Tôhoku Math. J.
42 (1990), 351–380

HYPERSURFACE SIMPLE $K3$ SINGULARITIES

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(Received May 31, 1989)

Introduction. In the theory of two-dimensional singularities, simple elliptic singularities and cusp singularities are regarded as the next most reasonable class of singularities after rational singularities. Cusp singularities appear on the Satake compactifications of Hilbert modular surfaces and have loops of rational curves as the exceptional sets of the minimal resolution. Simple elliptic singularities were investigated by Saito [11] in detail. By definition, each of them has a nonsingular elliptic curve as the exceptional set of the minimal resolution. Here we are interested especially in a hypersurface *simple elliptic* singularity (X, x). In this case, the defining equation of (X, x) is given by one of the following in some coordinates z_1, z_2, z_3 around x,

$$\tilde{E}_6 : z_1^3+z_2^3+z_3^3+\lambda_1 z_1 z_2 z_3=0 \qquad (E^2=-3)\,,$$

$$\tilde{E}_7 : z_1^2+z_2^4+z_3^4+\lambda_2 z_1 z_2 z_3=0 \qquad (E^2=-2)\,,$$

$$\tilde{E}_8 : z_1^2+z_2^3+z_3^6+\lambda_3 z_1 z_2 z_3=0 \qquad (E^2=-1)\,,$$

with the parameter satisfying $\lambda_1^3+27\neq 0$, $\lambda_2^4-64\neq 0$, $\lambda_3^6-432\neq 0$ and corresponding to the moduli of the elliptic curve E which appears as the exceptional set.

The purpose of this paper is to study similar properties for simple $K3$ singularities which we regard as natural generalizations in three-dimensional case of simple elliptic singularities.

The notion of a simple $K3$ singularity was defined by Watanabe [4] as a three-dimensional Gorenstein purely elliptic singularity of (0, 2)-type, whereas a simple elliptic singularity is a two-dimensional purely elliptic singularity of (0, 1)-type. Ishii [4] pointed out that a simple $K3$ singularity is characterized as a quasi-Gorenstein singularity such that the exceptional set of any minimal resolution is a normal $K3$ surface. Let $f\in \boldsymbol{C}[z_0, z_1, z_2, z_3]$ be a polynomial which is nondegenerate with respect to its Newton boundary $\Gamma(f)$ in the sense of [14], and whose zero locus $X=\{f=0\}$ in $\boldsymbol{C}^4$ has an isolated singularity at the origin $0\in \boldsymbol{C}^4$. Then the condition for $(X, 0)$ to be a simple $K3$ singularity is given by a property of the Newton boundary $\Gamma(f)$ of f (cf. Proposition 1.6). Tomari [12] showed that a minimal resolution $\pi: (\tilde{X}, E)\to(X, 0)$ of a simple $K3$ singularity is also obtained from $\Gamma(f)$. In this paper, we classify nondegenerate hypersurface simple $K3$ singularities and study the singularities on the $K3$ surface E through the minimal resolution π.

In §2, we classify nondegenerate hypersurface simple $K3$ singularities into ninety five classes in terms of the *weight* of f.

In § 3, we construct the minimal resolution π using the method of torus embeddings, and study the singularities on the weighted projective space $\boldsymbol{P}(p_1, p_2, p_3, p_4)$.

In § 4, we prove that the singularities on the normal $K3$ surface E are determined by the weight of f, and show the relation between the rank of singularities on E and the number of parameters in f.

We denote by $\boldsymbol{R}_+$ (resp. $\boldsymbol{R}_0$) the set of all positive (resp. nonnegative) real numbers. We define $\boldsymbol{Q}_0$, $\boldsymbol{Q}_+$, $\boldsymbol{Z}_0$, $\boldsymbol{Z}_+$ etc. similarly.

1. Preliminaries. In this section, we recall some definitions and results from [2], [4], [15] and [16].

First we define the plurigenera δ_m, $m \in \boldsymbol{N}$, for normal isolated singularities and define purely elliptic singularities. Let (X, x) be a normal isolated singularity in an n-dimensional analytic space X, and $\pi: (\tilde{X}, E) \to (X, x)$ a good resolution. In the following, we assume that X is a sufficiently small Stein neighbourhood of x.

DEFINITION 1.1 (Watanabe [15]). Let (X, x) be a normal isolated singularity. For any positive integer m,

$$\delta_m(X, x) := \dim_{\boldsymbol{C}} \Gamma(X - \{x\}, \mathcal{O}(mK)) / L^{2/m}(X - \{x\}),$$

where K is the canonical line bundle on $X - \{x\}$, and $L^{2/m}(X - \{x\})$ is the set of all $L^{2/m}$-integrable (at x) holomorphic m-ple n-forms on $X - \{x\}$.

Then δ_m is finite and does not depend on the choice of a Stein neighbourhood X.

DEFINITION 1.2 (Watanabe [15]). A singularity (X, x) is said to be purely elliptic if $\delta_m = 1$ for every $m \in \boldsymbol{N}$.

When X is a two-dimensional analytic space, purely elliptic singularities are quasi-Gorenstein singularities, i.e., there exists a non-vanishing holomorphic 2-form on $X - \{x\}$ (see [3]). But in higher dimension, purely elliptic singularities are not always quasi-Gorenstein (see [4], [17]).

In the following, we assume that (X, x) is quasi-Gorenstein. Let $E = \bigcup E_i$ be the decomposition of the exceptional set E into irreducible components, and write $K_{\tilde{X}} = \pi^* K_X + \sum_{i \in I} m_i E_i - \sum_{j \in J} m_j E_j$ with $m_i \geq 0$, $m_j > 0$. Ishii [2] defined the essential part of the exceptional set E as $E_J = \sum_{j \in J} m_j E_j$, and showed that if (X, x) is purely elliptic, then $m_j = 1$ for all $j \in J$.

DEFINITION 1.3. (Ishii [2]). A quasi-Gorenstein purely elliptic singularity (X, x) is of $(0, i)$-type if $H^{n-1}(E_J, \mathcal{O}_E)$ consists of the $(0, i)$-Hodge component $H^{0,i}(E_J)$, where

$$\boldsymbol{C} \simeq H^{n-1}(E_J, \mathcal{O}_E) = \mathrm{Gr}_F^0 H^{n-1}(E_J) = \bigoplus_{i=0}^{n-1} H^{0,i}(E_J).$$

DEFINITION-PROPOSITION 1.4 (Watanabe-Ishii [4]). *A three-dimensional singularity* (X, x) *is a simple K3 singularity if the following two equivalent conditions are satisfied:*

(1) (X, x) *is Gorenstein purely elliptic of* $(0, 2)$*-type.*

(2) (X, x) *is quasi-Gorenstein and the exceptional divisor E is a normal K3 surface for any minimal resolution* $\pi: (\tilde{X}, E) \to (X, x)$.

REMARK 1.5. A minimal resolution $\pi: (\tilde{X}, E) \to (X, x)$ is a proper morphism with $\tilde{X} - E \simeq X - \{x\}$, where $\tilde{X}$ has only terminal singularities and $K_{\tilde{X}}$ is numerically effective with respect to π.

Next we consider the case where (X, x) is a hypersurface singularity defined by a nondegenerate polynomial $f = \sum a_v z^v \in \boldsymbol{C}[z_0, z_1, \cdots, z_n]$, and $x = 0 \in \boldsymbol{C}^{n+1}$. Recall that the Newton boundary $\Gamma(f)$ of f is the union of the compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of $\bigcup_{a_v \neq 0} (v + \boldsymbol{R}_0^{n+1})$ in $\boldsymbol{R}^{n+1}$. For any face Δ of $\Gamma_+(f)$, set $f_\Delta := \sum_{v \in \Delta} a_v z^v$. We say f to be nondegenerate, if

$$\frac{\partial f_\Delta}{\partial z_0} = \frac{\partial f_\Delta}{\partial z_1} = \cdots = \frac{\partial f_\Delta}{\partial z_n} = 0$$

has no solution in $(\boldsymbol{C}^*)^{n+1}$ for any face Δ. When f is nondegenerate, the condition for (X, x) to be a purely elliptic singularity is given as follows:

THEOREM 1.6 (Watanabe [16]). *Let f be a nondegenerate polynomial and suppose* $X = \{f = 0\}$ *has an isolated singularity at* $x = 0 \in \boldsymbol{C}^{n+1}$.

(1) (X, x) *is purely elliptic if and only if* $(1, 1, \cdots, 1) \in \Gamma(f)$.

(2) *Let* $n = 3$ *and let* Δ_0 *be the face of* $\Gamma(f)$ *containing* $(1, 1, 1, 1)$ *in the relative interior of* Δ_0. *Then* (X, x) *is a simple K3 singularity if and only if* $\dim_{\boldsymbol{R}} \Delta_0 = 3$.

Thus if f is nondegenerate and defines a simple $K3$ singularity, then f_{Δ_0} is a quasi-homogeneous polynomial of a uniquely determined weight α called the weight of f and denoted $\alpha(f)$. Nemely, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \boldsymbol{Q}_+^4$ and $\deg_\alpha(v) := \sum_{i=1}^4 \alpha_i v_i = 1$ for any $v \in \Delta_0$. In particular, $\sum_{i=1}^4 \alpha_i = 1$, since $(1, 1, 1, 1)$ is always contained in Δ_0.

2. Weights of hypersurface simple $K3$ singularities. In this section, we calculate the weights of hypersurface simple $K3$ singularities defined by nondegenerate polynomials.

Let $W' := \{\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \boldsymbol{Q}_+^4 \mid \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1\}$ and for an element α of W', set $T(\alpha) := \{v \in \boldsymbol{Z}_0^4 \mid \alpha \cdot v = 1\}$ and $\langle T(\alpha) \rangle := \{\sum_{v \in T(\alpha)} t_v \cdot v \in \boldsymbol{R}^4 \mid t_v \in \boldsymbol{R}_0\}$. Then the set $\langle T(\alpha) \rangle$ is a closed cone in $\boldsymbol{R}^4$ spanned by $T(\alpha)$.

Let $W_4 := \{\alpha \in W' \mid (1, 1, 1, 1) \in \mathrm{Int}\langle T(\alpha), \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4\}$. By Theorem 1.6, W_4 is the set of weights of simple $K3$ singularities.

PROPOSITION 2.1. *The cardinality of W_4 is* 95.

In Table 2.2 can be found the complete list of weights $\alpha \in W_4$ and examples of $f=\sum_{v\in T(\alpha)} a_v z^v$ such that f is quasi-homogeneous and that $\{f=0\}\subset \boldsymbol{C}^4$ has a simple $K3$ singularity at the origin $0\in \boldsymbol{C}^4$. The polynomials f in Table 2.2 are chosen to satisfy the condition that $a_v \neq 0$ if and only if v is a extremal point of the convex hull of $T(\alpha)$ in $\boldsymbol{R}^4$. In particular, $\Gamma(f)=\Delta_0$ is the convex hull of $T(\alpha)$.

We express a weight α in W_4 as $\alpha=(p_1/p, p_2/p, p_3/p, p_4/p)$, where p, p_i are positive integers with $\boldsymbol{gcd}(p_1, p_2, p_3, p_4)=1$.

TABLE 2.2.

No.	weight α	f	$\sharp T(\alpha)$
1	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$	$x^4+y^4+z^4+w^4$	35
2	$\left(\frac{1}{3},\frac{1}{4},\frac{1}{4},\frac{1}{6}\right)$	$x^3+y^4+z^4+w^6$	15
3	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}\right)$	$x^3+y^3+z^6+w^6$	30
4	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{4},\frac{1}{12}\right)$	$x^3+y^3+z^4+w^{12}$	21
5	$\left(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)$	$x^2+y^6+z^6+w^6$	39
6	$\left(\frac{1}{2},\frac{1}{5},\frac{1}{5},\frac{1}{10}\right)$	$x^2+y^5+z^5+w^{10}$	28
7	$\left(\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{8}\right)$	$x^2+y^4+z^8+w^8$	35
8	$\left(\frac{1}{2},\frac{1}{4},\frac{1}{6},\frac{1}{12}\right)$	$x^2+y^4+z^6+w^{12}$	27
9	$\left(\frac{1}{2},\frac{1}{4},\frac{1}{5},\frac{1}{20}\right)$	$x^2+y^4+z^5+w^{20}$	23
10	$\left(\frac{1}{2},\frac{1}{3},\frac{1}{12},\frac{1}{12}\right)$	$x^2+y^3+z^{12}+w^{12}$	39
11	$\left(\frac{1}{2},\frac{1}{3},\frac{1}{10},\frac{1}{15}\right)$	$x^2+y^3+z^{10}+w^{15}$	18
12	$\left(\frac{1}{2},\frac{1}{3},\frac{1}{9},\frac{1}{18}\right)$	$x^2+y^3+z^9+w^{18}$	30
13	$\left(\frac{1}{2},\frac{1}{3},\frac{1}{8},\frac{1}{24}\right)$	$x^2+y^3+z^8+w^{24}$	27
14	$\left(\frac{1}{2},\frac{1}{3},\frac{1}{7},\frac{1}{42}\right)$	$x^2+y^3+z^7+w^{42}$	24

TABLE 2.2. (continued)

No.	weight α	f	$\sharp T(\alpha)$
15	$\left(\frac{1}{3},\frac{4}{15},\frac{1}{5},\frac{1}{5}\right)$	$x^3+y^3z+y^3w+z^5-w^5$	12
16	$\left(\frac{1}{3},\frac{7}{24},\frac{1}{4},\frac{1}{8}\right)$	$x^3+y^3w+z^4+w^8$	9
17	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{5},\frac{2}{15}\right)$	$x^3+y^3+z^5+xw^5+yw^5+zw^6$	14
18	$\left(\frac{1}{3},\frac{1}{3},\frac{2}{9},\frac{1}{9}\right)$	$x^3+y^3+xz^3+yz^3+z^4w+w^9$	23
19	$\left(\frac{3}{8},\frac{1}{4},\frac{1}{4},\frac{1}{8}\right)$	$x^2y+x^2z+x^2w^2+y^4+z^4+w^8$	24
20	$\left(\frac{3}{8},\frac{1}{3},\frac{1}{4},\frac{1}{24}\right)$	$x^2z+x^2w^6+y^3+z^4+w^{24}$	18
21	$\left(\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right)$	$x^2y+x^2z+x^2w+y^5+z^5+w^5$	34
22	$\left(\frac{2}{5},\frac{1}{3},\frac{1}{5},\frac{1}{15}\right)$	$x^2z+x^2w^3+y^3+z^5-w^{15}$	21
23	$\left(\frac{5}{12},\frac{1}{4},\frac{1}{6},\frac{1}{6}\right)$	$x^2z+x^2w+y^4+z^6+w^6$	17
24	$\left(\frac{5}{12},\frac{1}{3},\frac{1}{6},\frac{1}{12}\right)$	$x^2z+x^2w^2+y^3+z^6+w^{12}$	24
25	$\left(\frac{4}{9},\frac{1}{3},\frac{1}{9},\frac{1}{9}\right)$	$x^2z+x^2w+y^3+z^9-w^9$	33
26	$\left(\frac{9}{20},\frac{1}{4},\frac{1}{5},\frac{1}{10}\right)$	$x^2w+y^4+z^5+w^{10}$	13
27	$\left(\frac{11}{24},\frac{1}{3},\frac{1}{8},\frac{1}{12}\right)$	$x^2w+y^3+z^8+w^{12}$	15
28	$\left(\frac{10}{21},\frac{1}{3},\frac{1}{7},\frac{1}{21}\right)$	$x^2w+y^3+z^7+w^{21}$	24
29	$\left(\frac{1}{2},\frac{1}{5},\frac{1}{6},\frac{2}{15}\right)$	$x^2+y^5+z^6+yw^6+z^2w^5$	10
30	$\left(\frac{1}{2},\frac{1}{5},\frac{7}{40},\frac{1}{8}\right)$	$x^2+y^5+z^5w+w^8$	8
31	$\left(\frac{1}{2},\frac{5}{24},\frac{1}{6},\frac{1}{8}\right)$	$x^2+y^4z+y^3w^3+z^6+w^8$	12
32	$\left(\frac{1}{2},\frac{3}{14},\frac{1}{7},\frac{1}{7}\right)$	$x^2+y^4z+y^4w+z^7-w^7$	19
33	$\left(\frac{1}{2},\frac{2}{9},\frac{1}{6},\frac{1}{9}\right)$	$x^2+y^3z^2+y^4w+z^6+w^9$	16

TABLE 2.2. (continued)

No.	weight α	f	$\sharp T(\alpha)$
34	$\left(\frac{1}{2}, \frac{7}{30}, \frac{1}{5}, \frac{1}{15}\right)$	$x^2+y^4w+z^5+w^{15}$	13
35	$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{7}, \frac{3}{28}\right)$	$x^2+y^4+z^7+yw^7+zw^8$	12
36	$\left(\frac{1}{2}, \frac{1}{4}, \frac{3}{20}, \frac{1}{10}\right)$	$x^2+y^4+yz^5+z^6w+w^{10}$	16
37	$\left(\frac{1}{2}, \frac{1}{4}, \frac{3}{16}, \frac{1}{16}\right)$	$x^2+y^4+yz^4+z^5w+w^{16}$	24
38	$\left(\frac{1}{2}, \frac{4}{15}, \frac{1}{5}, \frac{1}{30}\right)$	$x^2+y^3z+y^3w^6+z^5-w^{30}$	21
39	$\left(\frac{1}{2}, \frac{5}{18}, \frac{1}{6}, \frac{1}{18}\right)$	$x^2+y^3z+y^3w^3+z^6+w^{18}$	24
40	$\left(\frac{1}{2}, \frac{2}{7}, \frac{1}{7}, \frac{1}{14}\right)$	$x^2+y^3z+y^3w^2+z^7-w^{14}$	27
41	$\left(\frac{1}{2}, \frac{7}{24}, \frac{1}{8}, \frac{1}{12}\right)$	$x^2+y^3z+y^2w^5+z^8+w^{12}$	16
42	$\left(\frac{1}{2}, \frac{3}{10}, \frac{1}{10}, \frac{1}{10}\right)$	$x^2+y^3z+y^3w+z^{10}+w^{10}$	36
43	$\left(\frac{1}{2}, \frac{11}{36}, \frac{1}{9}, \frac{1}{12}\right)$	$x^2+y^3w+z^9+w^{12}$	12
44	$\left(\frac{1}{2}, \frac{5}{16}, \frac{1}{8}, \frac{1}{16}\right)$	$x^2+y^2z^3+y^3w+z^8+w^{16}$	28
45	$\left(\frac{1}{2}, \frac{9}{28}, \frac{1}{7}, \frac{1}{28}\right)$	$x^2+y^3w+z^7+w^{28}$	24
46	$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{11}, \frac{5}{66}\right)$	$x^2+y^3+z^{11}+zw^{12}$	9
47	$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{21}, \frac{1}{14}\right)$	$x^2+y^3+yz^7+z^9w^2+w^{14}$	13
48	$\left(\frac{1}{2}, \frac{1}{3}, \frac{5}{48}, \frac{1}{16}\right)$	$x^2+y^3+z^9w+w^{16}$	12
49	$\left(\frac{1}{2}, \frac{1}{3}, \frac{5}{42}, \frac{1}{21}\right)$	$x^2+y^3+z^8w+w^{21}$	15
50	$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{15}, \frac{1}{30}\right)$	$x^2+y^3+yz^5+z^7w^2+w^{30}$	25
51	$\left(\frac{1}{2}, \frac{1}{3}, \frac{5}{36}, \frac{1}{36}\right)$	$x^2+y^3+z^7w+w^{36}$	24
52	$\left(\frac{1}{3}, \frac{1}{4}, \frac{2}{9}, \frac{7}{36}\right)$	$x^3+y^4+xz^3+zw^4$	5

TABLE 2.2. (continued)

No.	weight α	f	$\sharp T(\alpha)$
53	$\left(\frac{1}{3},\frac{5}{18},\frac{2}{9},\frac{1}{6}\right)$	$x^3+y^3w+y^2z^2+xz^3+z^3w^2+w^6$	10
54	$\left(\frac{1}{3},\frac{2}{7},\frac{5}{21},\frac{1}{7}\right)$	$x^3+y^3w+yz^3+z^3w^2-w^7$	9
55	$\left(\frac{7}{20},\frac{3}{10},\frac{1}{4},\frac{1}{10}\right)$	$x^2y+x^2w^3+y^3w+z^4-w^{10}$	11
56	$\left(\frac{11}{30},\frac{4}{15},\frac{1}{5},\frac{1}{6}\right)$	$x^2y+y^3z+z^5+w^6$	6
57	$\left(\frac{3}{8},\frac{1}{4},\frac{5}{24},\frac{1}{6}\right)$	$x^2y+y^4+xz^3+z^4w+w^6$	8
58	$\left(\frac{3}{8},\frac{5}{16},\frac{1}{4},\frac{1}{16}\right)$	$x^2z+x^2w^4+xy^2+y^3w+z^4+w^{16}$	19
59	$\left(\frac{8}{21},\frac{1}{3},\frac{5}{21},\frac{1}{21}\right)$	$x^2z+x^2w^5+y^3+z^4w-w^{21}$	18
60	$\left(\frac{7}{18},\frac{1}{3},\frac{2}{9},\frac{1}{18}\right)$	$x^2z+x^2w^4+y^3+yz^3+z^4w^2+w^{18}$	19
61	$\left(\frac{11}{28},\frac{1}{4},\frac{3}{14},\frac{1}{7}\right)$	$x^2z+y^4+z^4w+w^7$	7
62	$\left(\frac{2}{5},\frac{1}{4},\frac{1}{5},\frac{3}{20}\right)$	$x^2z+xw^4+y^4+yw^5+z^5+z^2w^4$	10
63	$\left(\frac{2}{5},\frac{3}{10},\frac{1}{5},\frac{1}{10}\right)$	$x^2z+x^2w^2+xy^2+y^2z^2+y^3w+z^5+w^{10}$	23
64	$\left(\frac{5}{12},\frac{7}{24},\frac{1}{6},\frac{1}{8}\right)$	$x^2z+xy^2+y^3w+z^6+w^8$	10
65	$\left(\frac{14}{33},\frac{1}{3},\frac{5}{33},\frac{1}{11}\right)$	$x^2z+y^3+z^6w+w^{11}$	9
66	$\left(\frac{3}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7}\right)$	$x^2z+x^2w+xy^2+y^3z+y^3w+z^7+w^7$	31
67	$\left(\frac{3}{7},\frac{1}{3},\frac{1}{7},\frac{2}{21}\right)$	$x^2z+xw^6+y^3+yw^7+z^7+zw^9$	14
68	$\left(\frac{13}{30},\frac{1}{3},\frac{2}{15},\frac{1}{10}\right)$	$x^2z+y^3+yz^5+z^6w^2+w^{10}$	10
69	$\left(\frac{7}{16},\frac{1}{4},\frac{3}{16},\frac{1}{8}\right)$	$x^2w+xz^3+y^4+yz^4+z^4w^2+w^8$	14
70	$\left(\frac{4}{9},\frac{5}{18},\frac{1}{6},\frac{1}{9}\right)$	$x^2w+xy^2+y^3z+y^2w^4+z^6+w^9$	14
71	$\left(\frac{7}{15},\frac{4}{15},\frac{1}{5},\frac{1}{15}\right)$	$x^2w+xy^2+y^3z+y^3w^3+z^5+w^{15}$	22

TABLE 2.2. (continued)

No.	weight α	f	$\sharp T(\alpha)$
72	$\left(\frac{7}{15},\frac{1}{3},\frac{2}{15},\frac{1}{15}\right)$	$x^2w+xz^4+y^3+yz^5+z^7w+w^{15}$	26
73	$\left(\frac{1}{2},\frac{1}{5},\frac{4}{25},\frac{7}{50}\right)$	$x^2+y^5+yz^5+zw^6$	6
74	$\left(\frac{1}{2},\frac{7}{32},\frac{5}{32},\frac{1}{8}\right)$	$x^2+y^4w+yz^5+z^4w^3+w^8$	9
75	$\left(\frac{1}{2},\frac{5}{22},\frac{2}{11},\frac{1}{11}\right)$	$x^2+y^4w+y^2z^3+z^5w+w^{11}$	14
76	$\left(\frac{1}{2},\frac{3}{13},\frac{5}{26},\frac{1}{13}\right)$	$x^2+y^4w+yz^4+z^4w^3+w^{13}$	13
77	$\left(\frac{1}{2},\frac{7}{26},\frac{5}{26},\frac{1}{26}\right)$	$x^2+y^3z+y^3w^5+z^5w+w^{26}$	21
78	$\left(\frac{1}{2},\frac{3}{11},\frac{2}{11},\frac{1}{22}\right)$	$x^2+y^3z+y^3w^4+yz^4+z^5w^2+w^{22}$	22
79	$\left(\frac{1}{2},\frac{9}{32},\frac{5}{32},\frac{1}{16}\right)$	$x^2+y^3z+y^2w^7+z^6w+w^{16}$	13
80	$\left(\frac{1}{2},\frac{13}{44},\frac{5}{44},\frac{1}{11}\right)$	$x^2+y^3z+z^8w+w^{11}$	9
81	$\left(\frac{1}{2},\frac{4}{13},\frac{3}{26},\frac{1}{13}\right)$	$x^2+y^3w+yz^6+z^8w+w^{13}$	16
82	$\left(\frac{1}{2},\frac{7}{22},\frac{3}{22},\frac{1}{22}\right)$	$x^2+y^3w+yz^5+z^7w+w^{22}$	25
83	$\left(\frac{1}{2},\frac{1}{3},\frac{5}{54},\frac{2}{27}\right)$	$x^2+y^3+yw^9+z^{10}w+z^2w^{11}$	10
84	$\left(\frac{1}{3},\frac{7}{27},\frac{2}{9},\frac{5}{27}\right)$	$x^3+xz^3+y^3z+yw^4+z^2w^3$	6
85	$\left(\frac{5}{14},\frac{2}{7},\frac{3}{14},\frac{1}{7}\right)$	$x^2y+x^2w^2+xz^3+y^3w+y^2z^2+z^4w+w^7$	13
86	$\left(\frac{9}{25},\frac{7}{25},\frac{1}{5},\frac{4}{25}\right)$	$x^2y+xw^4+y^3w+z^5+zw^5$	7
87	$\left(\frac{5}{13},\frac{4}{13},\frac{3}{13},\frac{1}{13}\right)$	$x^2z+x^2w^3+xy^2+y^3w+yz^3+z^4w+w^{13}$	20
88	$\left(\frac{11}{27},\frac{1}{3},\frac{5}{27},\frac{2}{27}\right)$	$x^2z+xw^8+y^3+yw^9+z^5w+zw^{11}$	11
89	$\left(\frac{5}{11},\frac{3}{11},\frac{2}{11},\frac{1}{11}\right)$	$x^2w+xy^2+xz^3+y^3z+y^3w^2+yz^4+z^5w+w^{11}$	24
90	$\left(\frac{1}{2},\frac{7}{34},\frac{3}{17},\frac{2}{17}\right)$	$x^2+y^4z+y^2w^5+z^5w+zw^7$	8

TABLE 2.2. (continued)

No.	weight α	f	$\sharp T(\alpha)$
91	$\left(\frac{1}{2}, \frac{4}{19}, \frac{3}{19}, \frac{5}{38}\right)$	$x^2+y^4z+yz^5+yw^6+z^3w^4$	7
92	$\left(\frac{1}{2}, \frac{11}{38}, \frac{5}{38}, \frac{3}{38}\right)$	$x^2+y^3z+yw^9+z^7w+zw^{11}$	10
93	$\left(\frac{1}{2}, \frac{5}{17}, \frac{2}{17}, \frac{3}{34}\right)$	$x^2+y^3z+yz^6+yw^8+z^7w^2+zw^{10}$	11
94	$\left(\frac{7}{19}, \frac{5}{19}, \frac{4}{19}, \frac{3}{19}\right)$	$x^2y+xz^3+xw^4+y^3z+y^2w^3+z^4w+zw^5$	9
95	$\left(\frac{7}{17}, \frac{5}{17}, \frac{3}{17}, \frac{2}{17}\right)$	$x^2z+xy^2+xw^5+y^3w+yz^4+yw^6+z^5w+zw^7$	13

Here we have the following proposition, essentially used in §3 and §4.

PROPOSITION 2.3. (1) *For any* $i=1, 2, 3, 4$, *one of the following is satisfied*:

(a) $p_i \mid p$,

(b) $p_i \mid (p-p_j)$ for some $j \neq i$.

(2) $\boldsymbol{gcd}(p_i, p_j, p_k)=1$ *for all distinct* i, j, k.

(3) *Let* $a_{ij} := \boldsymbol{gcd}(p_i, p_j)$ $(i \neq j)$, *then* $a_{ij} \mid p$.

(4) *If* $p_i \mid p$ *and* $p_i \mid (p-p_j)$, *then* $a_{ij}=p_i$ *and* $a_{ik}=a_{il}=1$, *where we set* $\{i, j, k, l\}=\{1, 2, 3, 4\}$.

PROOF. (1) The proof will be found in the proof of Proposition 2.1.

(2) Since $\boldsymbol{gcd}(p_1, p_2, p_3, p_4)=1$, if $\boldsymbol{gcd}(p_i, p_j, p_k)=d>1$, then $\boldsymbol{gcd}(p_l, d)=1$ and $\boldsymbol{gcd}(p, d)=1$. Thus $v_l \geq 1$ for every $v \in T(\alpha)$, a contradiction to the condition $(1, 1, 1, 1) \in \operatorname{Int}\langle T(\alpha)\rangle$.

(3) If there exists a_{ij} such that $a_{ij} \nmid p$, then $v_k \neq 0$ or $v_l \neq 0$ for every $v \in T(\alpha)$. Since the point $(1, 1, 1, 1)$ is contained the hyperplane $\{x_k+x_l=2\}$ in $\boldsymbol{R}^4$, we have $\{x_k+x_l<2\} \cap T(\alpha) \neq \varnothing$. Assume that $v_k=1$ and $v_l=0$, i.e., $v_i p_i+v_j p_j=p-p_k$. Then $a_{ij} \mid p_l$, hence we have $a_{ij}=1$ by (2), a contradiction.

(4) From the condition $p_i \mid (p-p_j)$, we have $p_i \mid (p_k+p_l)$, hence $a_{ik}=a_{il}=1$ by (2). The other assertion $a_{ij}=p_i$ follows from the fact $p_i \mid p_j$. Q.E.D.

REMARK 2.4. One can check the assertion (1) directly by Table 2.2.

PROOF OF PROPOSITION 2.1. Let $\alpha=(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be a weight in W_4, and let H_0 be the hyperplane in $\boldsymbol{R}^4$ containing $T(\alpha)$. Denote by δ the point $(1, 1, 1, 1)$ in $T(\alpha)$.

First, we explain the outline of the proof. By definition, there exist $\nu, \mu \in T(\alpha)$ with. $\nu_1 \geq 2$, $\mu_2 \geq 2$. Let H be the plane in H_0 through δ, ν, and μ. Then by definition again, there exists a point $\lambda \in T(\alpha)$ not contained in H. Conversely, for fixed ν, μ, and λ, we can calculate the weight α. Thus we may classify all the possible triples of points $\{\nu, \mu, \lambda\}$ as above and check the condition $\delta \in \mathrm{Int}\langle T(\alpha)\rangle$. We proceed in four steps.

Step 1. We classify points ν in $T(\alpha)$ with $\nu_1 \geq 2$. Since $\alpha_1 \geq 1/4$, we have $2 \leq \nu_1 \leq 4$.

Case 1. $\nu_1 = 3$ or 4.

Since $3\alpha_1 + \alpha_j \geq 1$ $(j = 1, 2, 3, 4)$, only possible cases are

$$\nu = (4, 0, 0, 0), (3, 1, 0, 0), (3, 0, 1, 0), \alpha = (1/4, 1/4, 1/4, 1/4) \quad \text{or}$$

$$\nu = (3, 0, 0, 1), \alpha_1 = \alpha_2 = \alpha_3 \quad \text{or}$$

$$\nu = (3, 0, 0, 0), \alpha_1 = 1/3.$$

Case 2. $\nu_1 = 2$ and $\nu_2 \neq 0$.

Since $2\alpha_1 + \alpha_2 + \alpha_j \geq 1$ $(j = 2, 3, ,4)$, we have

$$\nu = (2, 2, 0, 0), (2, 1, 1, 0), \alpha = (1/4, 1/4, 1/4, 1/4) \quad \text{or}$$

$$\nu = (2, 1, 0, 1), \alpha_1 = \alpha_2 = \alpha_3 \quad \text{or}$$

$$\nu = (2, 1, 0, 0), 2\alpha_1 + \alpha_2 = 1 .$$

Case 3. $\nu_1 = 2$, $\nu_2 = 0$ and $\nu_3 \neq 0$.

Since $2\alpha_1 + \alpha_3 + \alpha_j \geq 1$ $(j = 3, 4)$, we have

$$\nu = (2, 0, 2, 0), \alpha_1 = \alpha_2, \alpha_3 = \alpha_4 \quad \text{or}$$

$$\nu = (2, 0, 1, 1), \alpha_1 = \alpha_2 \quad \text{or}$$

$$\nu = (2, 0, 1, 0), 2\alpha_1 + \alpha_3 = 1 .$$

Case 4. $\nu = (2, 0, 0, n)$, $2\alpha_1 + n\alpha_4 = 1$ $(n \geq 0)$.

Thus if $\alpha \in W_4$, then α satisfies one of the following conditions:

(A) $\alpha_1 = \alpha_2$ (i.e., $\nu(2, 0, 1, 1) \in T(\alpha)$) .

(B) $\alpha_1 = 1/3$ (i.e., $\nu(3, 0, 0, 0) \in T(\alpha)$) .

(C) $2\alpha_1 + \alpha_2 = 1$ (i.e., $\nu(2, 1, 0, 0) \in T(\alpha)$) .

(D) $2\alpha_1 + \alpha_3 = 1$ (i.e., $\nu(2, 0, 1, 0) \in T(\alpha)$) .

(E) $2\alpha_1 + n\alpha_4 = 1$ for $n \geq 0$ (i.e., $\nu(2, 0, 0, n) \in T(\alpha)$) .

Steps 2 and 3. Next we classify points $\mu \in T(\alpha)$ with $\mu_2 \geq 2$, and determine weights by searching another point λ.

Step 2 of the case (A) $\alpha_1 = \alpha_2$ (i.e., $\nu(2, 0, 1, 1) \in T(\alpha)$). There exists $\mu \in T(\alpha)$ such that $\mu_1 = 0$ and $2 \leq \mu_2 \leq 4$. If $\mu_2 = 2$, then $\nu, \mu, \delta \in \{x_1 + x_2 = 2\}$, hence we may assume $3 \leq \mu_2$.

(A-1) Assume that $\mu_2 = 4$. Then $\alpha_1 = \alpha_2 = 1/4$ and $\alpha = (1/4, 1/4, 1/4, 1/4)$.

(A-2) Assume that $\mu_2 = 3$. Since $3\alpha_2 + \alpha_j \geq 1$ $(j = 3, 4)$, we have

$$\mu=(0,3,1,0) \quad \text{and} \quad \alpha=(1/4,\ 1/4,\ 1/4,\ 1/4) \quad \text{or}$$

$$\mu=(0,3,0,1) \quad \text{or}$$

$$\mu=(0,3,0,0)\,.$$

Step 3 of the case (A). If $\mu=(0,3,0,1)$, then $\alpha_1=\alpha_2=\alpha_3$ and $\nu, \mu, \delta\in\{x_4=1\}$. So there exists $(n,0,0,0)\in T(\alpha)$ with $n\geq 4$. Thus $n=4$ and $\alpha=(1/4,\ 1/4,\ 1/4,\ 1/4)$. If $\mu=(0,3,0,0)$, then $\alpha_1=\alpha_2=\alpha_3+\alpha_4$ and $\nu, \mu, \delta\in\{x_3=x_4\}$. So there exists $(0,0,m,n)\in T(\alpha)$ with $m>n$. Since $(0,0,3,3)\in T(\alpha)$, we have

$$\alpha_3=\alpha_4 \quad \text{and} \quad \alpha=\left(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}\right) \quad \text{or}$$

$$2\alpha_3=3\alpha_4 \quad \text{and} \quad \alpha=\left(\frac{1}{3},\frac{1}{3},\frac{1}{5},\frac{2}{15}\right) \quad \text{or}$$

$$\alpha_3=2\alpha_4 \quad \text{and} \quad \alpha=\left(\frac{1}{3},\frac{1}{3},\frac{2}{9},\frac{1}{9}\right) \quad \text{or}$$

$$\alpha_3=3\alpha_4 \quad \text{and} \quad \alpha=\left(\frac{1}{3},\frac{1}{3},\frac{1}{4},\frac{1}{12}\right).$$

Step 2 of the case (B) $\alpha_1=1/3$ (i.e., $\nu(3,0,0,0)\in T(\alpha)$). Since $\alpha_2+\alpha_3+\alpha_4=2/3$, we have $2/9\leq\alpha_2\leq 1/3$, so there exists $\mu\in T(\alpha)$ such that $2\leq\mu_2\leq 4$.

(B-1) Assume $\mu_2=4$. Since $1=3(\alpha_2+\alpha_3+\alpha_4)/2\leq 4\alpha_2+\alpha_4/2$,

(B-i) $\mu=(0,4,0,0)$ and $\alpha_1=1/3$, $\alpha_2=1/4$.

(B-2) Assume $\mu_2=3$. Since $1=3(\alpha_2+\alpha_3+\alpha_4)/2\leq 3\alpha_2+3\alpha_4/2$,

(B-ii) $\mu=(1,3,0,0)$ and $\alpha=(1/3,\ 2/9,\ 2/9,\ 2/9)$ or

(B-iii) $\mu=(0,3,1,0)$ and $3\alpha_2+\alpha_3=1$ or

(B-iv) $\mu=(0,3,0,1)$ and $3\alpha_2+\alpha_4=1$ or

(B-v) $\mu=(0,3,0,0)$ and $\alpha_1=\alpha_2=1/3$.

(B-3) Assume $\mu_2=2$. Since $1=3(\alpha_2+\alpha_3+\alpha_4)/2\leq 2\alpha_2+\alpha_3+3\alpha_4/2$ and $7/9\leq \alpha_1+2\alpha_2\leq 1$,

$\mu=(1,2,1,0)$ and $\alpha=(1/3,\ 2/9,\ 2/9,\ 2/9)$ (Case (B-ii)) or

(B-vi) $\mu=(1,2,0,1)$ and $\alpha_2=\alpha_3$ or

$\mu=(1,2,0,0)$ and $\alpha_1=\alpha_2=1/3$ (Case (B-v)) or

(B-vii) $\mu=(0,2,2,0)$ and $\alpha_2+\alpha_3=1/2$ or

$\mu=(0, 2, 1, 1)$ and $\alpha_1=\alpha_2=1/3$ (Case (B-v)) or

(B-viii) $\mu=(0, 2, 0, n)$ with $n\geq 2$ and $2\alpha_2+n\alpha_4=1$.

Step 3 of the case (B). We study the above eight cases in more detail.

(B-i). Assume $\alpha_1=1/3$, $\alpha_2=1/4$, $\nu=(3, 0, 0, 0)$ and $\mu=(0, 4, 0, 0)$. Then $\alpha_3=5/12-\alpha_4$ and $1/6\leq\alpha_4\leq 5/24$. If $\lambda\in T(\alpha)$, then

$$\frac{1}{3}\lambda_1+\frac{1}{4}\lambda_2+\left(\frac{5}{12}-\alpha_4\right)\lambda_3+\alpha_4\lambda_4=\frac{1}{3}\lambda_1+\frac{1}{4}\lambda_2+\frac{5}{12}\lambda_3+(-\lambda_3+\lambda_4)\alpha_4=1,$$

and since $\nu, \mu, \delta\in\{-x_3+x_4=0\}$, there exists $\lambda\in T(\alpha)$ with $-\lambda_3+\lambda_4<0$. For this λ, we have

$$\frac{1}{3}\lambda_1+\frac{1}{4}\lambda_2+\frac{5}{24}\lambda_3+\frac{5}{24}\lambda_4\leq 1\leq\frac{1}{3}\lambda_1+\frac{1}{4}\lambda_2+\frac{1}{4}\lambda_3+\frac{1}{6}\lambda_4,$$

and thus

$$\lambda=(0, 0, 4, 0), (0, 1, 3, 0), (0, 2, 2, 0), (1, 0, 3, 0), (1, 0, 2, 1), (1, 1, 2, 0),$$

$$\alpha=\left(\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}\right), \left(\frac{1}{3}, \frac{1}{4}, \frac{2}{9}, \frac{7}{36}\right), \left(\frac{1}{3}, \frac{1}{4}, \frac{5}{24}, \frac{5}{24}\right).$$

Similarly, from the cases (B-iii), $\cdots$, (B-vii), we obtain the following weights:

(B-iii) $$\alpha=\left(\frac{1}{3}, \frac{4}{15}, \frac{1}{5}, \frac{1}{5}\right), \left(\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}\right), \left(\frac{1}{3}, \frac{7}{27}, \frac{2}{9}, \frac{5}{27}\right).$$

(B-iv) $$\alpha=\left(\frac{1}{3}, \frac{4}{15}, \frac{1}{5}, \frac{1}{5}\right), \left(\frac{1}{3}, \frac{7}{24}, \frac{1}{4}, \frac{1}{8}\right), \left(\frac{1}{3}, \frac{2}{7}, \frac{5}{21}, \frac{1}{7}\right), \left(\frac{1}{3}, \frac{5}{18}, \frac{2}{9}, \frac{1}{6}\right).$$

(B-v) $$\alpha=\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{2}{15}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{9}, \frac{1}{9}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12}\right).$$

(B-vi) $$\alpha=\left(\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}\right), \left(\frac{1}{3}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}\right).$$

(B-vii) $$\alpha=\left(\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right), \left(\frac{1}{3}, \frac{5}{18}, \frac{2}{9}, \frac{1}{6}\right).$$

(B-viii). Assume $\alpha_1=1/3$, $2\alpha_2+n\alpha_4=1$ $(n\geq 2)$, $\nu=(3, 0, 0, 0)$, $\mu=(0, 2, 0, n)$. If $n=2$, then $\alpha_2+\alpha_4=1/2$ and $\alpha=(1/3, 1/3, 1/6, 1/6)$. Let us consider the case $n>2$. By $\alpha_2=1/2-n\alpha_4/2$ and $\alpha_3=1/6+(n-2)\alpha_4/2$, we have

$$\frac{1}{3}\lambda_1+\frac{1}{2}\lambda_2+\frac{1}{6}\lambda_3+\left(-\frac{n}{2}\lambda_2+\frac{n-2}{2}\lambda_3+\lambda_4\right)\alpha_4=1,$$

for any $\lambda \in T(\alpha)$. Since δ, ν, $\mu \in \{-nx_2/2-(n-2)x_3/2+x_4=0\}$, there exists $\lambda \in T(\alpha)$ with $n\lambda_2>(n-2)\lambda_3+2\lambda_4$. By $\alpha_1=1/3$, we may assume $\lambda_1 \leq 1$. But in this case, λ_2 is greater than 1, and the case (B-viii) is reduced to the cases (B-i), $\cdots$, (B-vii).

Step 2 of the case (C) $2\alpha_1+\alpha_2=1$ (i.e., $\nu(2, 1, 0, 0) \in T(\alpha)$). There exists $\mu \in T(\alpha)$ such that $\mu_2 \geq 2$. Since $\alpha_2+2\alpha_3+2\alpha_4=1$, we have $1/5 \leq \alpha_2$ and $2 \leq \mu_2 \leq 5$. As in Step 2 of the case (B), we have the following cases:

(C-i) $\mu=(0, 5, 0, 0), (0, 4, 1, 0)$ etc., and $\alpha=(2/5, 1/5, 1/5, 1/5)$.

(C-ii) $\mu=(0, 4, 0, 0)$ and $\alpha_1=3/8$, $\alpha_2=1/4$.

(C-iii) $\mu=(0, 3, 0, 2), (1, 2, 0, 1), (0, 2, 1, 2)$ and $\alpha_2=\alpha_3$ or

(C-iv) $\mu=(0, 3, 1, 0)$ and $3\alpha_2+\alpha_3=1$ or

(C-v) $\mu=(0, 3, 0, 1)$ and $3\alpha_2+\alpha_4=1$ or

(C-vi) $\mu=(0, 3, 0, 0), (1, 2, 0, 0), (0, 2, 1, 1)$ and $\alpha_1=\alpha_2=1/3$.

(C-vii) $\mu=(0, 2, 2, 0)$ and $\alpha_2+\alpha_3=1/2$ or

(C-viii) $\mu=(0, 2, 0, n)$ and $2\alpha_2+n\alpha_4=1$.

Step 3 of the case (C). Next, we determine the weights α for the above eight cases. By the same calculation stated in the case (B-i), we have the following weights α for the cases (C-ii), $\cdots$, (C-vii):

(C-ii) $$\alpha=\left(\frac{3}{8}, \frac{1}{4}, \frac{1}{5}, \frac{7}{40}\right), \left(\frac{3}{8}, \frac{1}{4}, \frac{5}{24}, \frac{1}{6}\right), \left(\frac{3}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right), \left(\frac{3}{8}, \frac{1}{4}, \frac{3}{16}, \frac{3}{16}\right).$$

(C-iii) $$\alpha=\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right), \left(\frac{3}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right).$$

(C-iv) $$\alpha=\left(\frac{11}{30}, \frac{4}{15}, \frac{1}{5}, \frac{1}{6}\right), \left(\frac{7}{19}, \frac{5}{19}, \frac{4}{19}, \frac{3}{19}\right), \left(\frac{3}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right),$$
$$\left(\frac{4}{11}, \frac{3}{11}, \frac{2}{11}, \frac{2}{11}\right).$$

(C-v) $$\alpha=\left(\frac{9}{25}, \frac{7}{25}, \frac{1}{5}, \frac{4}{25}\right), \left(\frac{5}{14}, \frac{2}{7}, \frac{3}{14}, \frac{1}{7}\right), \left(\frac{7}{20}, \frac{3}{10}, \frac{1}{4}, \frac{1}{10}\right),$$
$$\left(\frac{4}{11}, \frac{3}{11}, \frac{2}{11}, \frac{2}{11}\right), \left(\frac{6}{17}, \frac{5}{17}, \frac{4}{17}, \frac{2}{17}\right).$$

(C-vi) This case already appeared in the case (B).

(C-vii) $$\alpha=\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right), \left(\frac{7}{20}, \frac{3}{10}, \frac{1}{5}, \frac{3}{20}\right), \left(\frac{5}{14}, \frac{2}{7}, \frac{3}{14}, \frac{1}{7}\right), \left(\frac{3}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right).$$

(C-viii). Assume $2\alpha_1+\alpha_2=1$, $2\alpha_2+n\alpha_4=1$ $(n\geq 2)$, $\nu=(2,1,0,0)$, and $\mu=(0,2,0,n)$. If $n=2$, then $\alpha_2+\alpha_4=1/2$ and $\alpha=(1/3,1/3,1/6,1/6)$. Let us consider the case $n>2$. Since $\alpha_1=1/4+n\alpha_4/4$, $\alpha_2=1/2-\alpha_4/2$, $\alpha_3=1/4+(n-4)\alpha_4/4$, for any $\lambda\in T(\alpha)$, we have

$$\frac{1}{4}\lambda_1+\frac{1}{2}\lambda_2+\frac{1}{4}\lambda_3+\left(\frac{n}{4}\lambda_1-\frac{n}{2}\lambda_2+\frac{n-4}{4}\lambda_3+\lambda_4\right)\alpha_4=1\,.$$

Since δ, ν, $\mu\in\{n\lambda_1/4-n\lambda_2/2+(n-4)\lambda_3/4+\lambda_4=0\}$, there exists $\lambda\in T(\alpha)$ with $n\lambda_1-2n\lambda_2+(n-4)\lambda_3+4\lambda_4<0$. By assumption, we have $2\alpha_1=\alpha_2+n\alpha_4$, so we may assume $\lambda_1\leq 1$. But in this case, one can easily check that there exist no $\lambda\in T(\alpha)$ with $\lambda_2\leq 1$, or $\alpha=(2/5,1/5,1/5,1/5)$, $(1/3,1/3,2/9,1/9)$. Thus, the case (C-viii) is reduced to the cases (C-i), $\cdots$, (C-vii).

Step 2 of the case (D) $2\alpha_1+\alpha_3=1$ (i.e., $\nu(2,0,1,0)\in T(\alpha)$). There exists $\mu\in T(\alpha)$ such that $\mu_2\geq 2$. Since $2\alpha_2+\alpha_3+2\alpha_4=1$ and $\alpha_1<1/2$, we have $1/5\leq\alpha_2<1/2$ and $2\leq\mu_2\leq 5$. We may assume $\mu_1=0$ for $\alpha_1=\alpha_2+\alpha_4$. Then the only possibilities are the following ten cases:

(D-i) $\mu=(0,5,0,0)$, $(0,4,1,0)$ etc., and $\alpha=(2/5,1/5,1/5,1/5)$.

(D-ii) $\mu=(0,4,0,0)$ and $\alpha_2=1/4$.

(D-iii) $\mu=(0,3,0,2)$ and $\alpha_2=\alpha_3$ or

(D-iv) $\mu=(0,3,1,0)$ and $3\alpha_2+\alpha_3=1$ or

(D-v) $\mu=(0,3,0,1)$ and $3\alpha_2+\alpha_4=1$ or

(D-vi) $\mu=(0,3,0,0)$ and $\alpha_2=1/3$.

(D-vii) $\mu=(0,2,3,0)$, $(0,2,2,1)$, $(0,2,0,3)$ and $\alpha_3=\alpha_4$ or

(D-viii) $\mu=(0,2,1,2)$ (necessarily contained in $T(\alpha)$) or

(D-ix) $\mu=(0,2,2,0)$, $(0,2,0,4)$ and $\alpha_3=2\alpha_4$ or

(D-x) $\mu=(0,2,0,n)$ with $n\geq 3$ and $2\alpha_2+n\alpha_4=1$.

Step 3 of the case (D). Each case is investigated more precisely in the following. For the cases (D-ii), $\cdots$, (D-vi), we obtain the following weights:

(D-ii) $$\alpha=\left(\frac{5}{12},\frac{1}{4},\frac{1}{6},\frac{1}{6}\right),\left(\frac{2}{5},\frac{1}{4},\frac{1}{5},\frac{3}{20}\right),\left(\frac{11}{28},\frac{1}{4},\frac{3}{14},\frac{1}{7}\right),\left(\frac{3}{8},\frac{1}{4},\frac{1}{4},\frac{1}{8}\right),$$
$$\left(\frac{13}{32},\frac{1}{4},\frac{3}{16},\frac{5}{32}\right).$$

(D-iii). Assume $2\alpha_1+\alpha_3=1$, $\alpha_2=\alpha_3$, $\nu=(2,0,1,0)$, and $\mu=(0,3,0,2)$. Then $(2,1,0,0)$ is contained in $T(\alpha)$, and this case was already considered in (C).

(D-iv) $\alpha=\left(\frac{3}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7}\right),\left(\frac{5}{12},\frac{5}{18},\frac{1}{6},\frac{5}{36}\right),\left(\frac{12}{29},\frac{8}{29},\frac{5}{29},\frac{4}{29}\right),$

$\left(\frac{9}{22},\frac{3}{11},\frac{2}{11},\frac{3}{22}\right),\left(\frac{2}{5},\frac{4}{15},\frac{1}{5},\frac{2}{15}\right),\left(\frac{3}{8},\frac{1}{4},\frac{1}{4},\frac{1}{8}\right),$

$\left(\frac{9}{23},\frac{6}{23},\frac{5}{23},\frac{3}{23}\right).$

(D-v) $\alpha=\left(\frac{3}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7}\right),\left(\frac{5}{12},\frac{7}{24},\frac{1}{6},\frac{1}{8}\right),\left(\frac{7}{17},\frac{5}{17},\frac{3}{17},\frac{2}{17}\right),$

$\left(\frac{2}{5},\frac{3}{10},\frac{1}{5},\frac{1}{10}\right),\left(\frac{5}{13},\frac{4}{13},\frac{3}{13},\frac{1}{13}\right),\left(\frac{3}{8},\frac{5}{16},\frac{1}{4},\frac{1}{16}\right).$

(D-vi) $\alpha=\left(\frac{4}{9},\frac{1}{3},\frac{1}{9},\frac{1}{9}\right),\left(\frac{21}{48},\frac{1}{3},\frac{1}{8},\frac{5}{48}\right),\left(\frac{17}{39},\frac{1}{3},\frac{5}{39},\frac{4}{39}\right),$

$\left(\frac{13}{30},\frac{1}{3},\frac{2}{15},\frac{1}{10}\right),\left(\frac{3}{7},\frac{1}{3},\frac{1}{7},\frac{2}{21}\right),\left(\frac{5}{12},\frac{1}{3},\frac{1}{6},\frac{1}{12}\right),$

$\left(\frac{14}{33},\frac{1}{3},\frac{5}{33},\frac{1}{11}\right),\left(\frac{2}{5},\frac{1}{3},\frac{1}{5},\frac{1}{15}\right),\left(\frac{11}{27},\frac{1}{3},\frac{5}{27},\frac{2}{27}\right),$

$\left(\frac{7}{18},\frac{1}{3},\frac{2}{9},\frac{1}{18}\right),\left(\frac{8}{21},\frac{1}{3},\frac{5}{21},\frac{1}{21}\right),\left(\frac{3}{8},\frac{1}{3},\frac{1}{4},\frac{1}{24}\right).$

The cases (D-vii) and (D-ix) are special cases of the case (D-x) below.

(D-viii). Since $\delta,\ \nu,\ \mu\in\{x_1+3x_2-2x_3-2x_4=0\}=H$, and $\{(2,0,1,0),(1,1,1,1),(0,2,1,2)\}\subset H\cap T(\alpha)$, there exists $\lambda\in T(\alpha)$ such that $\lambda=(2,1,0,0)$ or $\lambda_2\geq 2$, $\lambda\neq\mu$.

(D-x). Assume $2\alpha_1+\alpha_3=1$, $2\alpha_2+n\alpha_4=1$ $(n\geq 3)$, $\nu=(2,0,1,0)$, and $\mu=(0,2,0,n)$. Then $\alpha_1=1/2-(n-2)\alpha_4/2$, $\alpha_2=1/2-n\alpha_4/2$, $\alpha_3=(n-2)\alpha_4$ and for any $\lambda\in T(\alpha)$,

$$\frac{1}{2}\lambda_1+\frac{1}{2}\lambda_2+\left\{-\frac{n-2}{2}\lambda_1-\frac{n}{2}\lambda_2+(n-2)\lambda_3+\lambda_4\right\}\alpha_4=1\,.$$

Since $\delta,\ \nu,\ \mu\in\{-(n-2)x_1/2-nx_2/2+(n-2)x_3+x_4=0\}=H$, there exists $\lambda\in T(\alpha)$ with $2(n-2)\lambda_3+2\lambda_4<(n-2)\lambda_1+n\lambda_2$. Then, $H\cap T(\alpha)=\{(2,0,0,n-2),\ (2,0,1,0),\ (1,1,0,n-1),\ (1,1,1,1),\ (0,2,0,n),\ (0,2,1,2),\cdots\}$ and thus $\lambda=(2,0,1,0)$ or $\lambda_2\geq 2$. So the cases (D-vii), (D-ix), (D-x) are reduced to the cases (D-i), $\cdots$, (D-vi).

(E) $2\alpha_1+n\alpha_4=1$ $(n\geq 0)$ (i.e., $\nu(2,0,0,n)\in T(\alpha)$).

(E-0) First we consider the case $\mathrm{n}=0$.

Step 2 of the case (E-0). Since $\alpha_1=\alpha_2+\alpha_3+\alpha_4=1/2$, there exists $\mu\in T(\alpha)$ with

$\mu_1=0$, $3\leq\mu_2\leq 6$. Thus we have the following cases:

(E-0-i) $\mu=(0, 0, 6, 0)$, $(0, 5, 1, 0)$ etc., and $\alpha=(1/2, 1/6, 1/6, 1/6)$.

(E-0-ii) $\mu=(0, 5, 0, 0)$ and $\alpha_2=1/5$.

(E-0-iii) $\mu=(0, 4, 0, 2)$, $(0, 3, 1, 2)$ and $\alpha_2=\alpha_3$ or

(E-0-iv) $\mu=(0, 4, 1, 0)$ and $4\alpha_2+\alpha_3=1$ or

(E-0-v) $\mu=(0, 4, 0, 1)$ and $4\alpha_2+\alpha_4=1$ or

(E-0-vi) $\mu=(0, 4, 0, 0)$, $(0, 3, 1, 1)$ and $\alpha_2=1/4$.

(E-0-vii) $\mu=(0, 3, 2, 0)$ and $\alpha_2=2\alpha_4$ or

(E-0-viii) $\mu=(0, 3, 1, 0)$ and $3\alpha_2+\alpha_3=1$ or

(E-0-ix) $\mu=(0, 3, 0, 0)$ and $\alpha_2=1/3$ or

(E-0-x) $\mu=(0, 3, 0, n)$ with $n\geq 1$ and $3\alpha_2+n\alpha_4=1$.

Step 3 of the case (E-0). We study the above ten cases in more detail. For the cases (E-0-ii), $\cdots$, (E-0-ix), we obtain the following weights:

(E-0-ii) $$\alpha=\left(\frac{1}{2},\frac{1}{5},\frac{1}{6},\frac{2}{15}\right),\left(\frac{1}{2},\frac{1}{5},\frac{7}{40},\frac{1}{8}\right),\left(\frac{1}{2},\frac{1}{5},\frac{1}{5},\frac{1}{10}\right),$$
$$\left(\frac{1}{2},\frac{1}{5},\frac{4}{25},\frac{7}{50}\right),\left(\frac{1}{2},\frac{1}{5},\frac{3}{20},\frac{3}{20}\right).$$

(E-0-iii) $$\alpha=\left(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{2},\frac{1}{5},\frac{1}{5},\frac{1}{10}\right).$$

(E-0-iv) $$\alpha=\left(\frac{1}{2},\frac{3}{14},\frac{1}{7},\frac{1}{7}\right),\left(\frac{1}{2},\frac{5}{24},\frac{1}{6},\frac{1}{8}\right),\left(\frac{1}{2},\frac{7}{34},\frac{3}{17},\frac{2}{17}\right),$$
$$\left(\frac{1}{2},\frac{1}{5},\frac{1}{5},\frac{1}{10}\right),\left(\frac{1}{2},\frac{4}{19},\frac{3}{19},\frac{5}{38}\right).$$

(E-0-v) $$\alpha=\left(\frac{1}{2},\frac{3}{14},\frac{1}{7},\frac{1}{7}\right),\left(\frac{1}{2},\frac{2}{9},\frac{1}{6},\frac{1}{9}\right),\left(\frac{1}{2},\frac{5}{22},\frac{2}{11},\frac{1}{11}\right),$$
$$\left(\frac{1}{2},\frac{7}{30},\frac{1}{5},\frac{1}{15}\right),\left(\frac{1}{2},\frac{7}{32},\frac{5}{32},\frac{1}{8}\right),\left(\frac{1}{2},\frac{3}{13},\frac{5}{26},\frac{1}{13}\right).$$

(E-0-vi) $$\alpha=\left(\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{8}\right),\left(\frac{1}{2},\frac{1}{4},\frac{1}{7},\frac{3}{28}\right),\left(\frac{1}{2},\frac{1}{4},\frac{3}{20},\frac{1}{10}\right),$$

$$\left(\frac{1}{2},\frac{1}{4},\frac{1}{6},\frac{1}{12}\right),\left(\frac{1}{2},\frac{1}{4},\frac{3}{16},\frac{1}{16}\right),\left(\frac{1}{2},\frac{1}{4},\frac{1}{5},\frac{1}{20}\right).$$

(E-0-vii) $$\alpha=\left(\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{8}\right),\left(\frac{1}{2},\frac{5}{21},\frac{1}{7},\frac{5}{42}\right),\left(\frac{1}{2},\frac{4}{17},\frac{5}{34},\frac{2}{17}\right),$$

$$\left(\frac{1}{2},\frac{3}{13},\frac{2}{13},\frac{3}{26}\right),\left(\frac{1}{2},\frac{2}{9},\frac{1}{6},\frac{1}{9}\right),\left(\frac{1}{2},\frac{1}{5},\frac{1}{5},\frac{1}{10}\right),$$

$$\left(\frac{1}{2},\frac{3}{14},\frac{5}{28},\frac{3}{28}\right).$$

(E-0-viii) $$\alpha=\left(\frac{1}{2},\frac{3}{10},\frac{1}{10},\frac{1}{10}\right),\left(\frac{1}{2},\frac{8}{27},\frac{1}{9},\frac{4}{54}\right),\left(\frac{1}{2},\frac{13}{44},\frac{5}{44},\frac{1}{11}\right),$$

$$\left(\frac{1}{2},\frac{5}{17},\frac{2}{17},\frac{3}{34}\right),\left(\frac{1}{2},\frac{7}{24},\frac{1}{8},\frac{1}{12}\right),\left(\frac{1}{2},\frac{2}{7},\frac{1}{7},\frac{1}{14}\right),$$

$$\left(\frac{1}{2},\frac{11}{38},\frac{5}{38},\frac{3}{38}\right),\left(\frac{1}{2},\frac{3}{11},\frac{2}{11},\frac{1}{22}\right),\left(\frac{1}{2},\frac{5}{18},\frac{1}{6},\frac{1}{18}\right),$$

$$\left(\frac{1}{2},\frac{7}{26},\frac{5}{26},\frac{1}{26}\right),\left(\frac{1}{2},\frac{4}{15},\frac{1}{5},\frac{1}{30}\right).$$

(E-0-ix) $$\alpha=\left(\frac{1}{2},\frac{1}{4},\frac{1}{12},\frac{1}{12}\right),\left(\frac{1}{2},\frac{1}{3},\frac{1}{11},\frac{5}{66}\right),\left(\frac{1}{2},\frac{1}{3},\frac{5}{54},\frac{2}{27}\right),$$

$$\left(\frac{1}{2},\frac{1}{3},\frac{2}{21},\frac{1}{14}\right),\left(\frac{1}{2},\frac{1}{3},\frac{1}{10},\frac{1}{15}\right),\left(\frac{1}{2},\frac{1}{3},\frac{1}{9},\frac{1}{18}\right),$$

$$\left(\frac{1}{2},\frac{1}{3},\frac{5}{48},\frac{1}{16}\right),\left(\frac{1}{2},\frac{1}{3},\frac{1}{8},\frac{1}{24}\right),\left(\frac{1}{2},\frac{1}{3},\frac{5}{42},\frac{1}{21}\right),$$

$$\left(\frac{1}{2},\frac{1}{3},\frac{2}{15},\frac{1}{30}\right),\left(\frac{1}{2},\frac{1}{3},\frac{5}{36},\frac{1}{36}\right),\left(\frac{1}{2},\frac{1}{3},\frac{1}{7},\frac{1}{42}\right).$$

(E-0-x). Assume $\alpha_1=1/2$, $3\alpha_2+n\alpha_4=1$, $\nu=(2,0,0,0)$, and $\mu=(0,3,0,n)$. Then $\alpha_2=1/3-n\alpha_4/3$, $\alpha_3=1/6+(n-3)\alpha_4/3$, and for any $\lambda\in T(\alpha)$,

$$\frac{1}{2}\lambda_1+\frac{1}{3}\lambda_2+\frac{1}{6}\lambda_3+\left(-\frac{n}{3}\lambda_2+\frac{n-3}{3}\lambda_3+\lambda_4\right)\alpha_4=1\,.$$

First, assume $n=1$. Then we have

$$\alpha=\left(\frac{1}{2}, \frac{3}{10}, \frac{1}{10}, \frac{1}{10}\right), \left(\frac{1}{2}, \frac{11}{36}, \frac{1}{9}, \frac{1}{12}\right), \left(\frac{1}{2}, \frac{4}{13}, \frac{3}{26}, \frac{1}{13}\right),$$

$$\left(\frac{1}{2}, \frac{5}{16}, \frac{1}{8}, \frac{1}{16}\right), \left(\frac{1}{2}, \frac{7}{22}, \frac{3}{22}, \frac{1}{22}\right), \left(\frac{1}{2}, \frac{9}{28}, \frac{1}{7}, \frac{1}{28}\right).$$

Next, assume $n \geq 2$. Let $H=\{-nx_1/3+(n-3)x_2/3+x_4=0\}$. Since δ, ν, $\mu \in H$, and $(0, 3, 0, n)$, $(0, 2, 2, 2) \in H \cap \{x_1=0\}$, there exists $\lambda \in T(\alpha)$ with $(n-3)\lambda_3+3\lambda_4<n\lambda_2$, $\lambda_2 \geq 3$, and thus the case (E-0-x) for $n \geq 2$ is reduced to the cases (E-0-i), $\cdots$, (E-0-ix).

(E-1) Next, we assume $n=1$.

Step 2 of the case (E-1). Since $2\alpha_2+2\alpha_3+2\alpha_4=1$, there exists $\mu \in T(\alpha)$ with $2 \leq \mu_2 \leq 5$. The possibilities are the nine cases below.

(E-1-i) $\mu=(0, 0, 5, 0)$, $(0, 4, 1, 0)$ etc., and $\alpha=(2/5, 1/5, 1/5, 1/5)$.

(E-1-ii) $\mu=(0, 4, 0, 1)$, $(0, 3, 1, 1)$, $(1, 2, 0, 1)$ and $\alpha_2=\alpha_3$ or

(E-1-iii) $\mu=(0, 4, 0, 0)$ and $\alpha_2=1/4$.

(E-1-iv) $\mu=(0, 3, 1, 0)$, $(1, 2, 0, 0)$ and $3\alpha_2+\alpha_3=1$ or

(E-1-v) $\mu=(0, 3, 0, n)$ with $n \geq 0$ and $3\alpha_2+n\alpha_4=1$ or

(E-1-vi) $\mu=(0, 2, 3, 0)$ and $\alpha_3=\alpha_4$ or

(E-1-vii) $\mu=(0, 2, 2, 1)$ (necessarily contained in $T(\alpha)$) or

(E-1-viii) $\mu=(0, 2, 1, n)$ with $n \geq 2$ and $2\alpha_2+\alpha_3+n\alpha_4=1$ or

(E-1-ix) $\mu=(0, 2, 0, n)$ with $n \geq 3$ and $2\alpha_2+n\alpha_4=1$.

Step 3 of the case (E-1). We now study the above nine cases in more detail.

(E-1-ii). Assume $2\alpha_1+\alpha_4=1$, $\alpha_2=\alpha_3$, $\nu=(2, 0, 0, 1)$, and $\mu=(0, 4, 0, 1)$. Then $\alpha_1=1/2-\alpha_4/2$, $\alpha_2=\alpha_3=1/4-\alpha_4/4$, $0<\alpha_4 \leq 1/5$, and for any $\lambda \in T(\alpha)$,

$$\frac{1}{2}\lambda_1+\frac{1}{4}\lambda_2+\frac{1}{4}\lambda_3+\left(-\frac{1}{2}\lambda_1-\frac{1}{4}\lambda_2-\frac{1}{4}\lambda_3+\lambda_4\right)\alpha_4=1.$$

Since δ, ν, $\mu \in \{-\lambda_1/2-\lambda_2/4-\lambda_3/4+\lambda_4=0\}=H$, there exists $\lambda \in T(\alpha)$ with $-\lambda_1/2-\lambda_2/4-\lambda_3/4+\lambda_4<0$. But one can easily check that $H \cap T(\alpha) \subset \{x_4=1\}$, and thus

$$\alpha=\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right).$$

(E-1-iii). By the same procedure as in the case (B-i), we have

$$\alpha=\left(\frac{5}{12}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}\right), \left(\frac{9}{20}, \frac{1}{4}, \frac{1}{5}, \frac{1}{10}\right), \left(\frac{7}{16}, \frac{1}{4}, \frac{3}{16}, \frac{1}{8}\right).$$

(E-1-iv). Similarly,

$$\alpha=\left(\frac{3}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7}\right),\left(\frac{4}{9},\frac{5}{18},\frac{1}{6},\frac{1}{9}\right),\left(\frac{5}{11},\frac{3}{11},\frac{2}{11},\frac{1}{11}\right),\left(\frac{7}{15},\frac{4}{15},\frac{1}{5},\frac{1}{15}\right).$$

(E-1-v). Assume $2\alpha_1+\alpha_4=1, 3\alpha_2+n\alpha_4=1, \nu=(2, 0, 0, 1)$, and $\mu=(0, 3, 0, n)$. Then $\alpha_1=1/2-\alpha_4/2$, $\alpha_2=1/3-n\alpha_4/3$, $\alpha_3=1/6+(2n-3)\alpha_4/6$, and for any $\lambda\in T(\alpha)$,

$$\frac{1}{2}\lambda_1+\frac{1}{3}\lambda_2+\frac{1}{6}\lambda_3+\left(-\frac{1}{2}\lambda_1-\frac{n}{3}\lambda_2+\frac{2n-3}{6}\lambda_3+\lambda_4\right)\alpha_4=1\,.$$

Since $\delta, \nu, \mu\in\{-\lambda_1/2-n\lambda_2/3+(2n-3)\lambda_3/6+\lambda_4=0\}$, there exists $\lambda\in T(\alpha)$ with $(2n-3)\lambda_3+6\lambda_4<3\lambda_1+2n\lambda_2$.

First, we consider the case $n=0$. Then we have

$$\alpha=\left(\frac{4}{9},\frac{1}{3},\frac{1}{9},\frac{1}{9}\right),\left(\frac{11}{24},\frac{1}{3},\frac{1}{8},\frac{1}{12}\right),\left(\frac{7}{15},\frac{1}{3},\frac{2}{15},\frac{1}{15}\right),\left(\frac{10}{21},\frac{1}{3},\frac{1}{7},\frac{1}{21}\right).$$

Next, assume $n=1$. Then $H\cap T(\alpha)\subset\{x_4=1\}$, and thus

$$\alpha=\left(\frac{3}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7}\right).$$

If $n\geq 2$, then one can find $\lambda\in T(\alpha)$ such that $\lambda_2\geq 2$ and $\lambda\neq(0, 3, 0, n)$, $(0, 2, 0, m)$ $(m\geq 3)$, $(0, 2, 2, 1)$. This case is thus reduced to the cases (E-1-i), $\cdots$, (E-1-iv).

(E-1-vi). In this case, the point $(2, 0, 1, 0)$ is in $T(\alpha)$, so we already considered this case in (D).

(E-1-vii). In this case, one can easily check that there exists another $\mu\in T(\alpha)$ with $\mu_2\geq 2$.

(E-1-viii). Assume $2\alpha_1+\alpha_4=1$, $2\alpha_2+\alpha_3+n\alpha_4=1$ $(n\geq 2)$. Then $\alpha_3=(n-1)\alpha_4$ and $(0, 2, 0, 2n-1)\in T(\alpha)$, so this case is reduced to the following case.

(E-1-ix). Assume $2\alpha_1+\alpha_4=1$, $2\alpha_2+n\alpha_4=1$ $(n\geq 3)$. Then $\alpha_1=1/2-\alpha_4/2$, $\alpha_2=1/2-n\alpha_4/2$, and $\alpha_3=(n-1)\alpha_4/2$. Let $H=\{-x_1/2-nx_2/2+(n-1)x_3/2+x_4=0\}$. Then $\delta, \nu, \mu\in H$ and

$$H\cap T(\alpha)\cap\{x_1=0\}\subset\{x_2=2\}\,,\quad H\cap T(\alpha)\cap\{x_1=1\}\subset\{x_2=1\}\,.$$

Thus, if $\lambda\in\{-x_1/2-nx_2/2+(n-1)x_3/2+x_4<0\}\cap T(\alpha)$, then $\lambda_2\geq 2$, and hence this case is reduced to the cases (E-1-i), $\cdots$, (E-1-v).

(E-2) Let $n\geq 2$.

Step 2 of the case (E-2). There exists $\mu\in T(\alpha)$ such that $\mu_1>\mu_4$. (This condition for μ is different from those in the above cases.) Because of the cases (A), $\cdots$, (D) already dealt with, we may assume that $\mu_1=1$ and $\mu_4=0$.

(E-2-i) $\quad\mu=(1, 3, 0, 0)$, $(1, 2, 1, 0)$ and $\alpha_2=\alpha_3=\alpha_4$.

(E-2-ii) $\mu=(1,1,2,0)$ and $\alpha_3=\alpha_4$.

(E-2-iii) $\mu=(1,2,0,0)$ and $\alpha_1+2\alpha_2=1$.

(E-2-iv) $\mu=(1,0,m,0)$ with $m\geq 3$ and $\alpha_1+m\alpha_3=1$.

Step 3 of the case (E-2). We determine weights for the above four cases.

(E-2-i) and (E-2-ii). In these cases, $(2,0,n,,0)\in T(\alpha)$. Then $n=2$, and $\alpha_1=\alpha_2$. This is the case (A).

(E-2-iii). Assume $2\alpha_1+n\alpha_4=1$ $(n\geq 2)$, $\alpha_1+2\alpha_2=1$, $\nu=(2,0,0,n)$, and $\mu=(1,2,0,0)$. Then $\alpha_1=1/2-n\alpha_4/2$, $\alpha_2=1/4+n\alpha_4/4$, $\alpha_3=1/4+(n-4)\alpha_4/4$, and for any $\lambda\in T(\alpha)$,

$$\frac{1}{2}\lambda_1+\frac{1}{4}\lambda_2+\frac{1}{4}\lambda_3+\left(-\frac{n}{2}\lambda_1+\frac{n}{4}\lambda_2+\frac{n-4}{4}\lambda_3+\lambda_4\right)\alpha_4=1\,.$$

Since $\delta,\nu,\mu\in\{-nx_1/2+nx_2/4+(n-4)x_3/4+x_4=0\}=H$, there exists $\lambda\in T(\alpha)$ with $n\lambda_2+(n-4)\lambda_3+4\lambda_4<2n\lambda_1$. If $n=2$, then $\{(0,2,2,0),\ (0,1,3,1),\ (0,0,4,2)\}=H\cap T(\alpha)\cap\{x_1=0\}$ and $\{(1,2,0,0),\ (1,1,1,1),\ (1,0,2,2)\}=H\cap T(\alpha)\cap\{x_1=1\}$. If $\lambda_1\leq 1$, then $\alpha_3=\alpha_4$ or $\alpha_3=2\alpha_4$, and thus

$$\alpha=\left(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}\right),\left(\frac{2}{5},\frac{3}{10},\frac{1}{5},\frac{1}{10}\right).$$

If $n\geq 3$, then $H\cap T(\alpha)\cap\{x_1=0\}=\{(1,0,4,1)\}$ or $\emptyset$, and $\{(1,2,0,0),\ (1,1,1,1),\ (1,0,2,2)\}=H\cap T(\alpha)\cap\{x_1=1\}$. So $\lambda_1\geq 2$, and this case is reduced to the cases (A), $\cdots$, (D).

(E-2-iv). Assume $2\alpha_1+n\alpha_4=1$ $(n\geq 2)$, $\alpha_1+m\alpha_3=1$ $(m\geq 3)$, $\nu=(2,0,0,n)$, and $\mu=(1,0,m,0)$. Then $\alpha_1=1/2-n\alpha_4/2$, $\alpha_2=(m-1)/2m+(mn-n-2m)\alpha_4/2m$, $\alpha_3=1/2m+n\alpha_4/2m$, and for any $\lambda\in T(\alpha)$

$$\frac{1}{2}\lambda_1+\frac{m-1}{2m}\lambda_2+\frac{1}{2m}\lambda_3+\left(-\frac{n}{2}\lambda_1+\frac{mn-n-2m}{2m}\lambda_2+\frac{n}{2m}\lambda_3+\lambda_4\right)\alpha_4=1\,.$$

Since $\delta,\nu,\mu\in\{-nx_1/2+(mn-n-2m)x_2/2+nx_3/2m+x_4=0\}=H$, there exists $\lambda\in T(\alpha)$ with $(mn-n-2m)\lambda_2+n\lambda_3+2m\lambda_4<mn\lambda_1$. We may assume that $\{\nu\}=T(\alpha)\cap\{x_2\geq 2\}$, for otherwise, this case is reduced to the cases (A), $\cdots$, (D). In particular, we obtain $\alpha_3>\alpha_4$. But then, one can easily check that $\lambda_1=1$, $\lambda_3=0$, and $\lambda_2\geq 2$. Hence $\lambda=(1,2,0,0)$ and this case is reduced to the case (E-2-iii).

Step 4. In conclusion, we check the condition $(1,1,1,1)\in\mathrm{Int}\langle T(\alpha)\rangle$ for α's obtained by the calculation in (A), $\cdots$, (E), and we obtain ninety five weights listed in Table 2.2. Q.E.D.

REMARK 2.5. The set of weights W_4 coinsides with the set A'_4 in Reid [10]. The inclusion $A'_4\subset W_4$ is obvious. For the list of A'_4, see also [1].

3. Minimal resolution. Here we study the minimal resolution of a hypersurface simple $K3$ singularity (X, x) which is defined by a nondegenerate polynomial f. Tomari [12], [13] proved the following theorem in terms of filtered rings without assuming f to be nondegenerate.

THEOREM 3.1. (Tomari [12], [13]). *Let (X, x) be as above with f nondegenerate and $\alpha(f)=(p_1/p, p_2/p, p_3/p, p_4/p)$. If $\pi: (\tilde{X}, E)\to(X, x)$ is the filtered blow-up with weight (p_1, p_2, p_3, p_4), then π is a minimal resolution of (X, x).*

Moreover, if f is semiquasi-homogeneous, i.e., $\{f_{\Delta_0}=0\}$ has an isolated singularity (which is also a simple $K3$ singularity) at the origin $0\in \boldsymbol{C}^4$, then π is a unique minimal resolution for (X, x).

To investigate the above filtered blow-up π, we begin with the filtered blow-up of $\boldsymbol{C}^4$ at the origin $0\in\boldsymbol{C}^4$. Let $\boldsymbol{p}=(p_1, p_2, p_3, p_4)$ be the 4-ple of positive integers with $\boldsymbol{gcd}(p_i, p_j, p_k)=1$ for all distinct i, j, k. Then the filtered blow-up $\Pi: (V, F)\to(\boldsymbol{C}^4, 0)$ with weight $\boldsymbol{p}$ is constructed as follows by using the method of torus embeddings.

First we introduce the notation (cf. [6], [8]). Let $N:=\boldsymbol{Z}^4$ and let $M:=\mathrm{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$ be the dual $\boldsymbol{Z}$-module of N. A subset σ of $N_{\boldsymbol{R}}=N\otimes_{\boldsymbol{Z}}\boldsymbol{R}$ is called a cone if there exists $n_1, \cdots, n_s\in N$ such that σ is written as $\sigma=\{\sum_{i=1}^{s} t_i n_i \mid t_i\in\boldsymbol{R}_0\}$, which we simply denote $\sigma=\langle n_1, \cdots, n_s\rangle$. We call $n_1, \cdots, n_s$ the generators of σ. For a cone σ in $N_{\boldsymbol{R}}$, we define the dual cone of σ by $\check{\sigma}=\{m\in M_{\boldsymbol{R}} \mid m(u)\geq 0 \text{ for any } u\in\sigma\}$, and associate a normal variety $X_\sigma=\mathrm{Spec}\,\boldsymbol{C}[M\cap\check{\sigma}]$ with the cone σ, where $\boldsymbol{C}[M\cap\check{\sigma}]$ is a $\boldsymbol{C}$-algebra generated by z^m for $m\in M\cap\check{\sigma}$.

REMARK 3.2. In this paper, we assume that the generators $n_1, \cdots, n_s$ of a cone σ consist of primitive elements of N, i.e., each n_i satisfies $n_i\boldsymbol{R}\cap N=n_i\boldsymbol{Z}$. We define the determinant $\det\sigma$ of a cone σ as the greatest common divisor of all (s, s) minors of the matrix (n_{ij}), where $n_i=(n_{i1}, \cdots, n_{i4})$ (cf. [9]).

Let $\sigma\subset N_{\boldsymbol{R}}=\boldsymbol{R}^4$ be the first quadrant of $\boldsymbol{R}^4$, i.e., $\sigma=\langle e_1, e_2, e_3, e_4\rangle$ where $e_1=(1, 0, 0, 0), \cdots, e_4=(0, 0, 0, 1)$. We divide the cone σ into four cones by adding the point $\boldsymbol{p}=(p_1, p_2, p_3, p_4)$ in σ.

$$\sigma=\bigcup_{i=1}^{4}\sigma_i, \qquad \sigma_i=\langle \boldsymbol{p}, e_j, e_k, e_l\rangle .$$

From the inclusions $\sigma_i\subset\sigma$, we obtain natural morphisms

$$\Pi_i:\ V_i=\mathrm{Spec}\,\boldsymbol{C}[\check{\sigma}_i\cap M]\longrightarrow \mathrm{Spec}\,\boldsymbol{C}[\check{\sigma}\cap M]=\boldsymbol{C}^4 .$$

Let V be the union of V_i $(i=1, 2, 3, 4)$ which is glued along the images of Π_i. Then we have a morphism

$$\Pi:\ V\to\boldsymbol{C}^4$$

and one can easily check that $V-\Pi^{-1}(0)\simeq\boldsymbol{C}^4-\{0\}$ and $\Pi^{-1}(0)$ is the weighted projective space $\boldsymbol{P}(p_1, p_2, p_3, p_4)$. We set $F := \Pi^{-1}(0) = \boldsymbol{P}(p_1, p_2, p_3, p_4)$.

REMARK 3.3. The normal variety V is a torus embedding associated to the fan $\Gamma^*(f)$ which is called the dual Newton boundary of f in [9], and denoted by $V = T_N\mathrm{emb}(\Gamma^*(f))$ (see [6], [8]).

We obtain the filtered blow-up of (X, x) by means of the above morphism Π as follows: Write $\alpha(f) = (p_1/p, p_2/p, p_3/p, p_4/p)$ as in §1 and construct Π for the weight $\boldsymbol{p} = (p_1, p_2, p_3, p_4)$. Let $\tilde{X}$ be the proper transform of X by Π, and set $\pi = \Pi|_{\tilde{X}}$, $E = \pi^{-1}(0)$. Then $\pi\colon (\tilde{X}, E) \to (X, x)$ is the filtered blow-up with weight $\boldsymbol{p}$.

Next we study the structure of V for a general weight $\boldsymbol{p} = (p_1\, p_2, p_3, p_4)$ with $\boldsymbol{gcd}(p_i, p_j, p_k) = 1$ for all distinct i, j, k. Let $a_{ij} = \boldsymbol{gcd}(p_i, p_j)$ and set $z_{ij} = z_i^{-(p_j/a_{ij})} z_j^{(p_i/a_{ij})}$. We can take $z_i, z_{ij}, z_{ik}, z_{il}$ to be a system of parameters on V_i.

PROPOSITION 3.4. (1) *$V_i \cap V_j \simeq \boldsymbol{C}^* \times \mathrm{Spec}\, \boldsymbol{C}[\check{\tau}_{ij} \cap \boldsymbol{Z}^3]$, where $\tau_{ij} = \langle (0, 1, 0), (0, 0, 1), (a_{ij}, p_k, p_l)\rangle$.*

(2) *$\{z_{ij} = 0\}$ in $V_i \simeq \mathrm{Spec}\, \boldsymbol{C}[\check{\rho}_{ij} \cap \boldsymbol{Z}^3]$, where $\rho_{ij} = \langle (0, 1, 0), (0, 0, 1), (p_i, p_k, p_l)\rangle$.*

PROOF. (1) Let $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3, \boldsymbol{m}_4$ be generators of M such that $\boldsymbol{m}_i(\boldsymbol{e}_j) = \delta_{i,j}$. Then the cone $\check{\sigma}_i$ is expressed as

$$\check{\sigma}_i = \left\langle \boldsymbol{m}_i,\ -\frac{p_j}{a_{ij}}\boldsymbol{m}_i + \frac{p_i}{a_{ij}}\boldsymbol{m}_j,\ -\frac{p_k}{a_{ik}}\boldsymbol{m}_i + \frac{p_i}{a_{ik}}\boldsymbol{m}_k,\ -\frac{p_l}{a_{il}}\boldsymbol{m}_i + \frac{p_i}{a_{il}}\boldsymbol{m}_l \right\rangle .$$

Since $\boldsymbol{gcd}(p_1, p_j) = a_{ij}$, there exist integers α, β such that $-p_j\alpha + p_i\beta = a_{ij}$. If we take another base $\{m_i', m_j', m_k', m_l'\}$ defined by

$$\boldsymbol{m}_i = \alpha\cdot\boldsymbol{m}_i' + \frac{p_i}{a_{ij}}\cdot\boldsymbol{m}_j', \quad \boldsymbol{m}_j = \beta\cdot\boldsymbol{m}_i' + \frac{p_j}{a_{ij}}\cdot\boldsymbol{m}_j', \quad \boldsymbol{m}_k = \boldsymbol{m}_k', \quad \boldsymbol{m}_l = \boldsymbol{m}_l',$$

then the cone $\check{\sigma}_i$ is expressed as

$$\check{\sigma}_i = \Big\langle \alpha\cdot\boldsymbol{m}_i' + \frac{p_i}{a_{ij}}\cdot\boldsymbol{m}_j',\ \boldsymbol{m}_i',\ -\frac{p_k}{a_{ik}}\cdot\alpha\cdot\boldsymbol{m}_i' - p_k\cdot a_{il}\cdot\boldsymbol{m}_j' + a_{ij}\cdot a_{il}\cdot\boldsymbol{m}_k',$$
$$-\frac{p_l}{a_{il}}\cdot\alpha\cdot\boldsymbol{m}_i' - p_l\cdot a_{ik}\cdot\boldsymbol{m}_j' + a_{ij}\cdot a_{ik}\cdot\boldsymbol{m}_l' \Big\rangle .$$

Since $V_i \cap V_j = V_i - \{z_{ij} = 0\}$ and $z_{ij} = z^{\boldsymbol{m}_i'}$, we get

$$V_i \cap V_j \simeq \boldsymbol{C}^* \times \mathrm{Spec}\, \boldsymbol{C}[\check{\tau}_{ij} \cap M'] ,$$

where M' is a free $\boldsymbol{Z}$-module generated by $\boldsymbol{m}_j', \boldsymbol{m}_k', \boldsymbol{m}_l'$ and

$$\tau_{ij} = \langle \boldsymbol{m}_j',\ -p_k\cdot\boldsymbol{m}_j' + a_{ij}\cdot\boldsymbol{m}_k',\ -p_l\cdot\boldsymbol{m}_j' + a_{ij}\cdot\boldsymbol{m}_l'\rangle .$$

If we use the base $\{\boldsymbol{e}_j', \boldsymbol{e}_k', \boldsymbol{e}_l'\}$ of $N' = \mathrm{Hom}(M', \boldsymbol{Z})$ such that $\boldsymbol{e}_s'(\boldsymbol{m}_t') = \delta_{s,t}$, then τ_{ij} is written in this new coordinate system as

$$\tau_{ij}=\langle(0,1,0),(0,0,1),(a_{ij},p_k,p_l)\rangle .$$

(2) We have

$$(\{z_{ij}=0\} \text{ in } V_i)\simeq \operatorname{Spec} \boldsymbol{C}\left[\left\langle \boldsymbol{m}_i,\ -\frac{p_k}{a_{ik}}\cdot\boldsymbol{m}_i+\frac{p_i}{a_{ik}}\cdot\boldsymbol{m}_k,\ -\frac{p_l}{a_{il}}\cdot\boldsymbol{m}_i+\frac{p_i}{a_{il}}\cdot\boldsymbol{m}_l\right\rangle\cap \boldsymbol{Z}^3\right].$$

Q.E.D.

COROLLARY 3.5. *If* $(p_1/p, p_2/p, p_3/p, p_4/p)\in W_4$, *then*
(1) $\operatorname{Spec} \boldsymbol{C}[\check{\tau}_{ij}\cap \boldsymbol{Z}^3]$ *has a terminal singularity at worst*;
(2) *if* $p_i\,|\,(p-p_j)$, *then* $\operatorname{Spec} \boldsymbol{C}[\check{\rho}_{ij}\cap \boldsymbol{Z}^3]$ *has a terminal singularity at worst.*

The above corollary follows directly from Lemma 3.6 below. First we recall the notion of cyclic quotient singularities for the case of dimension three. Let ξ be a primitive n-th root of unity, and let p, q be integers with $gcd(n, p, q)=1$. We define an equivalent relation on $\boldsymbol{C}^3$ by $(x_1, x_2, x_3)\sim(\xi x_1, \xi^p x_2, \xi^q x_3)$. Then $X=\boldsymbol{C}^3/\sim$ is expressed in terms of torus embeddings as

$$X=\operatorname{Spec} \boldsymbol{C}[\check{\sigma}\cap \boldsymbol{Z}^3] \quad\text{with}\quad \sigma=\langle(n,-p,-q),(0,1,0),(0,0,1)\rangle .$$

In particular, X has an isolated singularity if and only if $gcd(n,p)=gcd(n,q)=1$.

LEMMA 3.6 (Terminal lemma [5], [7]). *In the above situation, X has a terminal singularity if and only if X has an isolated singularity and one of the following conditions are satisfied*:
(1) $-p\equiv 1 \pmod n$,
(2) $-q\equiv 1 \pmod n$,
(3) $p+q\equiv 0 \pmod n$.

We need later the following result on the weighted projective space $\boldsymbol{P}(p_1, p_2, p_3, p_4)$. Let a_i be an integer defined by $p_i=a_i a_{ij} a_{ik} a_{il}$.

PROPOSITION 3.7. (1) *There exists a cone σ_{ij} in $\boldsymbol{R}^2$ with* $\det(\sigma_{ij})=a_{ij}$ *such that*

$$F\cap V_i\cap V_j\simeq \boldsymbol{C}^*\times \operatorname{Spec} \boldsymbol{C}[\check{\sigma}_{ij}\cap \boldsymbol{Z}^2],\ \text{where } F=\Pi^{-1}(0)=\boldsymbol{P}(p_1,p_2,p_3,p_4) .$$

Moreover, $\operatorname{Spec} \boldsymbol{C}[\check{\sigma}_{ij}\cap \boldsymbol{Z}^2]$ *has an $A_{a_{ij}-1}$ singularity if and only if* $a_{ij}\,|\,(p_k+p_l)$.
(2) *Let $D_i=F-V_i$. Then the following are equivalent to each other*:
  (a) *D_i-D_j has a singularity of type A,*
  (b) *D_i-D_j has an $A_{a_j a_{ij}-1}$ singularity,*
  (c) $p_j\,|\,(a_{jk}p_l+a_{jl}p_k)$.

In particular, if $(p_1/p, p_2/p, p_3/p, p_4/p)\in W_4$ *and* $p_j\,|\,(p-p_i)$, *then D_i-D_j has an A_{p_j-1} singularity.*

The following lemma is well-known:

LEMMA 3.8. *Let $\check{\tau}=\langle m_1, m_2\rangle$ be a two-dimensional cone in $M_{\boldsymbol{R}}$. Then the singularity* *of* Spec $C[\check{\tau}\cap M]$ *is of type A if and only if* $(m_1+m_2)/\det\check{\tau}$ *is an element of* M.

Proof of PROPOSITION 3.7. (1) The affine torus embedding $F\cap V_i$ is written as $F\cap V_i=\operatorname{Spec} C[\check{\sigma}'_i\cap M]$ where

$$\check{\sigma}'_i=\left\langle -\frac{p_j}{a_{ij}}\cdot \boldsymbol{m}_i+\frac{p_i}{a_{ij}}\cdot \boldsymbol{m}_j,\ -\frac{p_k}{a_{ik}}\cdot \boldsymbol{m}_i+\frac{p_i}{a_{ik}}\cdot \boldsymbol{m}_k,\ -\frac{p_l}{a_{il}}\cdot \boldsymbol{m}_i+\frac{p_i}{a_{il}}\cdot \boldsymbol{m}_l\right\rangle .$$

By the change of base for M defined in the proof of Proposition 3.4,

$$\check{\sigma}'_i=\left\langle \boldsymbol{m}_i,\ -\frac{p_k}{a_{ik}}\cdot\alpha\cdot \boldsymbol{m}_i-p_k a_{il}\cdot \boldsymbol{m}'_j+a_{ij}a_{il}\cdot \boldsymbol{m}'_k,\ -\frac{p_l}{a_{il}}\cdot\alpha\cdot \boldsymbol{m}'_i-p_l a_{ik}\cdot \boldsymbol{m}'_j+a_{ij}a_{ik}\cdot \boldsymbol{m}'_l\right\rangle .$$

Since $F\cap V_i\cap V_j=(F\cap V_i)-\{z_{ij}=0\}$, and $z_{ij}=z^{\boldsymbol{m}'_i}$, we have

$$F\cap V_i\cap V_j=\boldsymbol{C}^*\times\operatorname{Spec}\boldsymbol{C}[\check{\sigma}'_{ij}\cap M]\,,$$

where $\check{\sigma}'_{ij}=\langle -p_k\cdot \boldsymbol{m}'_j+a_{ij}\cdot \boldsymbol{m}'_k,\ -p_l\cdot \boldsymbol{m}'_j+a_{ij}\cdot \boldsymbol{m}_l\rangle$. Thus the assertion follows from Lemma 3.8.

(2) Similarly, we have $D_i-D_j=\operatorname{Spec}\boldsymbol{C}[\check{\tau}'_{ij}\cap M]$, where

$$\check{\tau}'_{ij}=\left\langle -\frac{p_k}{a_{ik}}\cdot \boldsymbol{m}_i+\frac{p_i}{a_{ik}}\cdot \boldsymbol{m}_k,\ -\frac{p_l}{a_{il}}\cdot \boldsymbol{m}_i+\frac{p_i}{a_{il}}\cdot \boldsymbol{m}_l\right\rangle .$$

Since $\boldsymbol{gcd}(p_j, p_k, p_l)=1$, we have $\det\check{\tau}'_{ij}=a_j a_{ij}$. This shows the equivalence between (a) and (b). By Lemma 3.8, the conditions (a) and (c) are also equivalent to each other.

Q.E.D.

4. Singularities on E. Let $f\in\boldsymbol{C}[z_1, z_2, z_3, z_4]$ be a nondegenerate polynomial which defines a simple $K3$ singularity at the origin $0\in\boldsymbol{C}^4$. We set $X=\{f=0\}\subset\boldsymbol{C}^4$, $x=0\in\boldsymbol{C}^4$ and let $\pi:(\tilde{X}, E)\to(X, x)$ be the minimal resolution of (X, x) constructed in § 3. In this section, we investigate the singularities on the normal $K3$ surface E.

In this section, we assume that f satisfies the following condition:

$(*)$ For any i, f_0 contains a term of the form z_i^n or $z_i^n z_j$ with a nonzero coefficient, where f is said to contain z^ν if $f=\sum_\nu a_\nu z^\nu$ and $a_\nu\neq 0$.

By Proposition 2.3, (1), there exists f which satisfies $(*)$ for any weight α of W_4. If f is semiquasi-homogeneous, then the condition $(*)$ is satisfied. In particular, the polynomials in Table 2.2 satisfy $(*)$.

LEMMA 4.1. (1) *E has γ_{ij} singular points of type $A_{a_{ij}-1}$, where*

$$\gamma_{ij}=\#\{\nu\in\Delta_0\cap\boldsymbol{Z}^4\mid \nu_k=\nu_l=0\}-1\,.$$

(2) *If f_0 contains $z_i^m z_j$ but not z_i^n, then E has a singular point of type A_{p_i-1}.*

(3) *Any singular point on E belongs to* (1) *or* (2).

PROOF. Here we use the notation introduced in § 3. Let $L_{ij}=D_k\cap D_l$ $(=\{z_k=z_l=0\}$

in F). Then L_{ij} is a projective line $\boldsymbol{P}^1(\boldsymbol{C})$ on F and any singular point on E is in L_{ij} for some i, j. We divide L_{ij} into two points $P_i = L_{ij} \cap D_j$, $P_j = L_{ij} \cap D_i$ and a one-dimensional complex torus $T_{ij} = L_{ij} - \{P_i, P_j\} \simeq \boldsymbol{C}^*$. Since f is nondegenerate, there is an embedded resolution $\rho: (Y, M) \to (X, x)$ such that Y is a torus embedding associated with a simplicial subdivision Σ^* of $\Gamma^*(f)$, i.e., Σ^* is a fan of cones generated by a part of a basis of N, and for any cone $\sigma \in \Sigma^*$ there exists a cone $\tau \in \Gamma^*(f)$ with $\sigma \subset \tau$. The resolution ρ dominates the minimal resolution π. Hence there exists a resolution $\varphi: (Y, M) \to (\tilde{X}, E)$ with $\rho = \pi \circ \varphi$.

$$\begin{tikzcd} (Y, M) \arrow[rr, "\varphi"] \arrow[dr, "\rho"'] & & (\tilde{X}, E) \arrow[dl, "\pi"] \\ & (X, x) & \end{tikzcd}$$

Choose a sufficiently small neighbourhood E_{ij} of T_{ij} in E, and let $\varphi_{ij}: (\varphi^{-1}(E_{ij}), C_{ij}) \to (E_{ij}, \mathrm{Sing}(E_{ij}))$ be the resolution of E_{ij} obtained by restriction of φ to $\varphi^{-1}(E_{ij})$. Then, by Oka [9, Lemma 4.8 and its proof], φ_{ij} is locally γ_{ij} copies of the resolution of $\mathrm{Spec}\,\boldsymbol{C}[\check{\sigma}_{ij} \cap \boldsymbol{Z}^3]$, where σ_{ij} is a cone in Proposition 3.7, (1). By Proposition 2.3, (3), we have $a_{ij} \mid (p_k + p_l)$, and hence the assertion (1) is proved. If f_0 contains $z_i^m z_j$ but not z_i^n, then $P_i \in E$. Let E_i be a sufficiently small neighbourhood of P_i in E, and let $\varphi_i: (\varphi^{-1}(E_i), C_i) \to (E_i, P_i)$ be the resolution as above. Then φ_i is locally isomorphic to the resolution of $D_j - D_i$. Thus the assertion (2) follows from Propostion 3.7, (2).

Q.E.D.

THEOREM 4.2. *Let f be a nondegenerate polynomial which defines a simple $K3$ singularity, and assume that f satisfies the condition $(*)$. Let $\pi: (\tilde{X}, E) \to (X, x)$ be the minimal resolution. Then the type and the number of singularities on E are determined by the weight $\alpha(f)$ independently of any particular choice of f. In particular, E has t_{ij} singular points of type $A_{a_{ij}-1}$ and σ_i singular points of type A_{p_i-1}, where*

$$t_{ij} = \#\{v \in T(\alpha) \mid v_k = v_l = 0\} - 1 \quad \text{and} \quad \sigma_i = \begin{cases} 0 & \text{if } p_i \mid p \\ 1 & \text{otherwise}. \end{cases}$$

PROOF. By Lemma 4.1, the singularities on E are determined by the Newton boundary of f_0. Let α be a weight in W_4 and let f be a polynomial such that $\alpha(f) = \alpha$ and that $\Gamma(f_0)$ is the convex hull of $T(\alpha)$. Then the singularities on E coincide with those stated in the theorem. Thus it suffices to show that if g is a polynomial with $\alpha(g) = \alpha$ and if E' is the exceptional set of the resolution of $\{g = 0\}$, then the singularities on E and E' are the same. By the condition $(*)$, one of the following cases occurs for each $i = 1, 2, 3, 4$. Let $\{i, j, k, l\}$ be the set of indices $\{1, 2, 3, 4\}$.

(1) $p_i \mid (p - p_j)$ and $p_i \mid (p - p_k)$.

(2) $p_i \mid (p - p_j)$, $p_i \nmid (p - p_k)$ and $p_i \nmid (p - p_l)$.

(3) $p_i | p$, $p_i \nmid (p-p_j)$, $p_i \nmid (p-p_k)$ and $p_i \nmid (p-p_l)$.

For each case, we study singularities on $E \cap V_i$ and $E' \cap V_i$ using Proposition 2.3 and Proposition 3.6. Recall that the singularities on $E \cap V_i$ are contained in $\{P_i\} \cup T_{ij} \cup T_{ik} \cup T_{il}$ in the same notation as in the proof of Lemma 4.1.

In the case (1), Both $E \cap V_i$ and $E' \cap V_i$ are nonsingular, or they have an A_{p_i-1}-singularity at the point P_i.

In the case (2), E (resp. E') has no singular points on $V_i - T_{ij}$ (resp. $V_i - L_{ij}$), and the type of singularities on T_{ij} and of a singularity P_i are the same. Now we consider the singularities on $E \cap V_j$ and $E' \cap V_j$. By the condition ($*$) again, we have the following three cases:

(2-1) Assume $p_j | (p-p_k)$ or $p_j | (p-p_l)$. Then E and E' have no singular points on T_{ij}, and hence $E \cap V_i$ and $E' \cap V_i$ are nonsingular.

(2-2) Assume $p_j | (p-p_i)$, $p_j \nmid (p-p_k)$ and $p_j \nmid (p-p_l)$. Then E and E' have no singular points on T_{jk} and T_{jl}. Since the type of the singularities on $\{P_i\} \cup T_{ij} \cup \{P_j\}$ are the same, both $E \cap (V_i \cup V_j)$ and $E' \cap (V_i \cup V_j)$ have t_{ij} singular points of type $A_{a_{ij}-1}$.

(2-3) Assume $p_j \nmid (p-p_i)$, $p_j \nmid (p-p_k)$ and $p_j \nmid (p-p_l)$. Then the point P_j is contained in neither E nor E'. Thus both $E \cap V_i$ and $E' \cap V_i$ have t_{ij} singular points of type $A_{a_{ij}-1}$.

In the case (3), the point P_i is on neither E nor E'. So we have $\mathrm{Sing}(E) = \mathrm{Sing}(E \cap (V_i \cup V_k \cup V_l))$ and $\mathrm{Sing}(E') = \mathrm{Sing}(E' \cap (V_j \cup V_k \cup V_l))$. If the case (3) occurs for i and j, then both E and E' have t_{ij} singular points of type $A_{a_{ij}-1}$ on T_{ij}.

Q.E.D.

REMARK 4.3. The condition ($*$) is essential to Theorem 4.2. For example, consider the polynomials

$$f = x^4 + y^4 + z^4 + (x^2 + y^2 + z^2)w^2 + w^5 ,$$

$$g = x^4 + y^4 + z^4 + (x^2 w + y^2 w + z^3)w + w^5 .$$

Then we have $\alpha(f) = \alpha(g) = \alpha = (1/4, 1/4, 1/4, 1/4)$, but the $K3$ surface E for f (resp. g) has A_1-singularity (resp. A_2-singularity) at the point P_4, while E for a polynomial h which satisfies ($*$) and $\alpha(h) = \alpha$ is nonsingular.

By Theorem 4.2, the singularities on the normal $K3$ surface E are determined by the weight $\alpha = \alpha(f)$. Let us define the rank of $\mathrm{Sing}(E)$ to be

$$r(\alpha) = \sum_{i<j} t_{ij}(a_{ij} - 1) + \sum_{i=1}^{4} \sigma_i(p_i - 1) .$$

Let $t(\alpha) = \# T(\alpha)$. Then the polynomial f with $\alpha(f) = \alpha$ has $t(\alpha)$ terms in general. In other words, f is of the form

$$f = \sum_{\nu \in T(\alpha)} a_\nu z^\nu \qquad \text{with} \quad a_\nu \neq 0 \qquad \text{for all} \quad \nu .$$

For a general f, we can choose a new variable w defined by

$$z_i = \sum_{\nu \in N_i(\alpha)} b_{i\nu} w^\nu \qquad \text{with} \quad N_i(\alpha) = \left\{ \nu \in T(\alpha) \;\middle|\; \sum_{j=1}^{4} \nu_j p_j = p_i \right\}$$

which preserves the weight α in such a way that $f(w)$ has $t(\alpha) - n(\alpha) + 4$ terms, four terms among which have coefficients 1, where

$$n(\alpha) = \sum_{i=1}^{4} \# N_i(\alpha) .$$

Then we obtain the following relation.

COROLLARY 4.4. *For any weight $\alpha \in W_4$, we have*

$$t(\alpha) - n(\alpha) + r(\alpha) = 19 .$$

In Table 4.6 can be found the list of $t(\alpha)$, $n(\alpha)$, $r(\alpha)$ and singularities on the K3 surface E for every weight $\alpha \in W_4$.

REMARK 4.5. We may regard the number $t(\alpha) - n(\alpha)$ as the number of parameters in a polynomial f with $\alpha(f) = \alpha$. Then Corollary 4.4 suggests $t(\alpha) - n(\alpha)$ to be the number of parameters associate to the moduli of the $K3$ surface E with a fixed singularity of rank $r(\alpha)$.

TABLE 4.6.

No.	$(p_1, p_2, p_3, p_4 : p)$	$t(\alpha)$	$n(\alpha)$	Sing(E)	$r(\alpha)$
1	(1, 1, 1, 1 : 4)	35	16	non-singular	0
2	(4, 3, 3, 2 : 12)	15	7	$3A_1 + 4A_2$	11
3	(2, 2, 1, 1 : 6)	30	14	$3A_1$	3
4	(4, 4, 3, 1 : 12)	21	11	$3A_3$	9
5	(3, 1, 1, 1 : 6)	39	20	non-singular	0
6	(5, 2, 2, 1 : 10)	28	14	$5A_1$	5
7	(4, 2, 1, 1 : 8)	35	18	$2A_1$	2
8	(6, 3, 2, 1 : 12)	27	14	$2A_1 + 2A_2$	6
9	(10, 5, 4, 1 : 20)	23	13	$A_1 + 2A_4$	9
10	(6, 4, 1, 1 : 12)	39	21	A_1	1
11	(15, 10, 3, 2 : 30)	18	10	$3A_1 + 2A_2 + A_4$	11
12	(9, 6, 2, 1 : 18)	30	16	$3A_1 + A_2$	5
13	(12, 8, 3, 1 : 24)	27	15	$2A_2 + A_3$	7
14	(21, 14, 6, 1 : 42)	24	14	$A_1 + A_2 + A_6$	9
15	(5, 4, 3, 3 : 15)	12	6	$5A_2 + A_3$	13
16	(8, 7, 6, 3 : 24)	9	5	$A_1 + 4A_2 + A_6$	15
17	(5, 5, 3, 2 : 15)	14	8	$A_1 + 3A_4$	13
18	(3, 3, 2, 1 : 9)	23	11	$A_1 + 3A_2$	7
19	(3, 2, 2, 1 : 8)	24	11	$4A_1 + A_2$	6
20	(9, 8, 6, 1 : 24)	18	10	$A_1 + A_2 + A_8$	11
21	(2, 1, 1, 1 : 5)	34	16	A_1	1

TABLE 4.6. (continued)

No.	$(p_1, p_2, p_3, p_4 : p)$	$t(\alpha)$	$n(\alpha)$	Sing(E)	$r(\alpha)$
22	(6, 5, 3, 1 : 15)	21	11	$2A_2+A_5$	9
23	(5, 3, 2, 2 : 12)	17	8	$6A_1+A_4$	10
24	(5, 4, 2, 1 : 12)	24	12	$3A_1+A_4$	7
25	(4, 3, 1, 1 : 9)	33	17	A_3	3
26	(9, 5, 4, 2 : 20)	13	7	$5A_1+A_8$	13
27	(11, 8, 3, 2 : 24)	15	9	$3A_1+A_{10}$	13
28	(10, 7, 3, 1 : 21)	24	14	A_9	9
29	(15, 6, 5, 4 : 30)	10	6	$2A_1+A_2+A_3+2A_4$	15
30	(20, 8, 7, 5 : 40)	8	6	$A_3+2A_4+A_6$	17
31	(12, 5, 4, 3 : 24)	12	7	$2A_2+2A_3+A_4$	14
32	(7, 3, 2, 2 : 14)	19	9	$7A_1+A_2$	9
33	(9, 4, 3, 2 : 18)	16	8	$4A_1+2A_2+A_3$	11
34	(15, 7, 6, 2 : 30)	13	7	$5A_1+A_2+A_6$	13
35	(14, 7, 4, 3 : 28)	12	8	$A_1+A_2+2A_6$	15
36	(10, 5, 3, 2 : 20)	16	9	$2A_1+A_2+2A_4$	12
37	(8, 4, 3, 1 : 16)	24	13	A_2+2A_3	8
38	(15, 8, 6, 1 : 30)	21	12	$A_1+A_2+A_7$	10
39	(9, 5, 3, 1 : 18)	24	13	$2A_2+A_4$	8
40	(7, 4, 2, 1 : 14)	27	14	$3A_1+A_3$	6
41	(12, 7, 3, 2 : 24)	16	9	$2A_1+2A_2+A_6$	12
42	(5, 3, 1, 1 : 10)	36	19	A_2	2
43	(18, 11, 4, 3 : 36)	12	8	$A_1+2A_2+A_{10}$	15
44	(8, 5, 2, 1 : 16)	28	15	$2A_1+A_4$	6
45	(14, 9, 4, 1 : 28)	24	14	A_1+A_8	9
46	(33, 22, 6, 5 : 66)	9	7	$A_1+A_2+A_4+A_{10}$	17
47	(21, 14, 4, 3 : 42)	13	8	$A_1+2A_2+A_3+A_6$	14
48	(24, 16, 5, 3 : 48)	12	8	$2A_2+A_4+A_7$	15
49	(21, 14, 5, 2 : 42)	15	9	$3A_1+A_4+A_6$	13
50	(15, 10, 4, 1 : 30)	25	14	$A_1+A_3+A_4$	8
51	(18, 12, 5, 1 : 36)	24	14	A_4+A_5	9
52	(12, 9, 8, 7 : 36)	5	4	$A_2+A_3+A_6+A_7$	18
53	(6, 5, 4, 3 : 18)	10	5	$A_1+3A_2+A_3+A_4$	14
54	(7, 6, 5, 3 : 21)	9	5	$3A_2+A_4+A_5$	15
55	(7, 6, 5, 2 : 20)	11	6	$3A_1+A_5+A_6$	14
56	(11, 8, 6, 5 : 30)	6	5	$A_1+A_7+A_{10}$	18
57	(9, 6, 5, 4 : 24)	8	5	$2A_1+A_2+A_4+A_8$	16
58	(6, 5, 4, 1 : 16)	19	10	$A_1+A_4+A_5$	10
59	(8, 7, 5, 1 : 21)	18	10	A_4+A_7	11
60	(7, 6, 4, 1 : 18)	19	10	$A_1+A_3+A_6$	10
61	(11, 7, 6, 4 : 28)	7	5	$2A_1+A_5+A_{10}$	17
62	(8, 5, 4, 3 : 20)	10	6	$A_2+2A_3+A_7$	15
63	(4, 3, 2, 1 : 10)	23	11	$2A_1+A_2+A_3$	7
64	(10, 7, 4, 3 : 24)	10	7	$A_1+A_6+A_9$	16
65	(14, 11, 5, 3 : 33)	9	7	A_4+A_{13}	17
66	(3, 2, 1, 1 : 7)	31	15	A_1+A_2	3
67	(9, 7, 3, 2 : 21)	14	8	$A_1+2A_2+A_8$	13
68	(13, 10, 4, 3 : 30)	10	7	$A_1+A_3+A_{12}$	16
69	(7, 4, 3, 2 : 16)	14	7	$4A_1+A_2+A_6$	12

TABLE 4.6. (continued)

No.	$(p_1, p_2, p_3, p_4 : p)$	$t(\alpha)$	$n(\alpha)$	Sing(E)	$r(\alpha)$
70	(8, 5, 3, 2 : 18)	14	8	$2A_1+A_4+A_7$	13
71	(7, 4, 3, 1 : 15)	22	12	A_3+A_6	9
72	(7, 5, 2, 1 : 15)	26	14	A_1+A_6	7
73	(25, 10, 8, 7 : 50)	6	5	$A_1+A_4+A_6+A_7$	18
74	(16, 7, 5, 4 : 32)	9	6	$2A_3+A_4+A_6$	16
75	(11, 5, 4, 2 : 22)	14	7	$5A_1+A_3+A_4$	12
76	(13, 6, 5, 2 : 26)	13	7	$4A_1+A_4+A_5$	13
77	(13, 7, 5, 1 : 26)	21	12	A_4+A_6	10
78	(11, 6, 4, 1 : 22)	22	12	$A_1+A_3+A_5$	9
79	(16, 9, 5, 2 : 32)	13	8	$2A_1+A_4+A_8$	14
80	(22, 13, 5, 4 : 44)	9	7	$A_1+A_4+A_{12}$	17
81	(13, 8, 3, 2 : 26)	16	9	$3A_1+A_2+A_7$	12
82	(11, 7, 3, 1 : 22)	25	14	A_2+A_6	8
83	(27, 18, 5, 4 : 54)	10	7	$A_1+A_3+A_4+A_8$	16
84	(9, 7, 6, 5 : 27)	6	4	$A_2+A_4+A_5+A_6$	17
85	(5, 4, 3, 2 : 14)	13	6	$3A_1+A_2+A_3+A_4$	12
86	(9, 7, 5, 4 : 25)	7	5	$A_3+A_6+A_8$	17
87	(5, 4, 3, 1 : 13)	20	10	$A_2+A_3+A_4$	9
88	(11, 9, 5, 2 : 27)	11	7	$A_1+A_4+A_{10}$	15
89	(5, 3, 2, 1 : 11)	24	12	$A_1+A_2+A_4$	7
90	(17, 7, 6, 4 : 34)	8	5	$2A_1+A_3+A_5+A_6$	16
91	(19, 8, 6, 5 : 38)	7	5	$A_1+A_4+A_5+A_7$	17
92	(19, 11, 5, 3 : 38)	10	7	$A_2+A_4+A_{10}$	16
93	(17, 10, 4, 3 : 34)	11	7	$A_1+A_2+A_3+A_9$	15
94	(7, 5, 4, 3 : 19)	9	5	$A_2+A_3+A_4+A_6$	15
95	(7, 5, 3, 2 : 17)	13	7	$A_1+A_2+A_4+A_6$	13

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