \varepsilon' + M|t-s| \tag{3.10}$$

M being a constant independent of m, s, t. Therefore it follows from (3.8), (3.9), (3.10) that for any $\varepsilon > 0$ there is a $\delta > 0$ with

$$|(u_m(t), \phi_j) - (u_m(s), \phi_j)| < \varepsilon \quad \text{if} \quad |t-s| < \delta , \quad m = 1, 2, \cdots . \tag{3.11}$$

Since ε is arbitrary positive number, (3.11) implies that the family $\{(u_m(t), \phi_j)\}$ is equicontinuous.

Now by the Ascoli-Arzelà theorem, and the usual diagonal argument, it follows from (3.7) and Lemma 3.2 that there is a subsequence $\{m_i\}$ of $\{m\}$ along which $\{u_m(t)\}$ converges to some $u(t)$, uniformly in $t \in [0, T]$, in the weak topology of $L^2_\sigma(\Omega)$: The uniform limit $u(t)$ of a sequence of continuous functions $u_m(t)$ is continuous for t, weakly (see Hopf [8]; and also Ladyzhenskaya [12]). On the other hand, since $\{u_m\}$ is bounded in $L^2((0, T); H^1_{0,\sigma})$ by (3.7), there is a subsequence of $\{m_i\}$ along which $\{u_{m_i}\}$ converges to some $\tilde{u}$ weakly in $L^2((0, T); H^1_{0,\sigma})$. It is easy to see that $\tilde{u} = u$; we shall assume that the original sequence $\{u_m(t)\}$ itself converges to u, for the sake of simplification of the notations. Since $\|a_m\| \leqq \|a\|$, taking the lim sup (in m) in (3.6), we see that the u satisfies the energy inequality (1.3). To show that the u is a desired solution, it remains only to show that it satisfies (1.2).

We claim:

$$\int_s^t (u_m \cdot \nabla u_m, \Phi) dt \to \int_s^t (u \cdot \nabla u, \Phi) dt , \quad \text{as} \quad m \to \infty \tag{3.12}$$

for every Φ in $\mathscr{F}_{s,t}$: $\mathscr{F}_{s,t}$ is the set of all Φ of the form

$$\Phi = \sum \lambda_l(\tau) \phi_l(x) \quad \text{(finite sum)} \tag{3.13}$$

where $\lambda_l(\tau)$ is arbitrary function in $H^1((s, t); \boldsymbol{R}^1)$. Indeed, we have

$$\int_s^t (u_m \cdot \nabla u_m, \Phi) dt - \int_s^t (u \cdot \nabla u, \Phi) dt$$
$$= \int_s^t ((u_m - u) \cdot \nabla u_m, \Phi) dt + \int_s^t (u \cdot \nabla (u_m - u), \Phi) dt \quad (\equiv I_1 + I_2) .$$

By (1.3), (3.7) and Lemma 2.5 (with $w_1 = w_m - u$, $w_2 = u_m$, $w_3 = \Phi$), we see that for any $\varepsilon > 0$ there is a constant $M = M_\varepsilon$, a positive integer

$N = N_\varepsilon$, and function $\psi_i(x)$ $(i = 1, 2, \cdots, N)$ in L^2, such that

$$|I_1| \leqq \varepsilon M' + M \sum_{i=1}^{N} \int_s^t (u_m - u, \psi_i)^2 dt \,. \tag{3.14}$$

M' being a constant independent of ε, m. Hence, letting $m \to \infty$, we get

$$\limsup_{m\to\infty} |I_1| \leqq \varepsilon M'$$

since $u_m(t) \to u(t)$, uniformly in t, in the weak topology of L^2_σ. Since ε is arbitrary, it follows that $I_1 \to 0$. We next show $I_2 \to 0$. If $w_i(x, t) = u^{(i)}(x, t)\Phi(x, t)$ ($u^{(i)}$: then i-th component of u), then $w_i \in L^2(\Omega \times (s, t))$ by Lemma 2.3. Hence there is a sequence $\{w_{i,k}\}_{k=1}^{\infty}$ $(i = 1, \cdots, n)$ in $C_0^\infty(\Omega \times (s, t))$ with $w_{i,k} \to w_i$ as $k \to \infty$ in $L^2(\Omega \times (s, t))$. For the $w_{i,k}$, we have, by partial integration,

$$|I_2| \leqq \sum_{i=1}^{n} \int_s^t |(u_m - u, \partial_i w_{i,k})| dt + \sum_{i=1}^{n} \left(\int_s^t \|\nabla u_m - \nabla u\|^2 dt\right)^{1/2} \left(\int_s^t \|w_{i,k} - w_i\|^2 dt\right)^{1/2} \quad (\partial_i = \partial/\partial x_i) \,.$$

Letting $m \to \infty$ and then $k \to \infty$ in the above inequality, we have, by (3.7), $I_2 \to 0$. Hence we have (3.12).

Taking a finite sum with respect to l and then letting $m \to \infty$ in (3.5), we obtain, by (3.12),

$$\int_s^t \{-(u, \Phi_t) + (\nabla u, \nabla \Phi) + (u \cdot \nabla u, \Phi)\} dt = \int_s^t (f, \Phi) dt - (u(t), \Phi(t)) + (u(s), \Phi(s)) \tag{3.15}$$

for every Φ in $\mathscr{F}_{s,t}$. We next show that (3.15) holds for every Φ in $C^1([s, t]; Y)$. Let $\Phi \in C^1([s, t]; Y)$. Let $\mathscr{F}_0$ be the set of all (finite) linear combination of the functions in the set $\{\phi_i\}$; $\mathscr{F}_0$ is dense in Y by definition. Hence by (2.3), there is a sequence $\{\Phi_N\}$ such that $\Phi_N \to \Phi$ in $H^1((s, t); Y)$, and which has the form

$$\Phi_N(\tau) = \psi_j + \Delta_N^{-1}(\tau - t_j)(\psi_{j+1} - \psi_j) \quad \text{if} \quad t_j \leqq \tau \leqq t_{j+1}$$

where $t_j = s + j\Delta_N$ $(j = 0, \cdots, N)$; and $\psi_j \in \mathscr{F}_0$. Applying (3.15) with $s = t_j$, $t = t_{j+1}$, one finds

$$\int_{t_j}^{t_{j+1}} \{-(u, \Phi_{N,t}) + (\nabla u, \nabla \Phi_N) + (u \cdot \nabla u, \Phi_N)\} dt = \int_{t_j}^{t_{j+1}} (f, \Phi_N) dt - (u(t_{j+1}), \Phi_N(t_{j+1})) + (u(t_j), \Phi_N(t_j)) \,.$$

Taking the summation with respect to j, we see that (3.15) holds for

$\Phi = \Phi_N$. Letting $N \to \infty$ in (3.15) with $\Phi = \Phi_N$, we can conclude that (3.15) holds for every Φ in $C^1([s, t]; Y)$. Since $C^1([s, t]; Y)$ is dense in $H^1((s, t); Y)$, it follows from Lemma 2.2 that (3.15) holds for every Φ in $H^1((s, t); Y)$. By taking $s = 0$, we can conclude that u satisfies (1.2). This completes the proof of Theorem 1.

4. The uniqueness of weak solutions; Proofs of Theorems 2 and 3. We follow Serrin [23]. Suppose u is a weak solution satisfying the assumptions of either Theorem 2 or Theorem 3. We then define

$$u_h(\tau) = \int_0^t \rho_h(\tau - \sigma)u(\sigma)d\sigma$$

for arbitrarily fixed t $(0 < t < T)$ and the weak solution u. Then $u_h \in H^1((0, T); Y)$. Hence we can take the u_h as a test function in (1.2) with u replaced by v, and there results: $(u_{h,t} = \partial_t u_h)$

$$\tag{4.1} \int_0^t \{-(v, u_{h,t}) + (\nabla v, \nabla u_h) + (v \cdot \nabla v, u_h)\}dt = \int_0^t (f, u_h)dt - (v(t), u_h(t)) + (a, u_h(0)) .$$

On the other hand, since $v \in L^2((0, t); H^1_{0,\sigma})$ by hypothesis, and since $C^\infty_{0,\sigma}$ is dense in $H^1_{0,\sigma}$, it follows from Lemma 2.2 that there is a sequence $\{v^k\}$ in $H^1((0, T); Y)$ with $v^k \to v$ in $L^2((0, T); H^1_{0,\sigma})$: note $C^\infty_{0,\sigma} \subset Y$. We then define v_h, v^k_h:

$$v_h(\tau) = \int_0^t \rho_h(\tau - \sigma)v(\sigma)d\sigma \, ; \qquad v^k_h(\tau) = \int_0^t \rho_h(\tau - \sigma)v^k(\sigma)d\sigma \, .$$

Then it follows from Lemma 2.1 that $v_h \in H^1((0, t); H^1_{0,\sigma})$, $v^k_h \in H^1((0, t); Y)$; and that $v_h \to v$ as $h \to 0$, $v^k_h \to v_h$ as $k \to \infty$ in the norm of $H^1((0, t); H^1_{0,\sigma})$. Now we take v^k_h as a test function in (1.2), and there results

$$\tag{4.2} \int_0^t \{-(u, v^k_{h,t}) + (\nabla u, \nabla v^k_h) + (u \cdot \nabla u, v^k_h)\}dt = \int_0^t (f, v^k_h)dt - (u(t), v^k_h(t)) + (u(0), v^k_h(0)) .$$

Letting $k \to \infty$ in the above identity, we get, by Lemma 2.1 and Lemma 2.4,

$$\tag{4.3} \int_0^t \{-(u, v_{h,t}) + (\nabla u, \nabla v_h) + (u \cdot \nabla u, v_h)\}dt = \int_0^t (f, v_h)dt - (u(t), v_h(t)) + (a, v_h(0)) .$$

Now by virtue of Fubini's theorem and the symmetry of the kernel ρ_h, it is easy to see that

$$\int_0^t (u, v_{h,t})dt = -\int_0^t (u_{h,t}, v)dt .$$

Consequently, addition of (4.1) and (4.3) yields

$$\int_0^t \{(\nabla v, \nabla u_h) + (\nabla u, \nabla v_h) + (v\cdot\nabla v, u_h) + (u\cdot\nabla u, v_h)\}dt$$
$$= \int_0^t \{(f, u_h) + (f, v_h)\}dt - (v(t), u_h(t)) - (u(t), v_h(t)) + (a, u_h(0)) + (a, v_h(0)) .$$

In the above identity we let $h\to 0$. Then it follows from Lemma 2.1 and Lemma 2.4 that

$$\int_0^t \{2(\nabla u, \nabla v) + (v\cdot\nabla v, u) + (u\cdot\nabla u, v)\}dt = \int_0^t \{(f, u) + (f, v)\}dt - (u(t), v(t)) + (a, a) . \tag{4.4}$$

By the theorem of Prodi [18] and Serrin [20], the u satisfies the energy equality:

$$\| n(t)\|^2 + 2\int_0^t \|\nabla u\|^2 dt = 2\int_0^t (f, u)dt + \|a\|^2 \tag{4.5}$$

since u is a weak solution in the class $L^{r,r'}(\Omega\times(0, T))$. On the other hand, by (1.6), it satisfies the energy inequality:

$$\| v(t)\|^2 + 2\int_0^t \|\nabla v\|^2 dt \leqq 2\int_0^t (f, v)dt + \|a\|^2 . \tag{4.6}$$

Addition of (4.4) (multiplied by -2), (4.5) and (4.6) yields

$$\| w(t)\|^2 + 2\int_0^t \|\nabla w\|^2 dt \leqq 2\int_0^t (w\cdot\nabla w, u)dt \tag{4.7}$$

where $w(t) = v(t) - u(t)$. Here we made use of the identity:

$$\int_s^t \{(u\cdot\nabla w, u) + (w, u\cdot\nabla u)\}dt = 0 ,$$

which can be seen from Lemma 2.4.

PROOF OF THEOREM 2. From Lemma 2.4 and the Hölder inequality it follows that for any $\varepsilon > 0$

$$\text{the RHS of (4.7)} \leqq \varepsilon\int_0^t \|\nabla w\|^2 dt + M\int_0^t \|u\|_{L^r}^{r'}\|w\|^2 dt ,$$

M being a constant independent of w. If we take ε so small that $\varepsilon \leqq 2$, then by (4.7) and the above inequality,

$$\|w(t)\|^2 \leqq M\int_0^t \|u\|_{L^r}^{r'}\|w\|^2 dt\,, \quad 0 \leqq t < T\,. \tag{4.8}$$

Since $\|w(t)\|^2$ is locally integrable on $[0, T)$, the above inequality (4.8) implies $w(t)=0$, a.e. in $(0, T)$, by the Gronwall inequality. (see Beckenbach-Bellmann [1; p. 134]). This completes the proof of Theorem 2.

PROOF OF THEOREM 3. Assume that there were not such a $\delta > 0$. Then $\int_s^t \|\nabla w\|^2 dt > 0$ for any $t > s$. Hence it follows from Lemma 2.7 that

$$\text{the RHS of (4.7)} \leqq \varepsilon\int_s^t \|\nabla w\|^2 dt + M\int_s^t \|w\|^2 dt\,,$$

M being a constant independent of w. Hence, similarly to the proof of Theorem 2, we can get

$$\|w(t)\|^2 \leqq M\int_s^t \|w\|^2 dt\,, \quad s \leqq t < T\,.$$

Hence we must have $w = 0$ on (s, T); a contradiction. This proves Theorem 3.

PROOF OF COROLLARY. Since u and v are both continuous in t in the weak topology of L^2_σ, Corollary easily follows from Theorem 3.

5. The decay of solutions; Proof of Theorem 4.

5.1. The proof of Theorem 4 is based on the following estimate to be proved in the next subsection.

$$\begin{aligned}\|(I + A)^{-\alpha}u(t)\|^2 \leqq &\ \|e^{-(t-s)A}(I + A)^{-\alpha}u(s)\|^2 \\ &+ M\int_s^t \|\nabla u\|^2 dt + \|a\|\int_s^t \|Pf\|dt \quad (0 \leqq s < t)\,,\end{aligned} \tag{5.1}$$

M being a constant independent of s, t.

For the moment we assume that (5.1) holds true. If $Av = 0$, then $\nabla v = 0$ by (1.8), from which it follows that $v = 0$. Hence the zero is not an eigenvalue of the positive self-adjoint operator A in L^2_σ. Thus

$$\|e^{-tA}\phi\| \to 0 \quad \text{as} \quad t \to \infty \tag{5.2}$$

for every ϕ in L^2_σ. Hence, letting t tend to infinity in (5.1), we see

$$\limsup_{t\to\infty} \|(I + A)^{-\alpha}u(t)\|^2 \leqq M\int_s^\infty \|\nabla u\|^2 dt + M\int_s^\infty \|Pf\|dt\,. \tag{5.3}$$

Letting s tend to infinity in the above inequality, we have Theorem 4 by hypothesis.

5.2. We shall show the estimate (5.1). Let s, t be fixed numbers such that $0 \leqq s < t < +\infty$. For positive numbers ε, h, we define

$$\Phi_{\varepsilon,h}(\tau) = U_\varepsilon(\tau)\int_s^t \rho_h(\tau-\sigma)U_\varepsilon(\sigma)u(\sigma)d\sigma\ , \quad s \leqq \tau \leqq t\ , \tag{5.4}$$

where u is a weak solution of the problem (N-S); $\rho_h = \rho_h(\tau)$ is a function defined in the section 4; and

$$U_\varepsilon(\tau) = e^{-(t+\varepsilon-\tau)A}(I+A)^{-\alpha}\ , \quad \tau \leqq t\ .$$

Then it is easy to see that for each fixed ε and h, $\Phi_{\varepsilon,h}$ has the following properties (i), (ii), (iii):

(i) $\Phi_{\varepsilon,h} \in C^1([s, t]; L^2_\sigma)$ and

$$\|\Phi_{\varepsilon,h}(\tau)\| \leqq M_2 \quad (M_2 \equiv \sup_{t>0} \|u(t)\|)\ ; \tag{5.5}$$

(ii) $\Phi_{\varepsilon,h}(\tau) \in D(A)$ and $A\Phi_{\varepsilon,h}(\tau)$ is continuous for τ $(s \leqq \tau \leqq t)$ in the norm of L^2_σ;

(iii) $\Phi_{\varepsilon,h}$ satisfies

$$\partial_\tau\Phi_{\varepsilon,h}(\tau) - A\Phi_{\varepsilon,h}(\tau) = U_\varepsilon(\tau)\int_s^t \partial_\tau\rho_h(\tau-\sigma)\cdot U_\varepsilon(\sigma)u(\sigma)d\sigma\ , \quad s \leqq \tau \leqq t\ . \tag{5.6}$$

$(\partial_\tau = \partial/\partial\tau)$. Moreover we have:

(iv) $\Phi_{\varepsilon,h} \in C([s, t]; L^n)$ and

$$\|\Phi_{\varepsilon,h}(\tau)\|_{L^n} \leqq M_0 M_2 \tag{5.7}$$

M_0 being a constant independent of ε, h, u.

Indeed, since by the closed graph theorem $(I+A)^{-2\alpha}$ is a bounded operator from L^2_σ into L^n (with a bound, say, M_0), it follows that $\Phi_{\varepsilon,h}(\tau)$ is continuous for τ in the norm of L^n, and that

$$\|\Phi_{\varepsilon,h}(\tau)\|_{L^n} \leqq M_0\int_s^t \rho_h(\tau-\sigma)\|u(\sigma)\|d\sigma \leqq M_0M_2$$

by hypothesis. Thus we have (iv).

Now we can take the $\Phi_{\varepsilon,h}$ as a test function Φ in (1.2) and there results

$$\int_s^t (u\cdot\nabla u, \Phi_{\varepsilon,h})dt = \int_s^t (f, \Phi_{\varepsilon,h})dt - (u(t), \Phi_{\varepsilon,h}(t)) + (u(s), \Phi_{\varepsilon,h}(s)) \tag{5.8}$$

since

$$\begin{aligned}\int_s^t &\{-(u, \partial_\tau\Phi_{\varepsilon,h}) + (\nabla u, \nabla\Phi_{\varepsilon,h})\}dt \\ &= \int_s^t \{-(u, \partial_\tau\Phi_{\varepsilon,h}) + (u, A\Phi_{\varepsilon,h})\}dt \\ &= \int_s^t\int_s^t \partial_\tau\rho_h(\tau-\sigma)(U_\varepsilon(\tau)u(\tau), U_\varepsilon(\sigma)u(\sigma))d\sigma d\tau \quad \text{(by (5.6))} \\ &= 0 \quad \text{(by the symmetry of } \rho_h(t))\ .\end{aligned}$$

We shall let $\varepsilon \to 0$ and then $h \to 0$ in (5.8). Since $((I+A)^{-2\alpha}e^{-(t-\sigma)A}u(t), u(\sigma))$ $(\equiv g(\sigma))$ is continuous for σ, we have, by Lemma 2.1,

$$\lim_{h\to 0}\lim_{\varepsilon\to 0}(u(t), \Phi_{\varepsilon,h}(t)) = \lim_{h\to 0}\int_s^t \rho_h(t-\sigma)g(\sigma)d\sigma = \frac{1}{2}(u(t), (I+A)^{-2\alpha}u(t)) .$$

Similarly,

$$\lim_{h\to 0}\lim_{\varepsilon\to 0}(u(s), \Phi_{\varepsilon,h}(s)) = \frac{1}{2}(u(s), e^{-2(t-s)A}(I+A)^{-2\alpha}u(s)) ;$$

and

$$\begin{aligned}\lim_{h\to 0}\lim_{\varepsilon\to 0}\int_s^t (f, \Phi_{\varepsilon,h})dt &= \int_s^t (f, e^{-2(t-\sigma)A}(I+A)^{-2\alpha}u(\sigma))d\sigma \\ &\leqq M_2\int_s^t \|Pf\|d\sigma \quad \text{(by (5.5))} .\end{aligned}$$

On the other hand, by Lemma 2.4 and (5.7)

$$\text{the LHS of (5.8)} \geqq -M\int_s^t \|\nabla u\|^2\|\Phi_{\varepsilon,h}\|_{L^n}dt \geqq -MM_0M_2\int_s^t \|\nabla u\|^2 dt .$$

Noting all the results obtained above, we let $\varepsilon \to 0$, and then $h \to 0$ in (5.8). Then we get the desired estimate (5.1). This completes the proof of Theorem 4.

5.3. Proof of Corollary 1. By the interpolation theorem,

$$\|\phi\| \leqq \|(I+A)^{-\alpha}\phi\|^{\beta}\|(I+A)^{1/2}\phi\|^{1-\beta}$$

where $\beta = 1/(1+2\alpha)$. Hence

$$\begin{aligned}(5.9)\qquad &\int_t^{t+1}\|u(s)\|^2 ds \\ &\leqq \left(\int_t^{t+1}\|(I+A)^{-\alpha}u(s)\|^2 ds\right)^{\beta}\left(\int_t^{t+1}\|(I+A)^{1/2}u(s)\|^2 ds\right)^{1-\beta} .\end{aligned}$$

Since

$$\begin{aligned}\int_t^{t+1}\|(I+A)^{1/2}u(s)\|^2 ds &= \int_t^{t+1}\|u(s)\|^2 ds + \int_t^{t+1}\|\nabla u(s)\|^2 ds \\ &\leqq M_2 + M_3 \quad \left(M_3 \equiv \int_0^{\infty}\|\nabla u\|dt < \infty\right)\end{aligned}$$

by hypothesis, it easily follows from Theorem 4 that the RHS of (5.9) tends to zero as $t \to \infty$. This proves Corollary 1.

5.4. Proof of Corollary 2. By the change of the variable and Corollary 1,

$$c=\lim_{t\to\infty}\int_0^1\|u(s+t)\|^2ds=\lim_{t\to\infty}\int_t^{t+1}\|u(s)\|^2ds=0\,.$$

This proves Corollary 2.

Appendix. PROOF OF PROPOSITION 1. We first recall the definition of a Hopf's weak solution ([8], [23]). Let $\mathscr{V}$ be the set of all C^∞ vector functions $\Phi=(\Phi^1,\cdots,\Phi^n)$ on $\Omega\times[0,T)$, which has its support in $\Omega\times[0,T)$, and are divergent free, i.e., $\sum_{i=1}^n(\partial/\partial x_i)\Phi^i(x,t)=0$. A function u on $\Omega\times(0,T)$ is called a Hopf's weak solution if

(H-1) for each T' $(0<T'<T)$, u is in the closure $V_{T'}$ of $\mathscr{V}$ under the norm of $L^2((0,T');H^1_{0,\sigma})$;

(H-2) the norm $\|u\|$ is uniformly bounded in t;

(H-3)

$$\int_0^T\{(u,\Phi_t)+(u,\Delta\Phi)+(u,u\cdot\nabla\Phi)\}dt=-\int_0^T(f,\Phi)dt-(a,\Phi(0))$$

for all Φ in $\mathscr{V}$.

Suppose that u is a weak solution in our sense. Since $C^\infty_{0,\sigma}$ is dense in $H^1_{0,\sigma}$, it follows from Lemma 2.2 that for any T' $(<T)$ u can be approximated by a sequence of functions u_N of the form: $u_N=\sum\lambda_j(t)\psi_j$ (finite sum) in the norm of $L^2((0,T');H^1_{0,\sigma})$, where $\lambda_j\in C^\infty([0,T'])$, $\psi_j\in C^\infty_{0,\sigma}$. Hence it is easy to see that $u_N\in V_{T''}$ and so $u\in V_{T''}$ for all T'' $(<T')$. Thus u satisfies the condition (H-1). Since (H-2), (H-3) are easily verified, u is a Hopf's weak solution. Under the assumption that $C^\infty_{0,\sigma}$ is dense in Y, we next show that a Hopf's weak solution u is a weak solution in our sense. By Lemma 2.2, any function Φ in $H^1((0,T);Y)$ such that for some T_0 $(<T)$ $\Phi(\cdot,t)=0$ on (T_0,T), can be approximated by a sequence of functions of the form $\sum\lambda_j(t)\psi_j$ (finite sum) in the norm of $L^2((0,T);Y)$ where $\lambda_j\in C_0^\infty([0,T))$, $\psi_j\in C^\infty_{0,\sigma}$. Hence it follows from Lemma 2.4 and (H-3) that (1.1) holds for such a Φ. It is now easy to see that a Hopf's solution is a weak solution in our sense. We next proceed to the proof of the latter part of Proposition 1. If $2\leqq n\leqq 4$, then by the Sobolev inequality, $H^1_{0,\sigma}\subset L^n$, and so $Y=H^1_{0,\sigma}$. Hence $C^\infty_{0,\sigma}$ is dense in Y. If Ω is a star-shaped bounded domain with respect to some point, say the origin, then for any u in Y, $u_\lambda\in Y$ and $u_\lambda\to u$ as $\lambda\to 1$ $(\lambda>1)$ in Y where $u_\lambda(x)=(Eu)(\lambda x)$ (E is defined in Lemma 3.1). We mollify u:

$$u_{\lambda,h}(x)=\int_{R^n}\rho_h(x-y)u_\lambda(y)dy=\rho_h*u_\lambda(x)$$

where $\rho_h *$ is the usual mollifier on $\boldsymbol{R}^n$. Then $u_{\lambda,h}\in C^\infty_{0,\sigma}$ and $u_{\lambda,h}\to u_\lambda$ as $h\to 0$ in Y. Thus $C^\infty_{0,\sigma}$ is dense in Y. Finally we consider the case $\Omega=\boldsymbol{R}^n$. If $f\in Y$, then we mollify f: $f_h=\rho_h*f$, $h>0$. Let B be the operator defined in Proposition 3, and $\{\zeta_N\}$ be a sequence of n-dimensional cut-off functions. We then set

$$f_{h,\mu,N}(x) = \left(-\delta_{jk}\Delta + \frac{\partial^2}{\partial x_j \partial x_k}\right)\zeta_N(x)(\mu + B)^{-1}f_h(x)\,, \quad \mu > 0\,.$$

It is easy to see that $f_{h,\mu,N} \in C_{0,\sigma}^\infty$. After letting $N \to \infty$, we let $\mu \to 0$, and then $h \to 0$; we see that $f_{h,\mu,N} \to f$ in Y. Thus $C_{0,\sigma}^\infty$ is dense in Y.

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ADDED IN PROOF. Professors J. Heywood and Y. Giga orally communicated to the author that $C_{0,\sigma}^{\infty}$ is dense in Y if Ω is a bounded or exterior domain. (See Proposition 1.)