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A MATSUMOTO-TYPE THEOREM FOR KAC-MOODY GROUPS

JUN MORITA AND ULF REHMANN

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Introduction. Let $A=(a_{ij})_{1\leq i,j\leq n}$ be an $n\times n$ generalized Cartan matrix, and $\mathfrak{g}$ the Kac-Moody algebra over the field $\boldsymbol{C}$ of complex numbers, defined by A, with simple roots $\Pi=\{\alpha_1,\cdots,\alpha_n\}$ and simple co-roots $\Pi^*=\{\alpha^*\mid\alpha\in\Pi\}$, where we denote by α^* the co-root of α (cf. [3]). Put $\alpha\beta^*=\alpha(\beta^*)$. Associated to $\mathfrak{g}$ and an arbitrary field F, we can construct a universal Kac-Moody group $G(A,F)$, and the Steinberg group $\mathrm{St}(A,F)$. Let $K_2(A,F)$ be the kernel of the canonical homomorphism of $\mathrm{St}(A,F)$ onto $G(A,F)$ (cf. Section 2). Matsumoto [4] has given a presentation of $K_2(A,F)$ if A is of finite type. As a natural generalization of his result, we will here give a presentation of $K_2(A,F)$ for arbitrary A.

Let L be the abelian group generated by the symbols $c_\alpha(u,v)$ for all $\alpha\in\Pi$ and u,v in the multiplicative group $F^\times$ of F with the following defining relations:

(M1) $c_\alpha(t,u)c_\alpha(tu,v)=c_\alpha(t,uv)c_\alpha(u,v)$

(M2) $c_\alpha(1,1)=1$

(M3) $c_\alpha(u,v)=c_\alpha(u^{-1},v^{-1})$

(M4) $c_\alpha(u,v)=c_\alpha(u,(1-u)v)$ with $u\neq 1$

(M5) $c_\alpha(u,v^{\alpha\beta^*})=c_\beta(u^{\beta\alpha^*},v)$

(M6) $c_{\alpha\beta}(tu,v)=c_{\alpha\beta}(t,v)c_{\alpha\beta}(u,v)$

(M7) $c_{\alpha\beta}(t,uv)=c_{\alpha\beta}(t,u)c_{\alpha\beta}(t,v)$

for all $\alpha,\beta\in\Pi$ with $\alpha\neq\beta$ and $t,u,v\in F^\times$, where $c_{\alpha\beta}(u,v)=c_\alpha(u,v^{\alpha\beta^*})=c_\beta(u^{\beta\alpha^*},v)$. Then we obtain the following:

THEOREM. $K_2(A,F)\simeq L$.

Our main technique is essentially due to Matsumoto [4]. Sometimes we can restrict the root parameter α to a subset Π' of Π. Indeed, we can omit $\alpha\in\Pi$ in generators by the relation (M5) if there exists $\beta\in\Pi$ such that $\alpha\beta^*=-1$. For example, it is enough to choose just one long root $\alpha\in\Pi$, say $\Pi'=\{\alpha\}$, if A is indecomposable and of finite type (i.e., one of A_n, B_n, $\cdots$, G_2). In Section 1, we will review the theory of Kac-Moody groups, and, in section 2, we will introduce the notion of Steinberg groups and K_2-groups. We will study some central extensions of the so-called monomial subgroups of Kac-Moody groups in Section 3, and, using this, we will establish our main result stated above and some related results in Section 4. We will present, in Section 5, some new classes of homologically simply connected Kac-Moody groups. In Section 6, we will give the details of many computations used in the previous sections.

For elements x, y of a group, $[x, y]=xyx^{-1}y^{-1}$ denotes the commutator of x and y. For groups G_1, G_2 such that G_1 acts on G_2, we let $G_1 \ltimes G_2$ denote the semidirect product of G_1 and G_2. We always use 1 as the trivial group. The symbol $\langle \cdots \rangle$ means the group generated by $\cdots$.

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1. **Kac-Moody groups.** Let $A=(a_{ij})_{1 \le i,j \le n}$ be an $n \times n$ generalized Cartan matrix, $\mathfrak{g}$ the Kac-Moody algebra over $\boldsymbol{C}$ defined by A, and Δ the root system of $\mathfrak{g}$ with simple roots $\Pi=\{\alpha_1, \cdots, \alpha_n\} \subset \Delta$ (cf. [2], [3], [7]). For the set of real roots (cf. [3], [9]), say Δ^{re}, we choose and fix a Chevalley basis $\mathscr{C}=\{e_\alpha \mid \alpha \in \Delta^{\mathrm{re}}\}$ (cf. [10], [15]). Using $\mathscr{C}$ and a suitable integrable representation of $\mathfrak{g}$, we can construct a universal Kac-Moody group

$$G=G(A, F)=\langle \exp se_\alpha \mid s \in F, \alpha \in \Delta^{\mathrm{re}} \rangle ,$$

over an arbitrary field F (cf. [12], [14], [16]).

Tits [15] has shown that G has a Steinberg-type presentation, that is, G is the group generated by $x_\alpha(s)$ for all $\alpha \in \Delta^{\mathrm{re}}$ and $s \in F$ with the following defining relations:

(A) $x_\alpha(r)x_\alpha(s)=x_\alpha(r+s)$;

(B) $[x_\alpha(r), x_\beta(s)]=\prod x_{i\alpha+j\beta}(N_{\alpha\beta ij}r^i s^j)$ if $(\boldsymbol{Z}_{>0}\alpha+\boldsymbol{Z}_{>0}\beta) \cap \Delta \subset \Delta^{\mathrm{re}}$;

(B$'$) $w_\alpha(u)x_\beta(s)w_\alpha(-u)=x_{\beta'}(\eta_{\alpha\beta}u^{-\beta\alpha^*}s)$;

(C) $h_\alpha(u)h_\alpha(v)=h_\alpha(uv)$

for all $\alpha, \beta \in \Delta^{\mathrm{re}}$, $r, s \in F$ and u, v in the multiplicative group $F^\times$ of F, where $x_\alpha(s)=\exp se_\alpha$, $w_\alpha(u)=x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u)$ and $h_\alpha(u)=w_\alpha(u)w_\alpha(-1)$, the product of the right hand side in (B) is taken over all real roots of the form $i\alpha+j\beta$ with $i, j \in \boldsymbol{Z}_{>0}$ in some fixed order, the $N_{\alpha\beta ij}$ are certain integers depending only on the structure of $\mathfrak{g}$ (cf. [10], [14]), $\eta_{\alpha\beta}=\pm 1$ is determined by

$$(\exp \operatorname{ad} e_\alpha)(\exp -\operatorname{ad} e_{-\alpha})(\exp \operatorname{ad} e_\alpha)e_\beta=\eta_{\alpha\beta}e_{\beta'} ,$$

α^* is the co-root of α, and $\beta'=\beta-(\beta\alpha^*)\alpha$.

Let Δ_+ be the set of positive roots defined by Π, and put $\Delta_+^{\mathrm{re}}=\Delta^{\mathrm{re}} \cap \Delta_+$, the set of positive real roots. Let U be the subgroup of G generated by $x_\alpha(s)$ for all $\alpha \in \Delta_+^{\mathrm{re}}$ and $s \in F$, and, for each $\alpha \in \Pi$, let V_α be the subgroup of G generated by $x_\alpha(r)x_\beta(s)x_\alpha(-r)$ for all $r, s \in F$ and $\beta \in \Delta_+^{\mathrm{re}} \setminus \{\alpha\}$. Put $U_\alpha=\langle x_\alpha(s) \mid s \in F \rangle \subset G$. Then $U=U_\alpha \ltimes V_\alpha$. Let H be the subgroup of G generated by $h_\alpha(u)$ for all $\alpha \in \Delta^{\mathrm{re}}$ and $u \in F^\times$. Put $B=\langle U, H \rangle \subset G$, then $B=H \ltimes U$. Let N be the subgroup of G generated by $w_\alpha(u)$ for all $\alpha \in \Delta^{\mathrm{re}}$ and $u \in F^\times$, and $S=\{w_\alpha(1) \mid \alpha \in \Pi\}$. Then (G, B, N, S) is a Tits system, $B \cap N=H \lhd N$, and N/H is isomorphic to the Weyl group W of $\mathfrak{g}$ (cf. [12], [15]). Note that W is a Coxeter group, whose Coxeter matrix $M=(m_{ij})_{1 \le i,j \le n}$ is given by $m_{ii}=1$ and $m_{ij}=2$ (resp. 3, 4, 6, ∞) with $i \ne j$ if $a_{ij}a_{ji}$ is 0 (resp. 1, 2, 3, ≥ 4) (cf. [1], [8]).

The structure of Tits system implies $G=UNU$, called the Bruhat decomposition.

Using the representation theory, one can easily see that the N-component in this decomposition is uniquely determined. For $g \in G$, we denote by $\nu(g)$ the N-component of g in the Bruhat decomposition $G = UNU$. This ν is just a well-defined map of G to N, but not a homomorphism. Sometimes, we call H a maximal torus, and N the monomial subgroup of G associated with H.

2. Steinberg groups and K_2-groups. Let $\mathrm{St}(A, F)$ be the group generated by the symbols $\hat{x}_\alpha(s)$ for all $\alpha \in \Delta^{\mathrm{re}}$ and $s \in F$ with the defining relations (A), (B) and (B′), where $x_\alpha(s)$ and $w_\alpha(u)$ are replaced by $\hat{x}_\alpha(s)$ and $\hat{w}_\alpha(u)$, respectively. We call $\mathrm{St}(A, F)$ the Steinberg group associated with G. Then there is a canonical homomorphism ρ of $\mathrm{St}(A, F)$ onto G such that $\rho(\hat{x}_\alpha(s)) = x_\alpha(s)$ for all $\alpha \in \Delta^{\mathrm{re}}$ and $s \in F$. Put $K_2(A, F) = \mathrm{Ker}\,\rho$. By Tits [15], $K_2(A, F)$ is generated by $\{u, v\}_\alpha$ for all $u, v \in F^\times$ and $\alpha \in \Delta^{\mathrm{re}}$, where

$$\{u, v\}_\alpha = \hat{h}_\alpha(u)\hat{h}_\alpha(v)\hat{h}_\alpha(uv)^{-1}, \quad \hat{h}_\alpha(u) = \hat{w}_\alpha(u)\hat{w}_\alpha(-1).$$

Then $\{u, v\}_\alpha$ is central, and

$$[\hat{h}_\alpha(u), \hat{h}_\beta(v)] = \{u, v^{\alpha\beta^*}\}_\alpha = \{u^{\beta\alpha^*}, v\}_\beta$$

for all $\alpha, \beta \in \Delta^{\mathrm{re}}$ and $u, v \in F^\times$. Furthermore,

$$\hat{w}_\alpha(u)\hat{h}_\beta(v)\hat{w}_\alpha(-u) = \hat{h}_{\beta'}(\eta_{\alpha\beta}u^{-\beta\alpha^*}v)\hat{h}_{\beta'}(\eta_{\alpha\beta}u^{-\beta\alpha^*})^{-1},$$

hence

$$\{u, v\}_{\beta'} = \begin{cases} \{u, v\}_\beta & \text{if} \quad \eta_{\alpha\beta} = 1\,; \\ \{v, u\}_\beta^{-1} & \text{if} \quad \eta_{\alpha\beta} = -1\,, \end{cases}$$

where $\beta' = \beta - (\beta\alpha^*)\alpha$. Therefore, $K_2(A, F)$ is generated by $\{u, v\}_\alpha$ for all $u, v \in F^\times$ and $\alpha \in \Pi$.

Let L be the abelian group generated by the symbols $c_\alpha(u, v)$ for all $\alpha \in \Pi$ and $u, v \in F^\times$ with the defining relations (M1)–(M7) as in the introduction. Then there is a homomorphism λ of L onto $K_2(A, F)$ such that $\lambda(c_\alpha(u, v)) = \{u, v\}_\alpha$ for all $\alpha \in \Pi$ and $u, v \in F^\times$ (cf. [4], [14]). We will show, in Section 4, that λ is an isomorphism. Now the following proposition is a direct consequence of the relation (B′) (cf. [4], [14]).

PROPOSITION 1. *Let $\alpha, \beta \in \Delta^{\mathrm{re}}$ be linearly independent real roots, and $u, v \in F^\times$.*

(1) $\hat{w}_\alpha(u)\hat{w}_\beta(v) = \hat{w}_\beta(v)\hat{w}_\alpha(u)$ *if* $\alpha\beta^* = \beta\alpha^* = 0$.

(2) $\hat{w}_\alpha(u)\hat{w}_\beta(v)\hat{w}_\alpha(u) = \hat{w}_\beta(v)\hat{w}_\alpha(u)\hat{w}_\beta(v)$ *if* $\alpha\beta^* = \beta\alpha^* = -1$.

(3) $(\hat{w}_\alpha(u)\hat{w}_\beta(v))^2 = (\hat{w}_\beta(v)\hat{w}_\alpha(u))^2$ *if* $\alpha\beta^* = -2, \quad \beta\alpha^* = -1$.

(4) $(\hat{w}_\alpha(u)\hat{w}_\beta(v))^3 = (\hat{w}_\beta(v)\hat{w}_\alpha(u))^3$ *if* $\alpha\beta^* = -3, \quad \beta\alpha^* = -1$.

3. Some central extensions of N. Let H_{α_i}, for each $\alpha_i \in \Pi$, be the subgroup of H generated by $h_{\alpha_i}(u)$ for all $u \in F^\times$. Then $H = H_{\alpha_1} \times \cdots \times H_{\alpha_n}$ and $H_{\alpha_i} \simeq F^\times$. Now we define a 2-cocycle $\xi\colon H \times H \to L$ by

$$\xi(x, y)=\prod_{1\le i\le n} c_{\alpha_i}(x_i, y_i) \prod_{1\le j<i\le n} c_{\alpha_i\alpha_j}(x_i, y_j)$$

for all

$$x=(x_1, \cdots, x_n), \qquad y=(y_1, \cdots, y_n)\in H .$$

Using this ξ, we construct a central extension $(\tilde{H}, \pi)$

$$1 \longrightarrow L \longrightarrow \tilde{H} \xrightarrow{\pi} H \longrightarrow 1$$

of H by L, where π denotes the associated homomorphism of $\tilde{H}$ onto H.

PROPOSITION 2. *$\tilde{H}$ is the group generated by the symbols $\tilde{h}_\alpha(u)$ and $z(l)$ for all $\alpha\in\Pi$, $u\in F^\times$ and $l\in L$ with the following defining relations*:

(H1) $\tilde{h}_\alpha(u)\tilde{h}_\alpha(v)=z(c_\alpha(u, v))\tilde{h}_\alpha(uv)$;

(H2) $\tilde{h}_\alpha(u)\tilde{h}_\beta(v)=z(c_{\alpha\beta}(u, v))\tilde{h}_\beta(v)\tilde{h}_\alpha(u)$;

(H3) $z(l_1)z(l_2)=z(l_1l_2)$;

(H4) $z(l)\tilde{h}_\alpha(u)=\tilde{h}_\alpha(u)z(l)$

for all α, $\beta\in\Pi$, u, $v\in F^\times$ and l_1, l_2, $l\in L$.

Let Z be the subgroup of $\tilde{H}$ generated by $z(l)$ for all $l\in L$. Then $Z\simeq L$, hence we identify L with Z.

In the remainder of this section, we will construct some central extension of the monomial subgroup N by L which is compatible with the extension $(\tilde{H}, \pi)$ of H. To do so, we first construct an action of N on $\tilde{H}$.

PROPOSITION 3. *N is the group generated by $w_\alpha(u)$ for all $\alpha\in\Pi$ and $u\in F^\times$ with the following defining relations*:

(N1) $w_\alpha(-u)=w_\alpha(u)^{-1}$;

(N2) $\underbrace{w_\alpha(1)w_\beta(1)\cdots}_{q}=\underbrace{w_\beta(1)w_\alpha(1)\cdots}_{q}$;

(N3) $w_\alpha(1)h_\gamma(v)w_\alpha(-1)=h_\gamma(v)h_\alpha(v^{-\alpha\gamma^*})$;

(N4) $h_\alpha(u)h_\alpha(v)=h_\alpha(uv)$;

(N5) $h_\alpha(u)h_\beta(v)=h_\beta(v)h_\alpha(u)$

for all α, β, $\gamma\in\Pi$ with $\alpha\neq\beta$ and u, $v\in F^\times$, where both sides of the equation in (N2) *consist of the product of q symbols as in Proposition* 1 *with $q=2$ (resp.* 3, 4, 6) *if $(\alpha\beta^*)(\beta\alpha^*)=0$ (resp.* 1, 2, 3), *and $h_\alpha(u)=w_\alpha(u)w_\alpha(-1)$.*

Using Propositions 2 and 3, we can confirm that $\tilde{H}$ becomes an N-group by

$$w_\alpha(u)\cdot\tilde{h}_\beta(v)=\tilde{h}_\beta(v)\tilde{h}_\alpha(v^{-\alpha\beta^*})c_{\alpha\beta}(u, v)^{-1}$$

for all α, $\beta\in\Pi$ and u, $v\in F^\times$ (cf. Section 6.I).

Let $\tilde{W}$ be the group generated by the symbols $\tilde{w}_\alpha$ for all $\alpha\in\Pi$ with the following defining relations:

(W1) $\tilde{h}_\alpha \tilde{w}_\gamma \tilde{h}_\alpha^{-1} = \tilde{w}_\gamma^c$;

(W2) $\underbrace{\tilde{w}_\alpha \tilde{w}_\beta \cdots}_{q} = \underbrace{\tilde{w}_\beta \tilde{w}_\alpha \cdots}_{q}$

for all $\alpha, \beta, \gamma \in \Pi$ with $\alpha \neq \beta$ and $\tilde{h}_\alpha = \tilde{w}_\alpha^2$, where $c = (-1)^{\gamma\alpha^*}$, and $q = 2$ (resp. 3, 4, 6) if $(\alpha\beta^*)(\beta\alpha^*) = 0$ (resp. 1, 2, 3). Put $\tilde{T} = \langle \tilde{h}_\alpha \mid \alpha \in \Pi \rangle \subset \tilde{W}$, and $N^* = \tilde{W} \ltimes \tilde{H}$, where $\tilde{W}$ acts on $\tilde{H}$ by

$$\tilde{w}_\alpha \cdot \tilde{h} = w_\alpha(-1) \cdot \tilde{h}$$

for all $\alpha \in \Pi$ and $\tilde{h} \in \tilde{H}$. Then $\tilde{T}$ is the group generated by $\tilde{h}_\alpha$ for all $\alpha \in \Pi$ with the following defining relations:

(T) $\tilde{h}_\alpha \tilde{h}_\beta \tilde{h}_\alpha^{-1} = \tilde{h}_\beta^c$ for all $\alpha \in \Pi$ with $c = (-1)^{\beta\alpha^*}$

(cf. Section 6.II). Hence, there is a canonical homomorphism ι of $\tilde{T}$ into $\tilde{H}$ such that $\iota(\tilde{h}_\alpha) = \tilde{h}_\alpha(-1)$ for all $\alpha \in \Pi$. Let J^* be the subgroup, which is normal in this case, of N^* generated by $(y, \iota(y)^{-1})$ for all $y \in \tilde{T}$, and $\tilde{N} = N^*/J^*$. Note that there is a canonical homomorphism ϕ of N^* onto the monomial subgroup N such that $\phi(\tilde{w}_\alpha) = w_\alpha(-1)$ and $\phi(\tilde{h}_\alpha(u)) = h_\alpha(u)$ for all $\alpha, \beta \in \Pi$ and $u \in F^\times$ and that $J^* \subset \operatorname{Ker} \phi$. Hence ϕ induces a homomorphism, again called ϕ, of $\tilde{N}$ onto N. Put $\tilde{w}_\alpha(u) = \phi^*(\tilde{h}_\alpha(u))\phi^*(\tilde{w}_\alpha)^{-1}$, where ϕ^* is the canonical homomorphism of N^* onto N.

PROPOSITION 4. (1) *The restriction of ϕ^* to $\tilde{H}$ is injective, hence we identify $\tilde{H}$ with $\phi^*(\tilde{H})$.*

(2) *The group $\tilde{N}$ is a central extension of N by L:*

$$1 \longrightarrow L \longrightarrow \tilde{N} \xrightarrow{\phi} N \longrightarrow 1$$

with $\phi(\tilde{w}_\alpha(u)) = w_\alpha(u)$ for all $\alpha \in \Pi$ and $u \in F^\times$.

(3) *The restriction of ϕ to $\tilde{H}$ coincides with π.*

Note that $\phi^*(\tilde{w}_\alpha) = \tilde{w}_\alpha(-1)$ and $\tilde{w}_\alpha(u)^{-1} = \tilde{w}_\alpha(-u)$.

4. Proof of Theorem. Let $\mathrm{St}(n+1, F) = \mathrm{St}(A_n, F)$ be the Steinberg group arising from $SL(n+1, F)$, and $K_2(n+1, F) = K_2(A_n, F)$ the associated K_2-group with the Steinberg symbol $\{\cdot, \cdot\}$ (cf. [6]). Then, by Matsumoto [4], $K_2(2, F)$ is the group generated by $\{u, v\}$ for all $u, v \in F^\times$ with the defining relations (M1)–(M4), where c_α is replaced by $\{\cdot, \cdot\}$. Hence, for each $\alpha \in \Pi$, there is a canonical homomorphism ζ_α of $K_2(2, F)$ into L such that $\zeta_\alpha(\{u, v\}) = c_\alpha(u, v)$ for all $u, v \in F^\times$. Put $M_\alpha = \operatorname{Ker} \zeta_\alpha$, and $\tilde{S}_\alpha = \mathrm{St}(2, F)/M_\alpha$. Let $\tilde{H}_\alpha$ be the subgroup of $\tilde{H}$ generated by $\tilde{h}_\alpha(u)$ for all $u \in F^\times$. Then there is a canonical monomorphism μ_α of $\tilde{H}_\alpha$ into $\tilde{S}_\alpha$. Let $J_\alpha = \langle (y, \mu_\alpha(y)^{-1}) \mid y \in \tilde{H}_\alpha \rangle \subset \tilde{H} \ltimes \tilde{S}_\alpha$ and

$$\tilde{P}_\alpha = \left(\frac{\tilde{H} \ltimes \tilde{S}_\alpha}{J_\alpha} \right) \ltimes V_\alpha$$

for each $\alpha\in\Pi$, where $\tilde{H}$ acts diagonally on $\tilde{S}_\alpha$ by $\tilde{h}_\beta(u)\cdot(\hat{x}_{12}(s) \bmod M_\alpha)=\hat{x}_{12}(u^{\alpha\beta^*}s) \bmod M_\alpha$ and $\tilde{h}_\beta(u)\cdot(\hat{x}_{21}(s) \bmod M_\alpha)=\hat{x}_{21}(u^{-\alpha\beta^*}s) \bmod M_\alpha$ for all $\alpha,\beta\in\Pi$, $s\in F$ and $u\in F^\times$, the group J_α is normal in $\tilde{H}\rtimes\tilde{S}_\alpha$, and the action on V_α can be defined since V_α is the unipotent radical of a rank one parabolic subgroup of G whose reductive part is the canonical image of $(\tilde{H}\rtimes\tilde{S}_\alpha)/J_\alpha$. Note that $\tilde{h}_\beta(u)\cdot M_\alpha=M_\alpha$. Put $\tilde{B}=\tilde{H}\rtimes U$. Then $\tilde{B}$ can be regarded as a subgroup of $\tilde{P}_\alpha$ for each $\alpha\in\Pi$. Let $\tilde{N}_\alpha$ be the subgroup of $\tilde{N}$ generated by $\tilde{w}_\alpha(u)$ for all $u\in F^\times$. Then $\tilde{N}_\alpha$ can also be regarded as a subgroup of $\tilde{P}_\alpha$ naturally. Taking

$$\tilde{N}\cap\tilde{P}_\alpha=\tilde{N}_\alpha$$

for all $\alpha\in\Pi$ and

$$\tilde{P}_\alpha\cap\tilde{P}_\beta=\tilde{B}$$

for all α, $\beta\in\Pi$ with $\alpha\neq\beta$, we let

$$\tilde{G}=*\langle\tilde{N},\tilde{P}_\alpha \mid \alpha\in\Pi\rangle$$

be the amalgamated free product of $\tilde{N}$ and $\tilde{P}_\alpha$ for all $\alpha\in\Pi$ along their intersections.

Let Γ be the subset of $G\times\tilde{N}$ consisting of all elements $(x,y)\in G\times\tilde{N}$ such that $\nu(x)=\phi(y)$. Then, as described in [4; p. 40ff], each $\tilde{P}_\alpha$ has a faithful action on Γ (cf. Section 6.III), which is compatible with our amalgamation here. Therefore, $\tilde{G}$ acts on Γ. In particular, L is embedded into $\tilde{G}$.

On the other hand, there is a natural homomorphism θ of $\tilde{G}$ onto $\mathrm{St}(A,F)$. In the standard way as in Steinberg [14], all the relations of $\mathrm{St}(A,F)$ can be lifted to $\tilde{G}$ using θ^{-1} since $\mathrm{St}(A,F)$ has an analogous decomposition (cf. [13]) and $\operatorname{Ker}\theta$ is central, which comes from the following:

PROPOSITION 5. *If A is a generalized Cartan matrix and F is an infinite field, then* $\mathrm{St}(A,F)$ *is homologically simply connected* (*cf. Section* 6.IV).

Note that $K_2(A,F)=L=0$ if F is a finite field. Hence θ is an isomorphism, and so is λ. Therefore, we have proved the following result:

THEOREM. $K_2(A,F)\simeq L$.

Sometimes we can restrict the root parameter α to a subset Π' of Π. Indeed, we can omit $\alpha\in\Pi$ in generators by the relation (M5) if there exists $\beta\in\Pi$ such that $\alpha\beta^*=-1$. Let

$$L_i=\begin{cases}K_2(3,F) & \text{if } a_{ki} \text{ is odd for some } 1\le k\le n\,;\\ K_2(2,F) & \text{if } a_{ki} \text{ is even for all } 1\le k\le n\,,\end{cases}$$

for each $1\le i\le n$. Then the Steinberg symbol corresponding to $\{\cdot,\cdot\}$ is denoted by $\{\cdot,\cdot\}_i$. Let J be the subgroup of $L_1\times L_2\times\cdots\times L_n$ generated by $\{u,v^{a_{ji}}\}_i\cdot\{v,u^{a_{ij}}\}_j$ for all $u,v\in F^\times$ and $1\le i<j\le n$. Put

$$L' = \frac{L_1 \times L_2 \times \cdots \times L_n}{J}.$$

Then, the theorem implies the following result:

COROLLARY 1. $K_2(A, F) \simeq L'$.

We say that a generalized Cartan matrix $A = (a_{ij})_{1 \le i,j \le n}$ is simply laced (in terms of Dynkin diagrams) if $a_{ij} = 0, -1$ for all $1 \le i \ne j \le n$.

COROLLARY 2. *Suppose that A is indecomposable and simply laced. If $n > 1$, then*

$$K_2(A, F) \simeq K_2(3, F).$$

Hence, we also see the following result, using the fact that every symmetrizable generalized Cartan matrix (cf. [2]) is obtained from a simply laced generalized Cartan matrix by foldings in terms of Dynkin diagrams.

COROLLARY 3. *Suppose A is symmetrizable. Then, $K_2(A, F) \ne 1$ for some field F.*

COROLLARY 4. *Suppose A is indecomposable and of finite type (i.e., one of A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2). Then, by Matsumoto* [4], *we have*

$$K_2(A, F) \simeq \begin{cases} K_2(2, F) & \text{if } A = C_n\,(n \ge 1); \\ K_2(3, F) & \text{if } A \ne C_n\,(n \ge 1). \end{cases}$$

COROLLARY 5. *Suppose A is of affine type $X_l^{(r)}$ (cf.* [2; *pp.* 44–45]).

(1) *Suppose that the tier number r is* 1. *Then*

$$K_2(X_l^{(1)}, F) \simeq \begin{cases} K_2(2, F) \oplus I^2(F) & \text{if } X_l^{(1)} = C_l^{(1)}\,(l \ge 1); \\ K_2(3, F) & \text{if } X_l^{(1)} \ne C_l^{(1)}\,(l \ge 1), \end{cases}$$

where $I(F)$ is the fundamental ideal of the Witt ring $W(F)$ of F (cf. [11]).

(2) *Suppose that the tier number r is* 2 *or* 3. *Then*

$$K_2(X_l^{(r)}, F) \simeq \begin{cases} K_2(2, F) & \text{if } X_l^{(r)} = A_l^{(2)}\,(l \ge 2); \\ K_2(3, F) & \text{if } X_l^{(r)} = D_l^{(2)}\,(l \ge 4),\ E_6^{(2)},\ D_4^{(3)}. \end{cases}$$

COROLLARY 6. *Let*

$$A = \begin{pmatrix} 2 & -a \\ -1 & 2 \end{pmatrix}$$

with $a \in \mathbf{Z}_{>0}$. Then

$$K_2(A, F) \simeq \begin{cases} K_2(2, F) & \text{if } a \text{ is even}; \\ K_2(3, F) & \text{if } a \text{ is odd}. \end{cases}$$

It is also possible to determine the group structure of $K_2(A, F)$ in many other cases.

5. Simply connected Kac-Moody groups. Here we will present some new classes of homologically simply connected groups. Let

$$A=\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Then, for $u, v \in F^{\times}$, we obtain

$$\{u, v^{-2}\}_{\alpha_1}=\{u^{-1}, v\}_{\alpha_2}=\{u, v^{-1}\}_{\alpha_3}=\{u^{-1}, v\}_{\alpha_1}=\{u, v^{-1}\}_{\alpha_1}$$

and $\{u, v\}_{\alpha_1}=1$, which implies $\{u, v\}_{\alpha}=1$ for all $u, v \in F^{\times}$ and $\alpha \in \Pi$. Hence $K_2(A, F)=1$ for *all* fields F. Furthermore, we obtain the following result:

EXAMPLE 1. Let F be an arbitrary field, and $A=(a_{ij})_{1 \le i, j \le n}$ an $n \times n$ generalized Cartan matrix with $a_{ij}=0$ unless $|i-j| \equiv 0, 1 \pmod{n}$. Put

$$d_A=|a_{12}a_{23}\cdots a_{n-1,n}a_{n1}-a_{21}a_{32}\cdots a_{n,n-1}a_{1n}|.$$

Then:

(1) $\{u^{d_A}, v\}_{\alpha}=1$ for all $u, v \in F^{\times}$ and $\alpha \in \Pi$;

(2) If $F=F^{d_A}$, then $K_2(A, F)=1$;

(3) If d_A is odd, then $K_2(A, F)$ is a d_A-torsion group, that is, $x^{d_A}=1$ for all

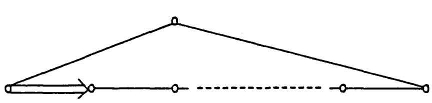

FIGURE 1

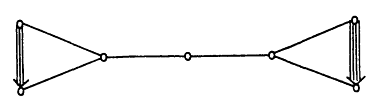

FIGURE 2

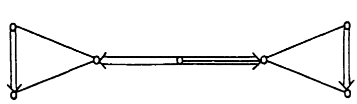

FIGURE 3

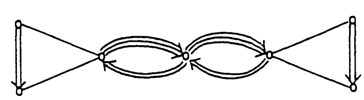

FIGURE 4

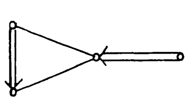

FIGURE 5

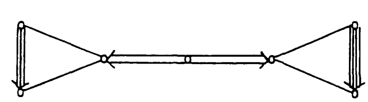

FIGURE 6

$x \in K_2(A, F)$;

(4) If $d_A = 1$, then $K_2(A, F) = 1$.

This is just a simple example, which we observed at first. In Figure 1, we shall draw a typical Dynkin diagram in this example, with $d_A = 1$.

Similarly we can construct lots of examples of generalized Cartan matrices A such that $K_2(A, F) = 1$ for every field F.

EXAMPLE 2. Let

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -3 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & -4 & 2 \end{pmatrix}$$

with the Dynkin diagram as in Figure 2. Then, $K_2(A, F) = 1$ for an arbitrary field F.

EXAMPLE 3. Let

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & -2 & 2 \end{pmatrix}$$

with the Dynkin diagram as in Figure 3. Then, $K_2(A, F) = 1$ for an arbitrary field F.

EXAMPLE 4. Let

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & -2 & 2 \end{pmatrix}$$

with the Dynkin diagram as in Figure 4. Then, $K_2(A, F) = 1$ for an arbitrary field F.

EXAMPLE 5. Let

$$A=\begin{pmatrix} 2 & -1 & -1 & 0 \\ -2 & 2 & -1 & 0 \\ -1 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

with the Dynkin diagram as in Figure 5. Then, $K_2(A, F)=I^2(F)$ for an arbitrary field F, where $I(F)$ is the fundamental ideal of the Witt ring $W(F)$ of F.

EXAMPLE 6. Let

$$A=\begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -3 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & -4 & 2 \end{pmatrix}$$

with the Dynkin diagram as in Figure 6. Then, $K_2(A, F)=I^2(F)$ for an arbitrary field F.

EXAMPLE 7. Let $m\in \boldsymbol{Z}_{>1}$, and put

$$A=\begin{pmatrix} 2 & -1 & -1 \\ -(m+1) & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Then, $K_2(A, F)=K_2(3, F)/K_2(3, F)^m$ for an arbitrary field F. Hence, $K_2(A, F)\simeq \mathrm{Br}_m(F)$ if char F is prime to m, and $K_2(A, F)\simeq \mu_m(F)$ if F is a local field, where $\mathrm{Br}_m(F)$ is the m-torsion part of the Brauer group $\mathrm{Br}(F)$ of F, while $\mu_m(F)=\{u\in F \mid u^m=1\}$ (cf. [5]).

As above, we get a lot of new examples of homologically simply connected groups which are matrix groups of infinite size.

6. Proofs.

I. Action of N on $\tilde{H}$. (i) We should first check that the action of N by

$$w_\alpha(t)\cdot \tilde{h}_\gamma(v)=\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{-\alpha\gamma^*})c_{\alpha\gamma}(t, v)^{-1}$$

preserves all the relations (H1)–(H4). Note that

$$w_\alpha(t)\cdot c_\gamma(u, v)=c_\gamma(u, v)\,.$$

Hence, (H3) and (H4) are easy.

(H1):

$$\begin{aligned} w_\alpha(t)\cdot(\tilde{h}_\gamma(u)\tilde{h}_\gamma(v)) &= \tilde{h}_\gamma(u)\tilde{h}_\alpha(u^{-\alpha\gamma^*})\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{-\alpha\gamma^*})c_{\alpha\gamma}(t,u)^{-1}c_{\alpha\gamma}(t,v)^{-1} \\ &= \tilde{h}_\gamma(uv)\tilde{h}_\alpha((uv)^{-\alpha\gamma^*})c_\gamma(u,v)c_{\alpha\gamma}(u^{-\alpha\gamma^*},v)c_\alpha(u^{-\alpha\gamma^*},v^{-\alpha\gamma^*})c_{\alpha\gamma}(t,uv)^{-1} \\ &= c_\gamma(u,v)(w_\alpha(t)\cdot\tilde{h}_\gamma(uv)) . \end{aligned}$$

(H2):

$$\begin{aligned} w_\alpha(t)\cdot(\tilde{h}_\beta(u)\tilde{h}_\gamma(v)) &= \tilde{h}_\beta(u)\tilde{h}_\alpha(u^{-\alpha\beta^*})c_{\alpha\beta}(t,u)^{-1}\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{-\alpha\gamma^*})c_{\alpha\gamma}(t,v)^{-1} \\ &= \tilde{h}_\gamma(v)\tilde{h}_\beta(u)\tilde{h}_\alpha(u^{-\alpha\beta^*}v^{-\alpha\gamma^*})c_{\beta\gamma}(u,v) \\ &\qquad \cdot c_\alpha(u^{-\alpha\beta^*},v^{-\alpha\gamma^*})c_{\alpha\gamma}(u^{-\alpha\beta^*},v)c_{\alpha\beta}(t,u)^{-1}c_{\alpha\gamma}(t,v)^{-1} \\ &= w_\alpha(t)\cdot(c_{\beta\gamma}(u,v)\tilde{h}_\gamma(v)\tilde{h}_\beta(u)) . \end{aligned}$$

Therefore, $w_\alpha(t)$ gives an automorphism of $\tilde{H}$.

(ii) We should, next, check that both sides in the relations (N1)–(N5) give the same effect on $\tilde{H}$. Note that

$$w_\alpha(t)\cdot\tilde{h}_\alpha(v)=\tilde{h}_\alpha(v^{-1})c_\alpha(t,v^2)^{-1} \quad \text{and} \quad h_\alpha(t)\cdot\tilde{h}_\gamma(v)=\tilde{h}_\gamma(v)c_{\alpha\gamma}(t,v) .$$

Hence, (N4) and (N5) are easy.

(N1):

$$\begin{aligned} w_\alpha(t)\cdot(w_\alpha(-t)\cdot\tilde{h}_\gamma(v)) &= w_\alpha(t)\cdot(\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{-\alpha\gamma^*})c_{\alpha\gamma}(-t,v)^{-1} \\ &= \tilde{h}_\gamma(v)c_\alpha(v^{-\alpha\gamma^*},v^{\alpha\gamma^*})c_\alpha(t,v^{-2\alpha\gamma^*})^{-1}c_{\alpha\gamma}(t,v)^{-1}c_{\alpha\gamma}(-t,v)^{-1}=\tilde{h}_\gamma(v) . \end{aligned}$$

(N3):

$$\begin{aligned} (w_\alpha(1)h_\beta(u)w_\alpha(-1))\cdot\tilde{h}_\gamma(v) &= (w_\alpha(1)h_\beta(u))\cdot(\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{-\alpha\gamma^*})c_{\alpha\gamma}(-1,v)^{-1}) \\ &= \tilde{h}_\gamma(v)c_{\beta\gamma}(u,v)c_{\beta\alpha}(u,v^{-\alpha\gamma^*})=(h_\beta(u)h_\alpha(u^{-\alpha\beta^*}))\cdot\tilde{h}_\gamma(v) . \end{aligned}$$

Finally, we check (N2). Let $\mathscr{L}$ (resp. $\mathscr{R}$) be the left (resp. right) hand side of the equation in (N2). If

$$\mathscr{L}\cdot\tilde{h}_\gamma(v)=\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{m_1})\tilde{h}_\beta(v^{m_2})l$$

and

$$\mathscr{R}\cdot\tilde{h}_\gamma(v)=\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{m'_1})\tilde{h}_\beta(v^{m'_2})l'$$

with $l, l'\in L$, then $m_1=m'_1$ and $m_2=m'_2$ by Proposition 1 and (N3). Therefore, it is enough to show $l=l'$.

(1): Suppose $\alpha\beta^*=\beta\alpha^*=0$ (hence $q=2$).Then

$$\tilde{h}_\alpha(u)\tilde{h}_\beta(v)=\tilde{h}_\beta(v)\tilde{h}_\alpha(u)$$

and

$$\mathscr{L}\cdot\tilde{h}_\gamma(v)=\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{-\alpha\gamma^*})\tilde{h}_\beta(v^{-\beta\gamma^*})=\mathscr{R}\cdot\tilde{h}_\gamma(v) .$$

(2): Suppose $\alpha\beta^*=\beta\alpha^*=-1$ (hence $q=3$). Then

$$\begin{aligned}\mathscr{L}\cdot\tilde{h}_\gamma(v)&=(w_\alpha(1)w_\beta(1))\cdot(\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{-\alpha\gamma^*}))=w_\alpha(1)\cdot(\tilde{h}_\gamma(v)\tilde{h}_\beta(v^{-\beta\gamma^*})\tilde{h}_\alpha(v^{-\alpha\gamma^*})\tilde{h}_\beta(v^{-\alpha\gamma^*}))\\&=w_\alpha(1)\cdot(\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{-\alpha\gamma^*})\tilde{h}_\beta(v^{-\beta\gamma^*})\tilde{h}_\beta(v^{-\alpha\gamma^*})c_{\beta\alpha}(v^{-\beta\gamma^*},v^{-\alpha\gamma^*}))\\&=w_\alpha(1)\cdot(\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{-\alpha\gamma^*})\tilde{h}_\beta(v^{(-\alpha-\beta)\gamma^*}))\\&=\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{-\alpha\gamma^*})\tilde{h}_\alpha(v^{\alpha\gamma^*})\tilde{h}_\beta(v^{(-\alpha-\beta)\gamma^*})\tilde{h}_\alpha(v^{(-\alpha-\beta)\gamma^*})\\&=\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{(-\alpha-\beta)\gamma^*})\tilde{h}_\beta(v^{(-\alpha-\beta)\gamma^*})c_\alpha(v^{-\alpha\gamma^*},v^{\alpha\gamma^*})c_{\beta\alpha}(v^{(-\alpha-\beta)\gamma^*},v^{(-\alpha-\beta)\gamma^*})\\&=\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{(-\alpha-\beta)\gamma^*})\tilde{h}_\beta(v^{(-\alpha-\beta)\gamma^*})c_\alpha(-1,v^{\beta\gamma^*})=\mathscr{R}\cdot\tilde{h}_\gamma(v)\,.\end{aligned}$$

(3): Suppose $\alpha\beta^*=-2$ and $\beta\alpha^*=-1$ (hence $q=4$). Then

$$\tilde{h}_\alpha(v^k)\tilde{h}_\alpha(v^{2m})=\tilde{h}_\alpha(v^{k+2m})\,,$$
$$\tilde{h}_\beta(v^k)\tilde{h}_\beta(v^m)=\tilde{h}_\beta(v^{k+m})\,,$$

and

$$\tilde{h}_\alpha(v^k)\tilde{h}_\beta(v^m)=\tilde{h}_\beta(v^m)\tilde{h}_\alpha(v^k)\,.$$

Put

$$[n_1,n_2,n_3]=\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{n_1})\tilde{h}_\beta(v^{n_2})c_{\gamma\alpha}(-1,v)^{n_3}\,.$$

Using the relations above, we see that

$$w_\alpha(1)\cdot[n_1,n_2,n_3]=[-n_1+2n_2-\alpha\gamma^*,n_2,n_1+n_3]\,,$$

and

$$w_\beta(1)\cdot[n_1,n_2,n_3]=[n_1,n_1-n_2-\beta\gamma^*,n_3]\,.$$

For our purpose, it is enough to check only the parity of the n_3-component, which allows us to consider $[n_1,n_2,n_3]$ taking $n_i \bmod 2$. Here we denote these two relations symbolically by

$$[n_1,n_2,n_3]\xrightarrow{\alpha}[n_1+\alpha,n_2,n_1+n_3]\,,$$

and

$$[n_1,n_2,n_3]\xrightarrow{\beta}[n_1,n_1+n_2+\beta,n_3]\,.$$

Then

$$[0,0,0]\xrightarrow{\alpha}[\alpha,0,0]\xrightarrow{\beta}[\alpha,\alpha+\beta,0]\xrightarrow{\alpha}[0,\alpha+\beta,\alpha]\xrightarrow{\beta}[0,\alpha,\alpha]\,,$$

and

$$[0,0,0]\xrightarrow{\beta}[0,\beta,0]\xrightarrow{\alpha}[\alpha,\beta,0]\xrightarrow{\beta}[\alpha,\alpha,0]\xrightarrow{\alpha}[0,\alpha,\alpha]\,.$$

Hence we have just confirmed

$$l=c_{\gamma\alpha}(-1,v)^{\alpha\gamma^*}=l'\,.$$

(4): Suppose $\alpha\beta^*=-3$ and $\beta\alpha^*=-1$ (hence $q=6$). Put

$$[n_1, n_2, n_3]=\tilde{h}_\gamma(v)\tilde{h}_\alpha(v^{n_1})\tilde{h}_\beta(v^{n_2})c_\alpha(-1, v)^{n_3}.$$

Then

$$w_\alpha(1)\cdot[n_1, n_2, n_3]=[n'_1, n'_2, n'_3]$$

with $n'_1=-n_1+n_2-\alpha\gamma^*$; $n'_2=n_2$; $n'_3=n_1n_2+\alpha\gamma^*n_1+(1+\alpha\gamma^*)n_2+n_3$, and

$$w_\beta(1)\cdot[n_1, n_2, n_3]=[n''_1, n''_2, n''_3]$$

with $n''_1=n_1$; $n''_2=n_1-n_2-\beta\gamma^*$; $n''_3=n_1n_2+\beta\gamma^*n_2+n_3$. For the same reason as above, we can take n_i mod 2, and use an anologous symbolical notation. Therefore, it is now easy to compute:

$$[0,0,0]\xrightarrow{\alpha}[\alpha,0,0]\xrightarrow{\beta}[\alpha,\alpha+\beta,0]\xrightarrow{\alpha}[\alpha+\beta,\alpha+\beta,\beta]$$
$$\xrightarrow{\beta}[\alpha+\beta,\beta,\alpha\beta+\alpha+\beta]\xrightarrow{\alpha}[0,\beta,\beta]\xrightarrow{\beta}[0,0,0],$$

and

$$[0,0,0]\xrightarrow{\beta}[0,\beta,0]\xrightarrow{\alpha}[\alpha+\beta,\beta,\alpha\beta+\beta]\xrightarrow{\beta}[\alpha+\beta,\alpha+\beta,\beta]$$
$$\xrightarrow{\alpha}[\alpha,\alpha+\beta,\beta]\xrightarrow{\beta}[\alpha,0,\alpha]\xrightarrow{\alpha}[0,0,0],$$

which leads to $l=l'=1$. Hence, $\tilde{H}$ is an N-group.

II. Presentation of $\tilde{T}$. Let T be the group generated by t_α for all $\alpha\in\Pi$ with the defining relations

(T) $\quad t_\alpha t_\beta t_\alpha^{-1}=t_\beta^c \qquad$ for all $\alpha, \beta\in\Pi$ with $c=(-1)^{\beta\alpha^*}$.

Then, t_α^2 $(\alpha\in\Pi)$ is central. For $\alpha, \beta\in\Pi$, using the relation (T), we see

$t_\alpha^4=1$ and $t_\alpha^2=t_\beta^2$	if	$\alpha\beta^*$ and $\beta\alpha^*$ are odd;
$t_\alpha^2=1$	if	$\alpha\beta^*$ is odd and $\beta\alpha^*$ is even;
$[t_\alpha, t_\beta]=1$	if	$\alpha\beta^*$ or $\beta\alpha^*$ is even.

We here prove that $\tilde{T}\simeq T$.

Let θ_γ be an automorphism of T defined by

$$\theta_\gamma\colon t_\alpha\mapsto t_\alpha t_\gamma^d$$

for all $\alpha\in\Pi$ with $d=0$ (resp. -1) if $\gamma\alpha^*$ is even (resp. odd). Indeed, one can easily check that θ_γ preserves the relation (T). For example, if $\beta\alpha^*$ is even, and if $\gamma\alpha^*$ and $\gamma\beta^*$ are odd, then

$$t_\alpha t_\beta t_\alpha^{-1}=t_\beta, \qquad t_\alpha t_\gamma t_\alpha^{-1}=t_\gamma^{-1}, \qquad t_\beta t_\gamma t_\beta^{-1}=t_\gamma^{-1}$$

and

$$\theta_\gamma(t_\alpha t_\beta t_\alpha^{-1})=t_\alpha t_\gamma^{-1}t_\beta t_\gamma^{-1}t_\gamma h_\alpha^{-1}=t_\alpha t_\gamma^{-1}t_\alpha^{-1}t_\alpha t_\beta t_\alpha^{-1}=t_\gamma t_\beta=t_\beta t_\gamma^{-1}=\theta_\gamma(t_\beta)\,.$$

Note that $\theta_\alpha^2(t)=t_\alpha t t_\alpha^{-1}$ for all $\alpha\in\Pi$ and $t\in T$. For convenience, put $c(m)=(-1)^m$ and $d(m)=(c(m)-1)/2$ for all $m\in\boldsymbol{Z}$, where $c(m)d(m)=-d(m)$.

(i) We now show that the θ_α $(\alpha\in\Pi)$ satisfy the relations corresponding to (W1) and (W2), where $\tilde{w}_\alpha$ is replaced by θ_α. (We do not need this part to show $\tilde{T}\simeq T$.)

(W1):

$$\theta_\alpha^2\theta_\gamma\theta_\alpha^{-2}(t)=t_\alpha t_\gamma^{-d(\gamma\alpha^*)}t_\alpha^{-1}\theta_\gamma(t)t_\alpha t_\gamma^{d(\gamma\alpha^*)}t_\alpha^{-1}=t_\gamma^{d(\gamma\alpha^*)}\theta_\gamma(t)t_\gamma^{-d(\gamma\alpha^*)}=\theta_\gamma^{2d(\gamma\alpha^*)}\theta_\gamma(t)=\theta_\gamma^{c(\gamma\alpha^*)}(t)\,.$$

(W2): Put $[n_1,n_2]=t_\gamma t_\alpha^{n_1}t_\beta^{n_2}$. Then we show

$$\underbrace{(\theta_\alpha\theta_\beta\cdots)}_{q}\cdot[0,0]=\underbrace{(\theta_\beta\theta_\alpha\cdots)}_{q}\cdot[0,0]\,.$$

Put

$$\mathscr{L}=\underbrace{\theta_\alpha\theta_\beta\cdots}_{q}\quad\text{and}\quad\mathscr{R}=\underbrace{\theta_\beta\theta_\alpha\cdots}_{q}\,.$$

If $\theta_\alpha\cdot[n_1,n_2]=[n'_1,n'_2]$, then we write symbolically:

$$[n_1,n_2]\xrightarrow{\alpha}[n'_1,n'_2]\,.$$

Put $a=\alpha\gamma^*$ and $b=\beta\gamma^*$.

(1): Suppose $\alpha\beta^*=\beta\alpha^*=0$. Then,

$$\mathscr{L}\cdot[0,0]=[d(\alpha\gamma^*),\,d(\beta\gamma^*)]=\mathscr{R}\cdot[0,0]\,.$$

(2): Suppose $\alpha\beta^*=\beta\alpha^*=-1$. Then

$$[n_1,n_2]\xrightarrow{\alpha}[d(\alpha\gamma^*)+n_1-n_2,\,d(n_2)]$$

and

$$[n_1,n_2]\xrightarrow{\beta}[n_1,\,d(\beta\gamma^*)c(n_1)+d(n_1)+n_2]\,.$$

If $(a,b)\equiv(0,0)$ mod 2, then

$$\begin{cases}[0,0]\xrightarrow{\alpha}[0,0]\xrightarrow{\beta}[0,0]\xrightarrow{\alpha}[0,0]\,;\\ [0,0]\xrightarrow{\beta}[0,0]\xrightarrow{\alpha}[0,0]\xrightarrow{\beta}[0,0]\,.\end{cases}$$

If $(a,b)\equiv(1,0)$ mod 2, then

$$\begin{cases}[0,0]\xrightarrow{\alpha}[-1,0]\xrightarrow{\beta}[-1,1]\xrightarrow{\alpha}[-3,-1]=[1,-1]\,;\\ [0,0]\xrightarrow{\beta}[0,0]\xrightarrow{\alpha}[1,0]\xrightarrow{\beta}[1,-1]\,.\end{cases}$$

If $(a, b) \equiv (0, 1) \bmod 2$, then

$$\begin{cases} [0,0] \xrightarrow{\alpha} [0,0] \xrightarrow{\beta} [0,-1] \xrightarrow{\alpha} [1,-1]\,; \\ [0,0] \xrightarrow{\beta} [0,-1] \xrightarrow{\alpha} [1,-1] \xrightarrow{\beta} [1,-1]\,. \end{cases}$$

If $(a, b) \equiv (1, 1) \bmod 2$, then

$$\begin{cases} [0,0] \xrightarrow{\alpha} [-1,0] \xrightarrow{\beta} [-1,0] \xrightarrow{\alpha} [-2,0]\,; \\ [0,0] \xrightarrow{\beta} [0,-1] \xrightarrow{\alpha} [0,-1] \xrightarrow{\beta} [0,-2] = [-2,0]\,. \end{cases}$$

(3): Suppose $\alpha\beta^* = -2$ and $\beta\alpha^* = -1$. Then

$$[n_1, n_2] \xrightarrow{\alpha} [d(\alpha\gamma^*) + n_1, n_2]$$

and

$$[n_1, n_2] \xrightarrow{\beta} [n_1, d(\beta\gamma^*) + d(n_1) + n_2]\,.$$

If $(a, b) \equiv (0, 0) \bmod 2$, then

$$\begin{cases} [0,0] \xrightarrow{\alpha} [0,0] \xrightarrow{\beta} [0,0] \xrightarrow{\alpha} [0,0] \xrightarrow{\beta} [0,0]\,; \\ [0,0] \xrightarrow{\beta} [0,0] \xrightarrow{\alpha} [0,0] \xrightarrow{\beta} [0,0] \xrightarrow{\alpha} [0,0]\,. \end{cases}$$

If $(a, b) \equiv (1, 0) \bmod 2$, then

$$\begin{cases} [0,0] \xrightarrow{\alpha} [-1,0] \xrightarrow{\beta} [-1,-1] \xrightarrow{\alpha} [-2,-1] \xrightarrow{\beta} [-2,-1]\,; \\ [0,0] \xrightarrow{\beta} [0,0] \xrightarrow{\alpha} [-1,0] \xrightarrow{\beta} [-1,-1] \xrightarrow{\alpha} [-2,-1]\,. \end{cases}$$

If $(a, b) \equiv (0, 1) \bmod 2$, then

$$\begin{cases} [0,0] \xrightarrow{\alpha} [0,0] \xrightarrow{\beta} [0,-1] \xrightarrow{\alpha} [0,-1] \xrightarrow{\beta} [0,-2]\,; \\ [0,0] \xrightarrow{\beta} [0,-1] \xrightarrow{\alpha} [0,-1] \xrightarrow{\beta} [0,-2] \xrightarrow{\alpha} [0,-2]\,. \end{cases}$$

If $(a, b) \equiv (1, 1) \bmod 2$, then

$$\begin{cases} [0,0] \xrightarrow{\alpha} [-1,0] \xrightarrow{\beta} [-1,-2] \xrightarrow{\alpha} [-2,-2] \xrightarrow{\beta} [-2,-3]\,; \\ [0,0] \xrightarrow{\beta} [0,-1] \xrightarrow{\alpha} [-1,-1] \xrightarrow{\beta} [-1,-3] \xrightarrow{\alpha} [-2,-3]\,. \end{cases}$$

(4): Suppose $\alpha\beta^* = -3$ and $\beta\alpha^* = -1$. Then the situation is very close to the case (2). The number $q = 6$ is the only difference. In particular, the calculation in (2) implies $\theta_\alpha\theta_\beta\theta_\alpha = \theta_\beta\theta_\alpha\theta_\beta$. Hence, $\mathscr{L} = \mathscr{R}$.

In any case, we obtain $\mathscr{L} = \mathscr{R}$. Therefore, the θ_α $(\alpha \in \Pi)$ satisfy the relation (W2).

Let W be the Weyl group of $\mathfrak{g}$ generated by simple reflections σ_α ($\alpha \in \Pi$), and l the length function on W (cf. [1]). Put $\Omega = T \times W$. Let λ_α be a transformation of Ω defined by

$$\lambda_\alpha \cdot (t, \sigma) = \begin{cases} (\theta_\alpha(t), \sigma_\alpha \sigma) & \text{if} \quad l(\sigma_\alpha \sigma) > l(\sigma)\,; \\ (\theta_\alpha(t t_\alpha), \sigma_\alpha \sigma) & \text{if} \quad l(\sigma_\alpha \sigma) < l(\sigma)\,, \end{cases}$$

and Λ the transformation group of Ω generated by λ_α for all $\alpha \in \Pi$. Then $\lambda_\alpha^2 \cdot (t, \sigma) = (t_\alpha t, \sigma)$. Hence the subgroup Λ_0 of Λ generated by λ_α^2 for all $\alpha \in \Pi$ is isomorphic to T.

(ii) We show that the λ_α ($\alpha \in \Pi$) satisfy the relations corresponding to (W1) and (W2), where $\tilde{w}_\alpha$ is replaced by λ_α (cf. [4]).

(W1): The relation

$$\lambda_\alpha^2 \lambda_\gamma \lambda_\alpha^{-2} = \lambda_\gamma^{c(\gamma\alpha^*)}$$

follows from a simple computation:

$$\theta_\gamma(t t_\gamma) = \theta_\gamma^{-1} \theta_\gamma^2(t t_\gamma) = \theta_\gamma^{-1}(t_\gamma t) = t_\gamma \theta_\gamma^{-1}(t)\,.$$

Let

$$\alpha^+(t) = \theta_\alpha(t) \quad \text{and} \quad \alpha^-(t) = \theta_\alpha(t t_\alpha)$$

for all $t \in T$. Symbolically we write

$$t \xrightarrow{\alpha^\pm} t'$$

if $\alpha^\pm(t) = t'$.

(W2): We should show

$$\underbrace{\lambda_\alpha \lambda_\beta \cdots}_{q} (t, \sigma) = \underbrace{\lambda_\beta \lambda_\alpha \cdots}_{q} (t, \sigma)$$

for all $(t, \sigma) \in \Omega$. Now we may assume that W is just the subgroup $W_{\alpha\beta}$ generated by σ_α and σ_β by the theory of general Coxeter groups (cf. [1; Chap. 4, §1, Ex. 3]). Since we have $\lambda_\gamma^2 \cdot (1, \sigma) = (t_\gamma, \sigma)$ together with the relation corresponding to (W1), we may also assume $t = 1$. Note that

$$\begin{cases} \underbrace{\lambda_\alpha^{-1} \lambda_\beta \cdots}_{q} = \lambda \underbrace{\lambda_\alpha \lambda_\beta \cdots}_{q} \\ \underbrace{\lambda_\beta \lambda_\alpha^{-1} \cdots}_{q} = \lambda \underbrace{\lambda_\beta \lambda_\alpha \cdots}_{q}\,, \end{cases}$$

where

$$\lambda = \lambda_\alpha^{n_1} \lambda_\beta^{n_2} \quad \text{with} \quad (n_1, n_2) = (-2, 0), (-2, -2), (-4, 2), (0, 0)$$

for

$$(\alpha\beta^*, \beta\alpha^*)=(0, 0), (-1, -1), (-2, -1), (-3, -1),$$

respectively,

$$\begin{cases} \underbrace{\lambda_\alpha\lambda_\beta^{-1}\cdots}_{q}=\lambda'\underbrace{\lambda_\alpha\lambda_\beta\cdots}_{q} \\ \underbrace{\lambda_\beta^{-1}\lambda_\alpha\cdots}_{q}=\lambda'\underbrace{\lambda_\beta\lambda_\alpha\cdots}_{q}, \end{cases}$$

where

$$\lambda'=\lambda_\alpha^{n_1'}\lambda_\beta^{n_2'} \qquad \text{with} \quad (n_1', n_2')=(0, -2), (-2, 2), (0, 0), (0, 0)$$

for

$$(\alpha\beta^*, \beta\alpha^*)=(0, 0), (-1, -1), (-2, -1), (-3, -1),$$

respectively, and

$$\begin{cases} \underbrace{\lambda_\alpha^{-1}\lambda_\beta^{-1}\cdots}_{q}=\lambda''\underbrace{\lambda_\alpha\lambda_\beta\cdots}_{q} \\ \underbrace{\lambda_\beta^{-1}\lambda_\alpha^{-1}\cdots}_{q}=\lambda''\underbrace{\lambda_\beta\lambda_\alpha\cdots}_{q}, \end{cases}$$

where

$$\lambda''=\lambda_\alpha^{n_1''}\lambda_\beta^{n_2''} \qquad \text{with} \quad (n_1'', n_2'')=(-2, -2), (4, 0), (-4, 2), (0, 0)$$

for

$$(\alpha\beta^*, \beta\alpha^*)=(0, 0), (-1, -1), (-2, -1), (-3, -1),$$

respectively. Put $[n_1, n_2]=t_\alpha^{n_1}t_\beta^{n_2}$.

(1) Suppose $\alpha\beta^*=\beta\alpha^*=0$. Then

$$\alpha^+\beta^+([0, 0])=[0, 0]=\beta^+\alpha^+([0, 0]);$$
$$\alpha^-\beta^+([0, 0])=[1, 0]=\beta^+\alpha^-([0, 0]);$$
$$\alpha^+\beta^-([0, 0])=[0, 1]=\beta^-\alpha^+([0, 0]);$$
$$\alpha^-\beta^-([0, 0])=[1, 1]=\beta^-\alpha^-([0, 0]),$$

which implies $\lambda_\alpha\lambda_\beta=\lambda_\beta\lambda_\alpha$.

(2) Suppose $\alpha\beta^*=\beta\alpha^*=-1$. Then it is enough to show the following:

$$(X_m) \qquad \underbrace{(\alpha^-\beta^-\cdots)}_{m}\underbrace{(\cdots\beta^+\alpha^+)}_{3-m}\cdot[0, 0]=\underbrace{(\beta^+\alpha^+\cdots)}_{3-m}\underbrace{(\cdots\alpha^-\beta^-)}_{m}\cdot[0, 0];$$

$$(\mathrm{Y}_m)\qquad \underbrace{(\alpha^+\beta^+\cdots)}_{3-m}\underbrace{(\cdots\beta^-\alpha^-)}_{m}\cdot[0,0]=\underbrace{(\beta^-\alpha^-\cdots)}_{m}\underbrace{(\cdots\alpha^+\beta^+)}_{3-m}\cdot[0,0]$$

for all $0\le m\le 3$.

Note that

$$[n_1,n_2]\xrightarrow{\alpha^+}[n_1-n_2,\, d(n_2)]\,;$$

$$[n_1,n_2]\xrightarrow{\alpha^-}[n_1+n_2+1,\, d(n_2)]\,;$$

$$[n_1,n_2]\xrightarrow{\beta^+}[n_1,\, d(n_1)+n_2]\,;$$

$$[n_1,n_2]\xrightarrow{\beta^-}[n_1,\, d(n_1)+n_2+1]\,.$$

Then

$$(\mathrm{X}_0)=(\mathrm{Y}_0)\quad \begin{cases}[0,0]\xrightarrow{\alpha^+}[0,0]\xrightarrow{\beta^+}[0,0]\xrightarrow{\alpha^+}[0,0]\\ [0,0]\xrightarrow{\beta^+}[0,0]\xrightarrow{\alpha^+}[0,0]\xrightarrow{\beta^+}[0,0]\,,\end{cases}$$

$$(\mathrm{X}_1)\quad \begin{cases}[0,0]\xrightarrow{\alpha^+}[0,0]\xrightarrow{\beta^+}[0,0]\xrightarrow{\alpha^-}[1,0]\\ [0,0]\xrightarrow{\beta^-}[0,1]\xrightarrow{\alpha^+}[-1,-1]\xrightarrow{\beta^+}[-1,-2]=[1,0]\,,\end{cases}$$

$$(\mathrm{X}_2)\quad \begin{cases}[0,0]\xrightarrow{\alpha^+}[0,0]\xrightarrow{\beta^-}[0,1]\xrightarrow{\alpha^-}[2,-1]\\ [0,0]\xrightarrow{\beta^-}[0,1]\xrightarrow{\alpha^-}[2,-1]\xrightarrow{\beta^+}[2,-1]\,,\end{cases}$$

$$(\mathrm{Y}_1)\quad \begin{cases}[0,0]\xrightarrow{\alpha^-}[1,0]\xrightarrow{\beta^+}[1,-1]\xrightarrow{\alpha^+}[2,-1]=[0,1]\\ [0,0]\xrightarrow{\beta^+}[0,0]\xrightarrow{\alpha^+}[0,0]\xrightarrow{\beta^-}[0,1]\,,\end{cases}$$

$$(\mathrm{Y}_2)\quad \begin{cases}[0,0]\xrightarrow{\alpha^-}[1,0]\xrightarrow{\beta^-}[1,0]\xrightarrow{\alpha^+}[1,0]\\ [0,0]\xrightarrow{\beta^+}[0,0]\xrightarrow{\alpha^-}[1,0]\xrightarrow{\beta^-}[1,0]\,,\end{cases}$$

$$(\mathrm{Y}_3)=(\mathrm{X}_3)\quad \begin{cases}[0,0]\xrightarrow{\alpha^-}[1,0]\xrightarrow{\beta^-}[1,0]\xrightarrow{\alpha^-}[2,0]\\ [0,0]\xrightarrow{\beta^-}[0,1]\xrightarrow{\alpha^-}[2,-1]\xrightarrow{\beta^-}[2,0]\,.\end{cases}$$

Hence, $\lambda_\alpha\lambda_\beta\lambda_\alpha=\lambda_\beta\lambda_\alpha\lambda_\beta$.

(3) Suppose $\alpha\beta^*=-2$ and $\beta\alpha^*=-1$. Then it is enough to show the following:

$$(\mathrm{X}_m)\qquad \underbrace{(\alpha^-\beta^-\cdots)}_{m}\underbrace{(\cdots\alpha^+\beta^+)}_{4-m}\cdot[0,0]=\underbrace{(\beta^+\alpha^+\cdots)}_{4-m}\underbrace{(\cdots\beta^-\alpha^-)}_{m}\cdot[0,0]\,;$$

$$(\mathrm{Y}_m)\qquad \underbrace{(\alpha^+\beta^+\cdots)}_{4-m}\underbrace{(\cdots\alpha^-\beta^-)}_{m}\cdot[0,0]=\underbrace{(\beta^-\alpha^-\cdots)}_{m}\underbrace{(\cdots\beta^+\alpha^+)}_{4-m}\cdot[0,0]$$

for all $0 \leq m \leq 4$.

Note that

$$[n_1, n_2] \xrightarrow{\alpha^+} [n_1, n_2]\,;$$

$$[n_1, n_2] \xrightarrow{\alpha^-} [n_1+1, n_2]\,;$$

$$[n_1, n_2] \xrightarrow{\beta^+} [n_1, d(n_1)+n_2]\,;$$

$$[n_1, n_2] \xrightarrow{\beta^-} [n_1, d(n_1)+n_2+1]\,.$$

Then

$$(\mathrm{X}_0)=(\mathrm{Y}_0) \quad \begin{cases} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \\ [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0]\,, \end{cases}$$

$$(\mathrm{X}_1) \quad \begin{cases} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^-} [1,0] \\ [0,0] \xrightarrow{\alpha^-} [1,0] \xrightarrow{\beta^+} [1,1] \xrightarrow{\alpha^+} [1,1] \xrightarrow{\beta^+} [1,0]\,, \end{cases}$$

$$(\mathrm{X}_2) \quad \begin{cases} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^-} [0,1] \xrightarrow{\alpha^-} [1,1] \\ [0,0] \xrightarrow{\alpha^-} [1,0] \xrightarrow{\beta^-} [1,0] \xrightarrow{\alpha^+} [1,0] \xrightarrow{\beta^+} [1,-1]=[1,1]\,, \end{cases}$$

$$(\mathrm{X}_3) \quad \begin{cases} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^-} [1,0] \xrightarrow{\beta^-} [1,0] \xrightarrow{\alpha^-} [2,0] \\ [0,0] \xrightarrow{\alpha^-} [1,0] \xrightarrow{\beta^-} [1,0] \xrightarrow{\alpha^-} [2,0] \xrightarrow{\beta^+} [2,0]\,, \end{cases}$$

$$(\mathrm{Y}_1) \quad \begin{cases} [0,0] \xrightarrow{\beta^-} [0,1] \xrightarrow{\alpha^+} [0,1] \xrightarrow{\beta^+} [0,1] \xrightarrow{\alpha^+} [0,1] \\ [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^-} [0,1]\,, \end{cases}$$

$$(\mathrm{Y}_2) \quad \begin{cases} [0,0] \xrightarrow{\beta^-} [0,1] \xrightarrow{\alpha^-} [1,1] \xrightarrow{\beta^+} [1,0] \xrightarrow{\alpha^+} [1,0] \\ [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^-} [1,0] \xrightarrow{\beta^-} [1,0]\,, \end{cases}$$

$$(\mathrm{Y}_3) \quad \begin{cases} [0,0] \xrightarrow{\beta^-} [0,1] \xrightarrow{\alpha^-} [1,1] \xrightarrow{\beta^-} [1,1] \xrightarrow{\alpha^+} [1,1] \\ [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^-} [0,1] \xrightarrow{\alpha^-} [1,1] \xrightarrow{\beta^-} [1,1]\,, \end{cases}$$

$$(\mathrm{Y}_4)=(\mathrm{X}_4) \quad \begin{cases} [0,0] \xrightarrow{\beta^-} [0,1] \xrightarrow{\alpha^-} [1,1] \xrightarrow{\beta^-} [1,1] \xrightarrow{\alpha^-} [2,1] \\ [0,0] \xrightarrow{\alpha^-} [0,0] \xrightarrow{\beta^-} [1,0] \xrightarrow{\alpha^-} [2,0] \xrightarrow{\beta^-} [2,1]\,. \end{cases}$$

Hence, $(\lambda_\alpha \lambda_\beta)^2 = (\lambda_\beta \lambda_\alpha)^2$.

(4) Suppose $\alpha\beta^* = -3$ and $\beta\alpha^* = -1$. Then it is enough to show the following:

$$(\mathrm{X}_m) \qquad (\underbrace{\alpha^-\beta^-\cdots}_{m})(\underbrace{\cdots\alpha^+\beta^+}_{6-m})\cdot[0,0] = (\underbrace{\beta^+\alpha^+\cdots}_{6-m})(\underbrace{\cdots\beta^-\alpha^-}_{m})\cdot[0,0];$$

$$(\mathrm{Y}_m) \qquad (\underbrace{\alpha^+\beta^+\cdots}_{6-m})(\underbrace{\cdots\alpha^-\beta^-}_{m})\cdot[0,0] = (\underbrace{\beta^-\alpha^-\cdots}_{m})(\underbrace{\cdots\beta^+\alpha^+}_{6-m})\cdot[0,0]$$

for all $0 \le m \le 6$. Note that

$$[n_1, n_2] \xrightarrow{\alpha^+} [n_1 - n_2, d(n_2)];$$

$$[n_1, n_2] \xrightarrow{\alpha^-} [n_1 + n_2 + 1, d(n_2)];$$

$$[n_1, n_2] \xrightarrow{\beta^+} [n_1, d(n_1) + n_2];$$

$$[n_1, n_2] \xrightarrow{\beta^-} [n_1, d(n_1) + n_2 + 1].$$

Then

$$(\mathrm{X}_0) = (\mathrm{Y}_0) \quad \left\{\begin{array}{l} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \\ [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0], \end{array}\right.$$

$$(\mathrm{X}_1) \quad \left\{\begin{array}{l} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^-} [1,0] \\ [0,0] \xrightarrow{\alpha^-} [1,0] \xrightarrow{\beta^+} [1,-1] \xrightarrow{\alpha^+} [2,-1] \xrightarrow{\beta^+} [2,-1] \xrightarrow{\alpha^+} [3,-1] \xrightarrow{\beta^+} [3,-2] = [1,0], \end{array}\right.$$

$$(\mathrm{X}_2) \quad \left\{\begin{array}{l} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^-} [0,1] \xrightarrow{\alpha^-} [2,-1] \\ [0,0] \xrightarrow{\alpha^-} [1,0] \xrightarrow{\beta^-} [1,0] \xrightarrow{\alpha^+} [1,0] \xrightarrow{\beta^+} [1,-1] \xrightarrow{\alpha^+} [2,-1] \xrightarrow{\beta^+} [2,-1], \end{array}\right.$$

$$(\mathrm{X}_3) \quad \left\{\begin{array}{l} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^-} [1,0] \xrightarrow{\beta^-} [1,0] \xrightarrow{\alpha^-} [2,0] \\ [0,0] \xrightarrow{\alpha^-} [1,0] \xrightarrow{\beta^-} [1,0] \xrightarrow{\alpha^-} [2,0] \xrightarrow{\beta^+} [2,0] \xrightarrow{\alpha^+} [2,0] \xrightarrow{\beta^+} [2,0], \end{array}\right.$$

$$(\mathrm{X}_4) \quad \left\{\begin{array}{l} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^-} [0,1] \xrightarrow{\alpha^-} [2,-1] \xrightarrow{\beta^-} [2,0] \xrightarrow{\alpha^-} [3,0] \\ [0,0] \xrightarrow{\alpha^-} [1,0] \xrightarrow{\beta^-} [1,0] \xrightarrow{\alpha^-} [2,0] \xrightarrow{\beta^-} [2,1] \xrightarrow{\alpha^+} [1,-1] \xrightarrow{\beta^+} [1,-2] = [3,0], \end{array}\right.$$

$$(\mathrm{X}_5) \quad \left\{\begin{array}{l} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^-} [1,0] \xrightarrow{\beta^-} [1,0] \xrightarrow{\alpha^-} [2,0] \xrightarrow{\beta^-} [2,1] \xrightarrow{\alpha^-} [4,-1] \\ [0,0] \xrightarrow{\alpha^-} [1,0] \xrightarrow{\beta^-} [1,0] \xrightarrow{\alpha^-} [2,0] \xrightarrow{\beta^-} [2,1] \xrightarrow{\alpha^-} [4,-1] \xrightarrow{\beta^+} [4,-1], \end{array}\right.$$

$$(\mathrm{Y}_1) \quad \left\{\begin{array}{l} [0,0] \xrightarrow{\beta^-} [0,1] \xrightarrow{\alpha^+} [-1,-1] \xrightarrow{\beta^+} [-1,-2] \xrightarrow{\alpha^+} [1,0] \xrightarrow{\beta^+} [1,-1] \xrightarrow{\alpha^+} [2,-1] \\ [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^-} [0,1] = [2,-1], \end{array}\right.$$

$$(\mathrm{Y}_2) \quad \left\{\begin{array}{l} [0,0] \xrightarrow{\beta^-} [0,1] \xrightarrow{\alpha^-} [2,-1] \xrightarrow{\beta^+} [2,-1] \xrightarrow{\alpha^+} [3,-1] \xrightarrow{\beta^+} [3,-2] \xrightarrow{\alpha^+} [5,0] = [1,0] \\ [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^+} [0,0] \xrightarrow{\beta^+} [0,0] \xrightarrow{\alpha^-} [1,0] \xrightarrow{\beta^-} [1,0], \end{array}\right.$$

$$(\mathrm{Y}_3)\qquad \begin{cases} [0,0]\xrightarrow{\beta^-}[0,1]\xrightarrow{\alpha^-}[2,-1]\xrightarrow{\beta^-}[2,0]\xrightarrow{\alpha^+}[2,0]\xrightarrow{\beta^+}[2,0]\xrightarrow{\alpha^+}[2,0] \\ [0,0]\xrightarrow{\alpha^+}[0,0]\xrightarrow{\beta^+}[0,0]\xrightarrow{\alpha^+}[0,0]\xrightarrow{\beta^-}[0,1]\xrightarrow{\alpha^-}[2,-1]\xrightarrow{\beta^-}[2,0]\,, \end{cases}$$

$$(\mathrm{Y}_4)\qquad \begin{cases} [0,0]\xrightarrow{\beta^-}[0,1]\xrightarrow{\alpha^-}[2,-1]\xrightarrow{\beta^-}[2,0]\xrightarrow{\alpha^-}[3,0]\xrightarrow{\beta^+}[3,-1]\xrightarrow{\alpha^+}[4,-1]=[2,1] \\ [0,0]\xrightarrow{\alpha^+}[0,0]\xrightarrow{\beta^+}[0,0]\xrightarrow{\alpha^-}[1,0]\xrightarrow{\beta^-}[1,0]\xrightarrow{\alpha^-}[2,0]\xrightarrow{\beta^-}[2,1]\,, \end{cases}$$

$$(\mathrm{Y}_5)\qquad \begin{cases} [0,0]\xrightarrow{\beta^-}[0,1]\xrightarrow{\alpha^-}[2,-1]\xrightarrow{\beta^-}[2,0]\xrightarrow{\alpha^-}[3,0]\xrightarrow{\beta^-}[3,0]\xrightarrow{\alpha^+}[3,0] \\ [0,0]\xrightarrow{\alpha^+}[0,0]\xrightarrow{\beta^-}[0,1]\xrightarrow{\alpha^-}[2,-1]\xrightarrow{\beta^-}[2,0]\xrightarrow{\alpha^-}[3,0]\xrightarrow{\beta^-}[3,0]\,, \end{cases}$$

$$(\mathrm{Y}_6)=(\mathrm{X}_6)\qquad \begin{cases} [0,0]\xrightarrow{\beta^-}[0,1]\xrightarrow{\alpha^-}[2,-1]\xrightarrow{\beta^-}[2,0]\xrightarrow{\alpha^-}[3,0]\xrightarrow{\beta^-}[3,0]\xrightarrow{\alpha^-}[4,0] \\ [0,0]\xrightarrow{\alpha^-}[1,0]\xrightarrow{\beta^-}[1,0]\xrightarrow{\alpha^-}[2,0]\xrightarrow{\beta^-}[2,1]\xrightarrow{\alpha^-}[4,-1]\xrightarrow{\beta^-}[4,0]\,. \end{cases}$$

Hence, $(\lambda_\alpha\lambda_\beta)^3=(\lambda_\beta\lambda_\alpha)^3$.

Hence there is a canonical homomorphism ψ of $\tilde{W}$ onto Λ such that $\psi(\tilde{w}_\alpha)=\lambda_\alpha$ for all $\alpha\in\Pi$. In particular, ψ gives a homomorphism of $\tilde{T}$ onto Λ_0, and $\tilde{T}\simeq T$.

III. Action of $\tilde{P}_\alpha$ on Γ. As a set of generators of $\tilde{P}_\alpha$, we take $\Upsilon_\alpha=\{\tilde{h}, u, \tilde{w}_\alpha(-1)\,|\,\tilde{h}\in\tilde{H}, u\in U\}$. For each element of Υ_α, we define the action on Γ as follows:

$$\tilde{h}\cdot(g,\tilde{n})=(\phi(\tilde{h})g,\tilde{h}\tilde{n})\,,$$

$$u\cdot(g,\tilde{n})=(ug,\tilde{n})\,,$$

$$\tilde{w}_\alpha(-1)\cdot(g,\tilde{n})=\begin{cases}(w_\alpha(-1)g,\tilde{w}_\alpha(-1)\tilde{n}) & \text{if}\quad v(w_\alpha(-1)g)=w_\alpha(-1)v(g)\,;\\ (w_\alpha(-1)g,\tilde{h}_\alpha(t^{-1})\tilde{n}) & \text{if}\quad v(w_\alpha(-1)g)=h_\alpha(t^{-1})v(g)\,.\end{cases}$$

These give an action of $\tilde{P}_\alpha$ on Γ, which can be confirmed by the fact that $\tilde{P}_\alpha$ is the group generated by Υ_α with the following defining relations:

(P1) $\tilde{H}$ is a subgroup;
(P2) U is a subgroup;
(P3) $\tilde{w}_\alpha(-1)^2=\tilde{h}_\alpha(-1)$;
(P4) $\tilde{h}_\beta(t)x_\gamma(s)\tilde{h}_\beta(t)^{-1}=x_\gamma(t^{\gamma\beta^*}s)$;
(P5) $\tilde{w}_\alpha(-1)^{-1}x_\alpha(t)\tilde{w}_\alpha(-1)=x_\alpha(-t^{-1})\tilde{h}_\alpha(t^{-1})\tilde{w}_\alpha(-1)^{-1}x_\alpha(-t^{-1})$;
(P6) $\tilde{w}_\alpha(-1)^{-1}x_{\gamma'}(s)\tilde{w}_\alpha(-1)=x_{\gamma''}(\eta_{\alpha\gamma'}s)$;
(P7) $\tilde{w}_\alpha(-1)^{-1}\tilde{h}_\beta(t)\tilde{w}_\alpha(-1)=\tilde{h}_\beta(t)\tilde{h}_\alpha(t^{-\alpha\beta^*})$

for all $\beta\in\Pi$, $\gamma\in\Delta_+^{\mathrm{re}}$, $\gamma'\in\Delta_+^{\mathrm{re}}\setminus\{\alpha\}$, $s\in F$, and $t\in F^\times$, where $\gamma''=\gamma'-(\gamma'\alpha^*)\alpha$ (cf. [4; p. 40ff]). In particular, L acts on Γ faithfully. Hence, $\tilde{P}_\alpha$ acts on Γ faithfully, since the kernel of this action is contained in L.

IV. Lifting of the relations (A), (B) and (B′). We proceed in the same way as in Steinberg [14]. Here we should also consider the case where $(\alpha\beta^*)(\beta\alpha^*)\geq 4$ with $\alpha,\beta\in\Pi$. If F is a finite field, then $K_2(A,F)=L=1$, and $\mathrm{St}(A,F)=\tilde{G}=G$. Hence, in this case, we need not prove anything. From now on, we assume that F is an infinite field.

Let E be a central extension of G with a homomorphism

$$\psi: E \to G .$$

For each $x \in G$, let

$$C(x) = \psi^{-1}(x) = \{x' \in E \mid \psi(x') = x\} ,$$

and put $C = C(1)$. Now we choose and fix an element $a \in F^{\times}$ such that $c = a^2 - 1 \neq 0$. Then, for each $\alpha \in \Delta^{\mathrm{re}}$ and $s \in F$, we define $x_\alpha(s)'$ by

$$x_\alpha(s)' = [y', x']$$

with $x' \in C(x_\alpha(c^{-1}s))$ and $y' \in C(h_\alpha(a))$. This definition is independent of the choice of x' and y'. Put

$$w_\alpha(u)' = x_\alpha(u)' x_{-\alpha}(-u^{-1})' x_\alpha(u)' ,$$

and

$$h_\alpha(u)' = w_\alpha(u)' w_\alpha(1)'^{-1} .$$

Then we will show that the relations (A), (B) and (B′) can be lifted to E using these ′-symbols. First we see the following two results as in [14].

(1) If $h \in H$, $h' \in C(h)$, $\alpha \in \Delta^{\mathrm{re}}$, $s \in F$ and $d \in F^{\times}$ with $h x_\alpha(s) h^{-1} = x_\alpha(ds)$, then

$$h' x_\alpha(s)' h'^{-1} = x_\alpha(ds)' .$$

(2) If $w \in N$, $w' \in C(w)$, $\alpha, \gamma \in \Delta^{\mathrm{re}}$, $s \in F$ and $d \in F^{\times}$ with $w x_\alpha(s) w^{-1} = x_\gamma(ds)$, then

$$w' x_\alpha(s)' w'^{-1} = x_\gamma(ds)' .$$

In particular, the relation (B′) can be lifted to E.

For $r, s \in F$, and for $\alpha, \beta \in \Delta^{\mathrm{re}}$ with $(\boldsymbol{Z}_{>0}\alpha + \boldsymbol{Z}_{>0}\beta) \cap \Delta \subset \Delta^{\mathrm{re}}$, let $f_{\alpha\beta}(r, s)$ be the element of C defined by

(F) $$x_\alpha(r)' x_\beta(s)' x_\alpha(r)'^{-1} = f_{\alpha\beta}(r, s) \prod x_{i\alpha + j\beta}(N_{\alpha\beta ij} r^i s^j)' x_\beta(s)' .$$

Then we consider the following three conditions.

(Dk) $f_{\alpha\beta}(r_1 + r_2, s) = f_{\alpha\beta}(r_1, s) f_{\alpha\beta}(r_2, s)$ if $m \leq k$;

(Ek) $f_{\alpha\beta}(r, s_1 + s_2) = f_{\alpha\beta}(r, s_1) f_{\alpha\beta}(r, s_2)$ if $m \leq k$;

(Fk) $f_{\alpha\beta} \equiv 1$ if $m \leq k$,

where m is the cardinality of the set $(\boldsymbol{Z}_{>0}\alpha + \boldsymbol{Z}_{>0}\beta) \cap \Delta$ of real roots appeared in the product $\prod$ of the right hand side in (F). In fact, $0 \leq k \leq 4$ (cf. [10], [14]). By the definition of $f_{\alpha\beta}(r, s)$, we see that (D0) and (E0) hold and that (F$(k-1)$) implies (Dk) and (Ek). Hence, we would like to show that (Dk) and (Ek) imply (Fk).

(3) Dk and E$k \Rightarrow$ Fk:

Taking the conjugate, by $h_\gamma(v)'$ with $\gamma \in \Delta^{\mathrm{re}}$ and $v \in F^{\times}$, in (F), we obtain $f_{\alpha\beta}(r, s) = f_{\alpha\beta}(r v^{\alpha\gamma^*}, s v^{\beta\gamma^*})$. Symbolically, we say:

$$f_{\alpha\beta}(r,s) \xrightarrow{(\gamma,v)} f_{\alpha\beta}(rv^{\alpha\gamma^*}, sv^{\beta\gamma^*}) .$$

Then

$$f_{\alpha\beta}(r,s) \xrightarrow{(\alpha,v^2)} f_{\alpha\beta}(rv^4, sv^{2\beta\alpha^*}) \xrightarrow{(\beta,v^{-\beta\alpha^*})} f_{\alpha\beta}(rv^d, s)$$

and $f_{\alpha\beta}(r(1-v^d), s)=1$, where $d=4-(\alpha\beta^*)(\beta\alpha^*)$. If $(\alpha\beta^*)(\beta\alpha^*)\neq 4$, then, choosing a suitable element $v\in F^\times$ such that $1-v^d\neq 0$, we obtain $f_{\alpha\beta}\equiv 1$. Suppose $(\alpha\beta^*)(\beta\alpha^*)=4$. Then $(\alpha\beta^*, \beta\alpha^*)=(2,2)$, $(1,4)$, $(4,1)$ because of the assumption in (B).

When $(\alpha\beta^*, \beta\alpha^*)=(2,2)$, we obtain

$$f_{\alpha\beta}(r,s) \xrightarrow{(\alpha,v)} f_{\alpha\beta}(rv^2, sv^2)$$

and $f_{\alpha\beta}(r,s)=f_{\alpha\beta}(rv^2, sv^2)$ for all $r, s\in F$ and $v\in F^*$. Then, as in [14], we obtain $f_{\alpha\beta}\equiv 1$.

When $(\alpha\beta^*, \beta\alpha^*)=(1,4)$, we get

$$f_{\alpha\beta}(r,s) \xrightarrow{(\beta,v)} f_{\alpha\beta}(rv, sv^2)$$

and $f_{\alpha\beta}(r,s)=f_{\alpha\beta}(rv, sv^2)$. If char $F\neq 2$, then $f_{\alpha\beta}(r,s)=f_{\alpha\beta}(-r,s)$ and $f_{\alpha\beta}(2r,s)=1$. Hence, $f_{\alpha\beta}\equiv 1$. If char $F=2$, then, choosing $v\in F^\times$ such that $v-v^2\neq 0$ and $1-v+v^2\neq 0$, we obtain

$$\begin{aligned} f_{\alpha\beta}(r(v-v^2), s) &= f_{\alpha\beta}(r, s/(v-v^2)^2) = f_{\alpha\beta}(r, s/v^2(1-v^2)) \\ &= f_{\alpha\beta}(r, s/v^2) f_{\alpha\beta}(r, s/(1-v)^2) = f_{\alpha\beta}(rv, s) f_{\alpha\beta}(r(1-v), s) = f_{\alpha\beta}(r,s) . \end{aligned}$$

Hence, $f_{\alpha\beta}\equiv 1$.

When $(\alpha\beta^*, \beta\alpha^*)=(4,1)$, we can also obtain $f_{\alpha\beta}\equiv 1$ similarly. We have just established that (Dk) and (Ek) imply (Fk) for all $0\le k\le 4$. Hence, the relation (B) can be also lifted to E.

(4) It follows from (F0) that the relation (A) can be lifted to E (cf. [14]).

Therefore, there is a canonical homomorphism $\hat{\psi}$ of $\mathrm{St}(A,F)$ to E such that $\hat{\psi}(\hat{x}_\alpha(s))=x_\alpha(s)'$. Hence, $\mathrm{St}(A,F)$ is homologically simply connected.

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF TSUKUBA
TSUKUBA, IBARAKI, 305
Japan

FAKULTÄT FÜR MATHEMATIK
UNIVERSITÄT BIELEFELD
4800 BIELEFELD 1
BUNDESREPUBLIK DEUTSCHLAND