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VARIETIES WHOSE SURFACE SECTIONS ARE ELLIPTIC

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Introduction. Let X be a complex projective manifold of dimension n. Let L be a *very ample* line bundle on X. Let S be the intersection of $n-2$ general members of $|L|$. Assume, moreover, that S is an *elliptic surface of Kodaira dimension* $\kappa(S)=1$ and that (X, L) is not a scroll over a surface. By [20], we know that there exists a reduction (X', L') of (X, L) such that $K_{X'}+(n-1)L'$ is very ample. Moreover, $K_{X'}+(n-2)L'$ is semi-ample and any smooth surface S', which is the intersection of $n-2$ general members of $|L'|$ is a *minimal model*, with $\kappa(S')\geq 0$ (see [9], [13], [19]). Let p: $X'\to C$ be the morphism associated to $|N(K_{X'}+(n-2)L')|$ for $N\gg 0$. N is chosen so that $C=p(X')$ is normal and p has connected fibres. It follows that dim $C=1$. We restrict ourselves to varieties of dimension $n\geq 4$, since the case $n=3$ has been considered by the first author in [2]. Note that the general fibre of p is a del Pezzo manifold of degree d, where $3\leq d\leq 8$. We classify X' in the cases $d=3, 4, 7, 8$. Since we have only partial results for $d=5, 6$ those will not be included here.

The paper is organized as follows. In Section 0, we give some background material and state, without proof, some of the needed results. In Section 1, we prove the results used later in the paper. In Section 2 we classify the possible singular fibres of the morphism p_3: $X'^3\to C$ in the case $d=3, 4, 7, 8$. In Sections 3 through 6 we analyze the structure of X' for these values of d.

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0. Notation and background material. Throughout this paper we let X be an irreducible complex projective manifold of dimension n, and L a very ample line bundle over X.

(0.1) Let L be a line bundle over X. We say that L *is nef* if $c_1(L)\cdot[C]\geq 0$, for all effective curves C on X. We say that a nef line bundle L is *big* if $c_1(L)^n>0$. We say

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that L is *spanned* if Bs $|L|$, the base locus of $|L|$, is empty. We say that L is *semiample* if there exists an $m>0$ such that Bs $|mL|$, the base locus of $|mL|$, is empty.

(0.2) $N_{A/B}$ is the normal bundle of A in B, and $N_{A/B,C}$ is its restriction to C.

(0.3) Let (X, L) be a polarized manifold. A *reduction* of (X, L) is a polarized manifold (X', L') such that:

(a) there exists a morphism $\pi: X \to X'$ expressing X as X' with a finite set F in X' blown up.

(b) $L=\pi^*(L')-[\pi^{-1}(F)]$, or equivalently, $K_X+(n-1)L=\pi^*(K_{X'}+(n-1)L')$.

(0.4) Let L be a line bundle over X. Let $A_1, \cdots, A_{n-2}$ be general members of the linear system $|L|$ and let $X^i=\bigcap_{1\leq j\leq n-i}A_j$. Then $\dim X^i=i$. We have the following descending chain $X\supset X^{n-1}\supset \cdots \supset X^3\supset X^2$, and we will often denote X^2 by S and L_{X^i} by L_i, and when no ambiguity exists, only by L.

(0.5) The following theorems ([20, (2.1)] and [19, (4.5), (5.1)]) play a major role in this paper. The first one is stated for smooth varieties, but it also remains true for varieties with "mild singularities".

(0.5.1) THEOREM. *Let L be a very ample line bundle on an n-dimensional complex projective manifold X with $n\geq 3$. Assume that $K_X+(n-1)L$ is nef and big. Then there exists a unique minimal reduction (X', L') of (X, L) such that $K_{X'}+(n-1)L'$ is very ample.*

(0.5.2) THEOREM. *Let L be an ample and spanned line bundle on an n-dimensional complex projective manifold X with $n\geq 2$. Assume that $K_X+(n-1)L$ is nef and big. Then there exists a unique minimal reduction (X', L') of (X, L) such that $K_{X'}+(n-1)L'$ is ample. If $h^0(N(K_X+(n-2)L))\neq 0$ for some $N>0$, then $K_{X'}+(n-2)L'$ is semi-ample, and any smooth surface S', which is the intersection of $n-2$ general members of $|L'|$, is a minimal model of non-negative Kodaira dimension.*

For the convenience of the reader, we recall the following definitions (see [8, (1.0)]).

(0.6) A *rung* of a polarized manifold (X, L) is an irreducible reduced member of $|L|$. A rung D is said to be *regular* if the homomorphism $\Gamma(X, L)\to\Gamma(D, L_D)$ is a surjection. A *ladder* of (X, L) is a sequence of subvarieties $X=X^n\supset X^{n-1}\supset \cdots \supset X^3\supset X^2\supset X^1$ of X, with $\dim X^i=i$, such that each X^i is a rung of $(X^{i+1}, L_{X^{i+1}})$. The ladder is said to be *regular* if each rung is regular. Note that $d=L\cdot L\cdot \cdots \cdot L(n \text{ times})=L_2\cdot L_2$ where L_2 is the restriction of L to a 2-dimensional rung X^2.

(0.7) By F_r with $r\geq 0$ we denote the r-th Hirzebruch surface, i.e. the unique $\boldsymbol{P}^1$-bundle $\pi: F_r\to \boldsymbol{P}^1$ over $\boldsymbol{P}^1$ with a section E satisfying $E\cdot E=-r$. For $r\geq 1$ we let $\tilde{F}_r$ denote the normal surface obtained from F_r by contracting E and let $\pi_1: F_r\to\tilde{F}_r$ be the contraction map. Given a line bundle L on $\tilde{F}_r$, the pullback of L to F_r is of the form $a[E+rf]$ for some integer a, where f is a fibre of π.

1. General Results.

(1.0) Throughout this paper, unless otherwise specified, X will denote a complex projective manifold of dimension n. Let L be a very ample line bundle on X. Assume that the intersection of $n-2$ general members of $|L|$ is an elliptic surface S of Kodaira dimension $\kappa(S)=1$ and that (X, L) is not a scroll over a surface. Then (see [13], [19]) there is a reduction (X', L') of (X, L) such that $K_{X'}+(n-2)L'$ is semi-ample. Let $p: X' \to C$ be the morphism associated to the linear system $|N(K_{X'}+(n-2)L')|$ for $N \gg 0$. Choose N big enough so that $C=p(X')$ is normal and the fibres of p are connected. It is clear that for $X'^i \in |L'_{X'^{i+1}}|$ we have $p_{X'^i}: X'^i \to C$. From now on we denote $p_{X'^i}$ by p_i. Whenever it is clear from the context we will write p instead of p_i.

(1.1) LEMMA. *Under the above assumptions* $\dim C=1$.

PROOF. We note that $\kappa(S')=1$. The elliptic fibration map on S' extends to X' giving the map $p: X' \to C$. Hence by [19, (0.3.2)] it follows that $\dim C=1$. ∎

(1.2) LEMMA. *L' is locally very ample with respect to p.*

PFOOF. $K_{X'}+(n-1)L'$ is very ample for $n \geq 3$ (see [20]). Moreover we have $K_{X'}+(n-2)L'=p^*(M)$ for some ample line bundle M on C. Hence $K_{X'}+(n-1)L'=p^*(M)+L'$, or equivalently, L' is locally very ample with respect to p. ∎

(1.3) REMARK. Let F^{n-1} denote a general fibre of p. Then there exists a *smooth regular ladder* of (F^{n-1}, L').

(1.4) REMARK. Since (F^{n-1}, L') has a ladder it follows easily (see [8, (1.0)]), that $d=(L'_{F^{n-1}})^{n-1}=(L'_{F^{n-2}})^{n-2}=\cdots=(L'_{F^{n-i}})^{n-i}=\cdots=(L'_{F^2})^2$, where $(L'_{F^{n-i}})^{n-i}=L'_{F^{n-i}}\cdot \cdots \cdot L'_{F^{n-i}}$ ($n-i$ times). But F^2 is a smooth del Pezzo surface. Hence $1 \leq d \leq 9$.

(1.5) LEMMA. *Let $p: X' \to C$ be as in* (1.0) *and $d=\deg L'_{F^{n-1}}$. Then*
(i) $3 \leq d \leq 4$, *if* $\dim X' \geq 8$;
(ii) $3 \leq d \leq 5$, *if* $\dim X'=6, 7$;
(iii) $3 \leq d \leq 6$, *if* $\dim X'=5$;
(iv) $3 \leq d \leq 8$, *if* $\dim X'=4$.

PROOF. By (1.4) we know that $1 \leq d \leq 9$. If $d=1$, it follows that $\mathrm{Bs}\,|L'_{F^{n-1}}|$, the base locus of $|L'_{F^{n-1}}|$, is nonempty which contradicts (1.2). Hence $d=1$ cannot occur. Since L' is locally very ample with respect to $p, d=2$ cannot occur either. The rest follows from [8]. ∎

(1.6) LEMMA. *$p_*(L')$ is a vector bundle of rank $d+n-2$ over C, where $d=\deg L'_{F^{n-1}}$ and $n=\dim X'$.*

PROOF. By (1.3)

$$0 \to \mathcal{O}_{F^{n-1}} \to L'_{F^{n-1}} \to L'_{F^{n-2}} \to 0\,.$$

Moreover $h^1(F^{n-1}, \mathcal{O}_{F^{n-1}})=0$. Hence $h^0(F^{n-1}, L')=h^0(F^{n-2}, L')+1$ or more generally

(*) $$h^0(F^{n-i}, L')=h^0(F^{n-i-1}, L')+1\ .$$

Thus we have

$$h^0(F^{n-1}, L')=h^0(F^2, L')+(n-3)\ .$$

Noting that $L'_{F^2}=-K_{F^2}$ and using the Riemann-Roch theorem we conclude that $h^0(L'_{F^2})=d+1$, where $d=K_{F^2}\cdot K_{F^2}$, whence $h^0(L'_{F^{n-1}})=d+n-2$. Hence in a neighbourhood of a smooth fibre, $p_*(L'_{F^{n-1}})$ is locally free of rank $d+n-2$, and since C is smooth it follows that $p_*(L')$ is everywhere torsion free of rank $d+n-2$. ∎

2. The possible singular fibres of $p_3: X'\to C$. Let F be a general fibre of $p_3: X'\to C$. Note that (F, L') is a del Pezzo surface and that L' is locally very ample with respct to p_3, see (1.2). Hence $3\le \deg L'_F\le 9$. Since the degree is preserved by flat maps, it follows that $3\le \deg L'_\Gamma\le 9$, where Γ is a possible singular fibre of p_3. Let S' be a general element of $|L'|$, and let $\gamma=\Gamma\cap S'$. Note that γ is a possible singular fibre of $p_2: S'\to C$. Moreover, S' is an elliptic surface with no multiple fibres (see [2]). Hence the possible types for γ are: I_b with $1\le b\le 9$, II, III, IV and I_b^* with $b=0, 1$ (see [14, (6.2)]).

(2.0.0) REMARK. Let $\Gamma=\sum \Gamma^\alpha$ and let $S'\in |L'|$. If γ is one of the following types I_b with $1\le b\le 9$, III, IV and I_b^* with $b=0, 1$, then the general element of $|L'|$ cannot pass through a possible singular point of a component Γ^α of a singular fibre. This follows, since γ^α is smooth and Γ^α $(\supset\gamma^\alpha)$ is a local complete intersection.

(2.0.1) LEMMA. *Let $\Gamma=\sum\Gamma^\alpha$ denote a possible reducible fibre of p_3. Let $\gamma=\Gamma\cap S'$ with S' a general member of $|L'|$. If γ is one of the types* I_b *with* $2\le b\le 9$, III, IV *and* I_b^* *with $b=0, 1$, then either $\Gamma^\alpha\cong F_r$ with $r\ge 0$ or $\tilde{F}_r$ with $r\ge 1$.*

PROOF. Note that if γ is of one of the types I_b with $2\le b\le 9$, III, IV, then $\gamma=\sum\gamma^\alpha$. Or else $\gamma=\gamma^0+\gamma^1+\gamma^2+\gamma^3+2\gamma^4+\cdots+2\gamma^{4+b}$ with $b=0, 1$, where γ^α are smooth rational curves in both cases (see [14, (6.2)]. Also $\gamma^\alpha=\Gamma^\alpha\cap S'$ is ample on Γ^α. Hence $\Gamma^\alpha\cong F_r$ with $r\ge 0$ or $\tilde{F}_r$ with $r\ge 1$ (see [17, (0.6.1)]). ∎

(2.0.2) LEMMA. *Let $\Gamma=\sum\Gamma^\alpha$ denote a possible reducible fibre of p_3. If $\Gamma^\alpha\cap\Gamma^\beta$ is nonempty and if $L'\cdot\Gamma^\alpha\cdot\Gamma^\beta=1$, then Γ^α and Γ^β meet transversely in a smooth rational curve B.*

PROOF. Let $B=\Gamma^\alpha\cap\Gamma^\beta$. Since L'_B is very ample and $L'\cdot B=1$, it follows that B is a linear $\boldsymbol{P}^1$. Hence B is a smooth rational curve. Therefore the intersection of Γ^α and Γ^β is transverse. Also note that $B\in|[\Gamma^1]_{\Gamma^0}|$. Hence B is a *Cartier divisor* on Γ^α and Γ^β. ∎

Since $B=\Gamma^\alpha\cap\Gamma^\beta$, $L'\cdot B=L'\cdot\Gamma^\alpha\cdot\Gamma^\beta=\gamma^\alpha\cdot\gamma^\beta$, and $\gamma^\alpha\cdot\gamma^\beta$ is either 1 or 2, it follows that $L'\cdot B$ is either 1 or 2.

(2.0.3) REMARK. If Γ^α and Γ^β meet transversely, then

(2.0.3.1) $$N_B = N_{B/\Gamma^0} \oplus N_{B/\Gamma^1}\,.$$

Moreover, since $(K_{X'}+L')_B = \mathcal{O}_B$ and $L' \cdot B = d$ with $d=1$ or 2, it follows that $K_{X'} \cdot B = -d$. Therefore from the adjunction formula and (2.0.3.1) we have

(2.0.3.2) $$\deg N_B = \deg N_{B/\Gamma^0} + \deg N_{B/\Gamma^1} = -2+d\,.$$

We also have for $\alpha \neq \beta$

(2.0.3.3) $$N_{B/\Gamma^\alpha} = N_{\Gamma^\beta/X'B}\,.$$

(2.0.4) LEMMA. *Let Γ^α, Γ^β, B and L' be as in* (2.0.1) *and* (2.0.2).

(i) *If $\Gamma^\alpha \cong \boldsymbol{P}^2$, then $[B] = \mathcal{O}_{\boldsymbol{P}^2}(e)$ with $e=1$ or 2. Moreover, we cannot have $\Gamma^\alpha \cong \boldsymbol{P}^2 \cong \Gamma^\beta$.*

(ii) *If $\Gamma^\alpha \cong \tilde{F}_r$, then $\pi_1^*([B]) = a[E+rf]$, $\pi_1^*(L') = b[E+rf]$ for some $a, b \in \boldsymbol{Z}$. Moreover, $r=1$ if $L' \cdot B = 1$, and $r=1$ or 2 if $L' \cdot B = 2$.*

(iii) *If $\Gamma^\alpha \cong F_r$, then $L' = [E+kf]$ with $k \geq r+1$ and $[B] = [aE + (d+ar-ak)f]$, where $d = L' \cdot B$.*

PROOF. (i) follows from $d = L' \cdot B = \mathcal{O}_{\boldsymbol{P}^2}(a) \cdot \mathcal{O}_{\boldsymbol{P}^2}(e) = ae$ and the fact that $d=1$ or 2. For the remaining see ([18, (1.4)]).

(ii) Note that if N is a line bundle on $\tilde{F}_r$, then $\pi_1^*(N) = a[E+rf]$ for some $a \in \boldsymbol{Z}$, see (0.7). Moreover, if M is another line bundle on $\tilde{F}_r$, then $N \cdot M = \pi_1^*(N) \cdot \pi_1^*(M) = (a[E+rf]) \cdot (b[E+rf]) = abr$. Thus $d = L' \cdot \Gamma^\alpha \cdot \Gamma^\beta = L'_{\Gamma^\alpha} \cdot [\Gamma^\beta]_{\Gamma^\alpha} = abr$. Whence $r=1$ or $r=2$, since $d=1$ or 2.

(iii) Note that since L' is ample, $L' = [E+kf]$ with $k \geq r+1$. Moreover, $[B] = [aE+bf]$ for some $a, b \in \boldsymbol{Z}$. Hence by $d = L' \cdot B = [E+kf] \cdot [aE+bf] = -ar+b+ak$, we have $b = d+ar-ak$.

(2.0.5) REMARK. If $\Gamma \cong \tilde{F}_2$, then $\deg L'_\Gamma = 2a^2$ with $a \in \boldsymbol{Z}$. Hence L'_Γ has even degree. Moreover, $(\deg L'_\Gamma)/2$ is the square of an integer.

In (2.1) through (2.4) we classify possible types of singular fibres.

(2.1) The case $\deg L'_\Gamma = 3$.

(2.1.1) CLAIM. *Γ has at most two components.*

PROOF. It is easy to see that in this case the only possible types for γ are I_1, I_2, I_3, II, III, IV. If γ is of type I_3, i.e. $\gamma = \gamma^0 + \gamma^1 + \gamma^2$, then $\Gamma = \sum_{\alpha=0}^{2} \Gamma^\alpha$. By (2.0.1) we have either $\Gamma^\alpha \cong F_r$ with $r \geq 0$ or $\tilde{F}_r$ with $r \geq 1$. Note that $3 = \deg L'_\Gamma$. Since L' is very ample on Γ^α, and $\deg L'_{\Gamma^\alpha} = 1$ for $0 \leq \alpha \leq 2$, it follows that $\Gamma^\alpha \cong \boldsymbol{P}^2$ for each α. By (2.0.4), (i) this cannot happen. Similarly type IV is ruled out. This proves the claim.

(2.1.2) CLAIM. *If γ is type* I_2 *or* III, *then either $\Gamma = \boldsymbol{P}^2 + F_0$ or $\Gamma = \boldsymbol{P}^2 + \tilde{F}_2$.*

PROOF. If γ is of type I_2 or III, i.e. $\gamma = \gamma^0 + \gamma^1$ with $\gamma^0 \cdot \gamma^1 = 2$ then $\Gamma = \Gamma^0 + \Gamma^1$, and by (2.0.1) $\Gamma^\alpha \cong F_r$, $r \geq 0$ or $\tilde{F}_r$, $r \geq 1$. We choose a generic $S' \in |L'|$ such that S' meets Γ^0 and Γ^1 respectively in γ^0 and γ^1. Since deg $L'_\Gamma = 3$, if follows that on Γ^1 say, deg $L'_{\Gamma^1} = 1$ and deg $L'_{\Gamma^0} = 2$. Hence $\Gamma^1 \cong \boldsymbol{P}^2$. Moreover since L' is very ample on Γ^0, it follows that Γ^0 is either F_0 or $\tilde{F}_2$. This proves the claim.

Since we need an explicit description of L' and B on each component in eventual proofs, we have the following remark.

(2.1.3) REMARK. (i) Let $B = \Gamma^0 \cap \Gamma^1$ in (2.1.2). Note that on $\Gamma^1 \cong \boldsymbol{P}^2$, $B \in |\mathcal{O}_{\boldsymbol{P}^2}(2)|$.

(ii) Suppose γ is of type I_2, in (2.1.2).

If $\Gamma^0 = F_0$, then $L' = [E+f]$. Using $(K_{X'} + L')_{F_0} = \mathcal{O}_{F_0}$ and the adjunction formula we have $N_{F_0/X'} = -[E+f]$. Note that B is an effective divisor in F_0. Hence $2p_a(B) - 2 = B \cdot (K+B)$, where $K = K_{F_0}$. Using $L' \cdot B = 2$ we get $B = E+f$. We observe that the curve B cannot be irreducible, for otherwise $\boldsymbol{P}^2$ and F_0 would intersect transversely and therefore by (2.0.3.3) we get a contradiction.

If $\Gamma^0 = \tilde{F}_2$, then $\pi_1^*(L') = [E+2f]$. *We also have* $N_{\tilde{F}_2/X'} = -L'$. A similar reasoning gives $\pi_1^*(B) = E + 2f$, and thus $B = 2f$, where f denotes the ruling of the quadric cone $\tilde{F}_2$.

(iii) Suppose γ is of type III, in (2.1.2). Let $B = \Gamma^0 \cap \Gamma^1$. Note that Γ^0 and Γ^1 are tangent along B *since* $L' \cdot B = L' \cdot \Gamma^\alpha \cdot \Gamma^\beta = \gamma^\alpha \cdot \gamma^\beta = 2p$. Let $\bar{B} = B_{\mathrm{red}}$. Note that $L' \cdot \bar{B} = 1$.

If $\Gamma^0 = F_0$, then $L' = [E+f]$, $N_{F_0/X'} = -[E+f]$ and either $\bar{B} = E$ or $\bar{B} = f$.

If $\Gamma^0 = \tilde{F}_2$, then $\pi_1^*(L') = [E+2f]$, $N_{\tilde{F}_2/X'} = -L'$ and $\bar{B} = f$, where f denotes the ruling of the quadric cone $\tilde{F}_2$.

(2.1.4) CLAIM. *Let Γ be as in* (2.1.2). *Neither of the two types of Γ can occur.*

PROOF. Let $\Gamma = \boldsymbol{P}^2 + F_0$ and suppose that $\gamma = \Gamma \cap S'$ is of type I_2. By (2.1.3) on F_0, $B = E+f$. As we have observed in (2.1.3) (ii), B is reducible. Consider the component f of B. Note that f is a smooth rational curve satisfying

$$F_0 \cdot f = -1 . \tag{2.1.4.1}$$

Hence f is not numerically effective (nef for short).

(2.1.4.a) If $\boldsymbol{R}_+[f]$ is an extremal ray then, since f is not nef by [15, (3.3)] it follows that the contraction morphism $\Phi: X' \to Y$ associated to the extremal ray $\boldsymbol{R}_+[f]$ has a three dimensional image. Note that dim $\Phi(F_0) = 0$ or 1 and dim $\Phi(f) = 0$ in each case. Note also that f is contained in the other component, $\boldsymbol{P}^2$, of Γ. Hence dim $\Phi(\boldsymbol{P}^2) = 0$. This contradicts the list in [15, (3.3)].

(2.1.4.b) If $\boldsymbol{R}_+[f]$ is not an extremal ray it follows (see [15, (1.4)]) that for an arbitrary positive ε, there exist a finite number r of rational curves $l_1, \cdots, l_r$ in X' such that

$$f = \sum_{i=1}^{r} a_i l_i + C \tag{2.1.4.2}$$

in $\overline{NE}(X')$, where $a_i \in \boldsymbol{R}_+$ and $C \in \overline{NE}_\varepsilon(X', L')$. We choose $\varepsilon < 1$. Since $K' + L'$ is nef and $(K'+L')\cdot f = 0$ from (2.1.4.2) it follows that

(2.1.4.3) $$f = \sum_{i=1}^{r} a_i l_i .$$

Using (2.1.4.1) and (2.1.4.3) we conclude that $F_0 \cdot l_i < 0$ for some i. Let $\Phi_i\colon X' \to Y$ be the contraction morphism associated to the extremal ray $\boldsymbol{R}_+[l_i]$. As we have seen previously, Y has a three-dimensional image and $\dim \Phi_i(F_0) = 0$ or 1.

(2.1.4.b$_1$) Let $\dim \Phi_i(F_0) = 0$. Hence $f (\subset F_0)$ is contracted by Φ_i, and thus $[f] \in \boldsymbol{R}_+[l_i]$. Note that f is a smooth rational curve with $[f] = b[l_i]$ for some $b \in \boldsymbol{R}_+$. Thus $\boldsymbol{R}_+[f] = \boldsymbol{R}_+[l_i]$. Note that f is also contained in the other component, $\boldsymbol{P}^2$, of Γ. Hence $\dim \Phi_i(\boldsymbol{P}^2) = 0$. Either case contradicts [15, (3.1), (3.3)].

(2.1.4.b$_2$) If $\dim \Phi_i(F_0) = 1$, either $\Phi_i(f) = y \in Y$ or $\Phi_i(f) = \boldsymbol{P}^1$.

If $\Phi_i(f) = y$, then $[f] \in \boldsymbol{R}_+[l_i]$. As in (2.1.4.b$_1$) we get a contradiction.

If $\Phi_i(f) = \boldsymbol{P}^1$, then $\Phi_i(E) = y'$, where E denotes the other ruling of F_0. Thus $[E] \in \boldsymbol{R}_+[l_i]$. Replacing f by E in (2.1.4.b$_1$), again we get a contradiction.

Let $\Gamma = \boldsymbol{P}^2 + F_0$ and suppose that $\gamma = \Gamma \cap S'$ is of type III. Note that $F_0 \cdot \bar{B} = -1$, where $\bar{B}$ is as in (2.1.3) (iii). Suppose that on F_0, $\bar{B} = f$ (or E).

(2.1.4.c) If $\boldsymbol{R}_+[\bar{B}]$ is an extremal ray, then since $\bar{B}$ is not nef, the same proof as in (2.1.4.a) with f replaced by $\bar{B}$ rules out this case.

(2.1.4.d) If $\boldsymbol{R}_+[\bar{B}]$ is not an extremal ray then as in (2.1.4.b), $f = \sum_{i=1}^{r} a_i l_i$ and $F_0 \cdot l_i < 0$ for some i. Let $\Phi_i\colon X' \to Y$ be the contraction morphism associated to the extremal ray $\boldsymbol{R}_+[l_i]$. As seen earlier, Y has a three-dimensional image and $\dim \Phi_i(F_0) = 0$ or 1.

(2.1.4.d$_1$) Let $\dim \Phi_i(F_0) = 0$. Hence $f (\subset F_0)$ is contracted by Φ_i. But $f \subset \boldsymbol{P}^2$, and thus $\dim \Phi_i(\boldsymbol{P}^2) = 0$. In either case this contradicts [15, (3.1), (3.3)].

(2.1.4.d$_2$) If $\dim \Phi_i(F_0) = 1$, either $\Phi_i(f) = y \in Y$ or $\Phi_i(f) = \boldsymbol{P}^1$.

If $\Phi_i(f) = y$, then as in (2.1.4.d$_1$) we get a contradiction.

If $\Phi_i(f) = \boldsymbol{P}^1$, then F_0 is smoothly contracted by Φ_i. Hence Y is a smooth threefold and there is a morphism $p'\colon Y \to C$ such that $\Phi_i \circ p' = p$. Let $L'' = (\Phi_i)_*(L' + F_0)$. Note that $\Phi_i(\Gamma) = \boldsymbol{P}^2$ and $(L''_{\boldsymbol{P}^2})^2 = 3$. But there are no line bundles of self-intersection 3 on $\boldsymbol{P}^2$. Hence this case is also ruled out.

Let $\Gamma = \boldsymbol{P}^2 + \tilde{F}_2$. If $\gamma = \Gamma \cap S'$ is of type I_2, then by (2.1.3) (ii) on $\tilde{F}_2$, $B = 2f$, where f is the ruling of the quadric cone $\tilde{F}_2$. Now $\tilde{F}_2 \cdot f = -1$. On the other hand, if $\gamma = \Gamma \cap S'$ is of type III, then $\tilde{F}_2 \cdot \bar{B} = -1 = \tilde{F}_2 \cdot f$.

(2.1.4.e) If $\boldsymbol{R}_+[f]$ is an extremal ray, then since f is not nef, it follows that the contraction morphism Φ associated to the extremal ray $\boldsymbol{R}_+[f]$ has a three-dimensional image and $\dim \Phi(\tilde{F}_2) = 0$. Hence $\dim \Phi(\boldsymbol{P}^2) = 0$ or 1. But as is well known, $\dim \Phi(\boldsymbol{P}^2) = 1$ is impossible. Thus $\dim \Phi(\boldsymbol{P}^2) = 0$. This contradicts [15, (3.3)].

(2.1.4.f) If $\boldsymbol{R}_+[f]$ is not an extremal ray, then $[f] = b[l_i]$ as in (2.1.4.b$_2$). And again we have a contradiction.

(2.1.5) Suppose γ is of type I_1 or II. Since γ is irreducible, *L' is ample and $\gamma=\Gamma\cap S'$*, where S' is a generic element of $|L'|$, we conclude that Γ is irreducible and reduced. ∎

(2.2) The case $\deg L'_\Gamma=4$.

(2.2.1) CLAIM. *Γ has at most two components.*

PROOF. In this case the possible types for γ are I_b with $1\leq b\leq 4$, II, III, IV. Since two adjacent components of Γ necessarily have $\deg L'_{\Gamma^\alpha}=1$, type I_4 is impossible, as in (2.1.1) as also are type I_3 and type IV. This proves the claim.

(2.2.2) CLAIM. *If γ is of type I_2 or III, then either $\Gamma=\tilde{F}_2+\tilde{F}_2$, or $\Gamma=\tilde{F}_2+F_0$, or $\Gamma=F_0+F_0$, or $\Gamma=\boldsymbol{P}^2+F_1$.*

PROOF. If γ is of type I_2 or III, then $\Gamma=\Gamma^0+\Gamma^1$. Since $4=\deg L'_{\Gamma^0}+\deg L'_{\Gamma^1}$, we have either i) $\deg L'_{\Gamma^0}=2=\deg L'_{\Gamma^1}$ or ii) $\deg L'_{\Gamma^0}=1$ and $\deg L'_{\Gamma^1}=3$ (and vice versa). Using these numerical data, (2.0.4) and (2.0.5), it is easy to see that the possibilities for Γ in i) are: $\tilde{F}_2+\tilde{F}_2$, or $\tilde{F}_2+F_0$, or F_0+F_0, and in ii) $\boldsymbol{P}^2+F_r$ for $r\geq 0$. Using $3=\deg L'=-r+2k$, with $k\geq r+1$ we have $r=1$. This proves our claim.

As in (2.1.3) we need the following remark.

(2.2.3) REMARK. (i) Suppose γ is of type I_2, in (2.2.2).

If $\Gamma=\tilde{F}_2+\tilde{F}_2$, then as in (2.1.3) (ii), we have $\pi_1^*(L')=[E+2f]$, $N_{\tilde{F}_2/X'}=-L'$ and $B=2f$, where f denotes the ruling of the quadric cone $\tilde{F}_2$.

If $\Gamma=\tilde{F}_2+F_0$, then on F_0, $L'=[E+f]$, $N_{F_0/X'}=-[E+f]$, and $B=E+f$; while on $\tilde{F}_2$, $B=2f$.

If $\Gamma=F_0+F_0$, then $L'=[E+f]$ and $N_{F_0/X'}=-[E+f]$ as seen in (2.1.3) (ii). Note that B is an effective divisor in F_0. Hence $2p_a(B)-2=B\cdot(K+B)$, where $K=K_{F_0}$. Using $L'\cdot B=2$ we get $B=E+f$.

If $\Gamma=\boldsymbol{P}^2+F_r$, then using $3=\deg L'_{F_1}$, we have $L'_{F_1}=[E+2f]$. From $(K_{X'}+L')_{\Gamma^\alpha}=\mathcal{O}_{\Gamma^\alpha}$ and the adjunction formula we have $N_{F_1/X'}=-[E+f]$. Moreover on F_1, $B=E+f$ and on $\boldsymbol{P}^2$, $B=l_1+l_2$.

(ii) Suppose γ is of type III, in (2.2.2). Let $B=\Gamma^0\cap\Gamma^1$. Note that Γ^0 and Γ^1 are tangent along B, since $L'\cdot B=L'\cdot\Gamma^\alpha\cdot\Gamma^\beta=\gamma^\alpha\cdot\gamma^\beta=2p$. Let $\bar{B}=B_{\mathrm{red}}$. Note that $L'\cdot\bar{B}=1$. As in (i) we can calculate L'_{Γ^α} and $N_{\Gamma^\alpha/X'}$. So we only need to compute $\bar{B}$.

If $\Gamma=\tilde{F}_2+\tilde{F}_2$, then $\bar{B}=f$, where f denotes the ruling of $\tilde{F}_2$. If $\Gamma=\tilde{F}_2+F_0$, then on F_0, either $\bar{B}=E$, or $\bar{B}=f$; while on $\tilde{F}_2$, $\bar{B}=f$. If $\Gamma=F_0+F_0$, then either $\bar{B}=E$, or $\bar{B}=f$. If $\Gamma=\boldsymbol{P}^2+F_1$, then on F_1, either $\bar{B}=E$, or $\bar{B}=f$; while on $\boldsymbol{P}^2$, $\bar{B}=l$. Note that $\bar{B}=E$ cannot occur, for if it did, then $B=2\bar{B}$ and we have $p_a(B)=0$ in $\boldsymbol{P}^2$ and $p_a(B)=-2$ in F_1.

(2.2.4) CLAIM. *Let Γ be as in (2.2.2). None of the types of Γ can occur.*

PROOF. Let $\Gamma=\tilde{F}_2+\tilde{F}_2$. Reasoning as in (2.1.4), we rule out this possibility.

Let $\Gamma = \tilde{F}_2 + F_0$. Suppose that γ is of type I_2. By (2.2.3) (i), $B = 2f$ on $\tilde{F}_2$. Now $\tilde{F}_2 \cdot f = -1$. On the other hand, if γ is of type III, then $\tilde{F}_2 \cdot \bar{B} = -1 = \tilde{F}_2 \cdot f$. Reasoning as in (2.1.4.e), (2.1.4.f), we get a contradiction.

Let $\Gamma = \boldsymbol{P}^2 + F_1$. If γ is of type I_2, then $B = E + f$ on F_1. Moreover $F_1 \cdot f = -1$. This is ruled out as in the case $\Gamma = \boldsymbol{P}^2 + F_0$ and γ is of type I_2 in (2.1.4).

If $\Gamma = \boldsymbol{P}^2 + F_1$ and γ is of type III, then this is ruled out as in the case $\Gamma = \boldsymbol{P}^2 + F_0$ and γ is of type III in (2.1.4).

The case $\Gamma = F_0 + F_0$ is ruled out similarly.

If γ is of type I_1 or II, then Γ is irreducible and reduced as seen in (2.1.5). ∎

(2.3) The case $\deg L'_\Gamma = 8$.

CLAIM. *Γ has at most three components.*

PROOF. We will show that either $\Gamma = F_0 + F_1 + F_1$ or Γ is as in Table 1. Note that the only possible types for γ are I_b with $1 \le b \le 8$, II, III, IV and I_b^* with $b = 0, 1$.

(2.3.1) If γ is of type I_b with $6 \le b \le 8$, then $\Gamma = \sum_{\alpha=0}^{b-1} \Gamma^\alpha$. Since $\sum_{\alpha=0}^{b-1} \deg L'_{\Gamma^\alpha} = 8$ and L' is ample, we necessarily have two adjacent components of Γ, say Γ^α and Γ^β with $\deg L'_{\Gamma^\alpha} = 1 = \deg L'_{\Gamma^\beta}$, and this cannot happen by (2.0.4), (i).

(2.3.2) If γ is of type I_5,then we can easily see that L' has degree 1 on at least two components, say Γ^0 and Γ^1. By (2.0.4) it cannot happen that they are adjacent to one another. Hence the adjacent component of Γ^0, say Γ^α, has degree two. By (2.0.4) (ii), the only possibility is $\Gamma^0 \cong \boldsymbol{P}^2$ and $\Gamma^\alpha \cong F_0$. Note that $N_{B/\boldsymbol{P}^2} = \mathcal{O}_{\boldsymbol{P}^2}(1)$, and on F_0, $[B] = [aE + (1-a)f]$. Hence by (2.0.3.2) we have $a^2 - a - 1 = 0$, which is impossible.

(2.3.3) If γ is of type I_4, then in view of (2.3.2), and since $\sum_{\alpha=0}^{3} \deg L'_{\Gamma^\alpha} = 8$, it suffices to consider the case when Γ^α and Γ^β are adjacent and having $\deg L'_{\Gamma^\alpha} = 2 = \deg L'_{\Gamma^\beta}$. By (2.0.4), (ii) the only possibility is $\Gamma^\alpha \cong F_0 \cong \Gamma^\beta$. From $\deg L'_{\Gamma^\alpha} = 2$ and (2.0.4) it follows that $[B] = [aE + (1-a)f]$ and $L' = [E + f]$. Moreover, $N_{\Gamma^\alpha/X'} = -[E + f]$. Hence by (2.0.3.3) we have $2a^2 - 2a - 1 = 0$ which is impossible.

(2.3.4) If γ is of type I_3, then in view of (2.3.2) and (2.3.3), and since $\sum_{\alpha=0}^{2} \deg L'_{\Gamma^\alpha} = 8$ it suffices to consider the case when Γ^α and Γ^β are adjacent and having: (a) $\deg L'_{\Gamma^\alpha} = 1$, and $\deg L'_{\Gamma^\beta} = 3$; (b) $\deg L'_{\Gamma^\alpha} = 2$, and $\deg L'_{\Gamma^\beta} = 3$.

In (a) by (2.0.4), (ii), $\Gamma^\alpha \cong \boldsymbol{P}^2$, and $\Gamma^\beta \cong F_1$ is the only possibility. Hence by (2.0.4) and (2.0.3.2) it follows that $3a^2 - 2a - 2 = 0$, which is impossible since $a \in \boldsymbol{Z}$. In (b) $\Gamma^\alpha \cong F_0$, and $\Gamma^\beta \cong F_1$ is the only possibility. Then from (2.0.4) and (2.0.3.3) it follows that the curve $B = F_0 \cap F_1$ is either a fibre on F_0 and the unique curve on F_1 of self-intersection -1, or the curve E on F_0 and F_1 where E is as in (0.7). Note that in this case the third component, say Γ^2, is isomorphic to F_1 and is adjacent to Γ^β. From (2.0.4) and (2.0.3.2) it follows that the curve $B' = F_1 \cap F_1$ is a fibre on one of the F_1 and is the unique curve of self-intersection -1 on the other F_1. Note that the same is obtained if γ is of type IV.

(2.3.5) If γ is of type I_0^*, then $\Gamma = \sum_{\alpha=0}^{4} a_\alpha \Gamma^\alpha$ with $a_\alpha = 1$ for $0 \le \alpha \le 3$ and $a_4 = 2$.

Since $\sum_{\alpha=0}^{4} a_\alpha \deg L'_{\Gamma^\alpha}=8$ we have either (i) $\deg L'_{\Gamma^\alpha}=1$ for $0\le\alpha\le3$, and $\deg L'_{\Gamma^4}=2$, or (ii) $\deg L'_{\Gamma^\alpha}=1$ for $\alpha=0, 1, 4$ and $\deg L'_{\Gamma^2}=2=\deg L'_{\Gamma^3}$, or (iii) $\deg L'_{\Gamma^\alpha}=1$ for $\alpha=0, 2, 4$ and $\deg L'_{\Gamma^1}=2=\deg L'_{\Gamma^3}$, or (iv) $\deg L'_{\Gamma^\alpha}=1$ for $1\le\alpha\le4$, and $\deg L'_{\Gamma^0}=3$.

Note that (ii), (iii), (iv) cannot occur by (2.0.4) (i). So it remains to rule out case (i).

In case (i) $\Gamma^\alpha\cong \boldsymbol{P}^2$ for $0\le\alpha\le3$ and Γ^4 is either $\tilde{F}_2$, or F_0. Note that the $\boldsymbol{P}^2$'s are disjoint and each of the $\boldsymbol{P}^2$'s intersect Γ^4. Let $B=\boldsymbol{P}^2\cap\Gamma^4$. Since $L'\cdot B=L'\cdot\boldsymbol{P}^2\cdot\Gamma^4=\gamma^\alpha\cdot\gamma^4=1$ for $0\le\alpha\le3$ it follows that B is a smooth rational curve. Therefore the intersection of $\boldsymbol{P}^2$ and Γ^4 is transverse. Hence B is a Cartier divisor on $\boldsymbol{P}^2$ and Γ^4. On $\Gamma^4\cong\tilde{F}_2$, $\pi_1^*(L')=[E+2f]$, $N_{\tilde{F}_2/X'}=-L'$ and $B=f$, where f denotes the ruling of the quadric cone $\tilde{F}_2$. But f is not a Cartier divisor on $\tilde{F}_2$. Hence this case cannot occur. Whereas if $\Gamma^4\cong F_0$, then $L'=[E+f]$, $N_{F_0/X'}=-[E+f]$ and either $B=E$ or $B=f$. As for $\boldsymbol{P}^2$, $L'_{\boldsymbol{P}^2}=\mathcal{O}_{\boldsymbol{P}^2}(1)$ and $N_{\boldsymbol{P}^2/X'}=\mathcal{O}_{\boldsymbol{P}^2}(-2)$. An easy calculation shows that (2.0.3.3) does not hold. Hence this possibility is also ruled out.

If γ is of type I_1^*, then $\Gamma=\sum_{\alpha=0}^{4} a_\alpha\Gamma^\alpha$ with $a_\alpha=1$ for $0\le\alpha\le3$ and $a_4=2=a_5$. This forces only one possibility, namely $\deg L'_{\Gamma^\alpha}=1$ for all α. This violates (2.0.4), (i).

(2.3.6) CLAIM. *If γ is of type* I_2 *or* III, *then Γ has one of the following possible type* : $\boldsymbol{P}_1^2+F_r$, *with* $r=1, 3, 5$; *or* $\tilde{F}_2+F_r$, *with* $r=0, 2, 4$; *or* F_r+F_s, *with* $(r, s)=(0, 0)$, $(0, 2)$, $(0. 4)$, $(1, 1)$, $(1, 3)$, $(2, 2)$.

PROOF. If γ is of type I_2 or III then $\Gamma=\Gamma^0+\Gamma^1$. By (2.0.4) $\Gamma^\alpha\cong\tilde{F}_r$, $r\ge1$ or F_r, $r\ge0$. If both $\Gamma^\alpha\cong\tilde{F}_r$, then by (2.0.4) (ii), $r=1$ or 2, and since $8=\deg L'_{\Gamma^0}+\deg L'_{\Gamma^1}$, it is easy to see that the only possibility is $\Gamma^0\cong\boldsymbol{P}_2^2\cong\Gamma^1$, i.e. $\Gamma^\alpha\cong\boldsymbol{P}^2$ and $L'_{\boldsymbol{P}^2}=\mathcal{O}_{\boldsymbol{P}^2}(2)$. The other possibility is $\Gamma^0\cong\tilde{F}_r$ and $\Gamma^1\cong F_r$ or both $\Gamma^\alpha\cong F_r$. So overall we have:

(1) $\boldsymbol{P}_2^2+\boldsymbol{P}_2^2$; or (2) $\boldsymbol{P}_2^2+F_r$; or (3) $\boldsymbol{P}_1^2+F_r$; or (4) $\tilde{F}_2+F_r$; or (5) F_r+F_s.

In (1), B is a linear $\boldsymbol{P}^1$, since $L'=\mathcal{O}_{\boldsymbol{P}^2}(2)$ and $2=L'\cdot B$. This argument goes through for types I_2 as well as III. Now using (2.0.3.2) we get a contradiction.

In (2), on $\boldsymbol{P}^2$, $[B]=\mathcal{O}_{\boldsymbol{P}^2}(1)$ and this does not depend on the type of γ. We also know that $\deg L'_{F_r}=4$. The same reasoning as above gives $r=0, 2$. Hence by (2.0.4) and (2.0.3.2) we have, in both cases, $a^2-a-1=0$ which is impossible.

In (3) since $7=\deg L'_{F_r}=-r+2k$ and $k\ge r+1$, it follows that $r=1, 3, 5$.

In (4) since $6=\deg L'_{F_r}$ it follows that $r=0, 2, 4$.

In (5) we have a symmetrical situation with respect to r and s. Moreover $\deg L'_{\Gamma^\alpha}\ge2$. Hence we consider only the possibilities $\deg L'_{\Gamma^0}=2, 3, 4$ and $\deg L'_{\Gamma^1}=6, 5, 4$, respectively. Hence as seen easily we must have (r, s) as in the claim.

As in (2.1.3) we need the following remark.

(2.3.7) REMARK. (i) Suppose γ is of type I_2, in (2.3.6).
If $\Gamma=\boldsymbol{P}_1^2+F_r$, then by (2.3.6), $r=1, 3, 5$. Note that on $\boldsymbol{P}^2$, $[B]=\mathcal{O}_{\boldsymbol{P}^2}(2)$.

Let $r=5$. Then $L'=[E+6f]$, $N_{F_5/X'}=-[E+f]$. Note that B is an effective divisor in F_5, hence $2p_a(B')-2=[(-2E-7f)+(aE+(2-a)f)]\cdot[aE+(2-a)f]=-7a^2+9a-4$. Since $p_a(B')\geq 0$ we have $7a^2-9a+2\leq 0$ and thus $a=1$, i.e. $B=E+f$. By [11, V, 2.18], B is a reducible curve.

Let $r=3$. Then as in the case $r=5$, $L'=[E+5f]$, $N_{F_3/X'}=-[E]$ and $B=E$. An easy calculation shows that (2.0.3.3) does not hold.

Let $r=1$. Then $L'=[E+4f]$, $N_{F_1/X'}=[-E+f]$. Since B is effective and since $2=L'\cdot B=(E+4f)\cdot(aE+bf)=3a+b$, we have $B=2f$.

If $\Gamma=\tilde{F}_2+F_r$, then by (2.3.6), $r=0,2,4$. If $r=4$, then $L'_{F_4}=[E+5f]$, $N_{F_4/X'}=-[E+f]$ and $B=E+f$ on F_4. If $r=2$, then $L'_{F_2}=[E+4f]$, $N_{F_2/X'}=-[E]$ and $B=E$ on F_2. If $r=0$, then $L'_{F_4}=[E+3f]$, $N_{F_4/X'}=-[E+f]$ and $B=2f$ on F_0.

If $\Gamma=F_r+F_s$, then (r,s) is as in (2.3.6). In the usual way we compute L'_{Γ^α}, $N_{\Gamma^\alpha/X'}$ and B.

(ii) If γ is of type III, in (2.3.6), then $\Gamma=\Gamma^0+\Gamma^1$, and L'_{Γ^α}, $N_{\Gamma^\alpha/X'}$ are as in (2.3.7) (i), whereas B is computed as in (2.1.3) (iii) or (2.2.3) (ii).

(2.3.8) CLAIM. *Let Γ be as in* (2.3.6). *Then the types of Γ are as in Table* 1.

PROOF. Going over the possible types for Γ and reasoning as in (2.1.4) and (2.2.4) we can rule out the following cases: $\boldsymbol{P}_1^2+F_r$, with $r=1,3,5$; $\tilde{F}_2+F_r$, with $r=0,4$; F_r+F_s, with $(r,s)=(0,4)$, $(1,3)$. Hence the possible types for Γ are as in Table 1.

In Table 1, (a,b) stands for $[aE+bf]$. *If* $\Gamma^0\cong\tilde{F}_2$, then in the table we should read $\pi_1^*(L'_{\Gamma^0})$ instead of L'_{Γ^0}.

TABLE 1

$\Gamma^0+\Gamma^1$	L'_{Γ^0}	L'_{Γ^1}	$N_{\Gamma^0/X'}$	$N_{\Gamma^1/X'}$
$\tilde{F}_2+F_2$	$(1,2)$	$(1,4)$	$(-1,-2)$	$(-1,\ 0)$
F_0+F_2	$(1,1)$	$(1,4)$	$(-1,-1)$	$(-1,\ 0)$
F_1+F_1	$(1,2)$	$(1,3)$	$(-1,-1)$	$(-1,\ 0)$
F_0+F_0	$(1,2)$	$(1,2)$	$(-1,\ 0)$	$(-1,\ 0)$
F_2+F_2	$(1,3)$	$(1,3)$	$(-1,-1)$	$(-1,-1)$

If γ is of type I_1 or II, then Γ is irreducible and reduced as we have seen in (2.1.5). ■

(2.4) The case $\deg L'_\Gamma=7$.

CLAIM. *Γ has at most three components. Moreover if Γ has three components then $\Gamma=F_0+F_1+F_0$, if Γ has two components then either $\Gamma=\tilde{F}_2+F_1$,* or *$\Gamma=F_1+F_2$.*

The proof will be omitted since it is similar to that in (2.3).

3. The case $d=3$.

(3.1) THEOREM. *Let X, L, X', L' and $p: X' \to C$ be as in* (1.0). *Then there exists a morphism $\Phi: X' \to \boldsymbol{P}_C(V)$ with $V = p_*(L')$ such that the following diagram commutes*:

$$\begin{tikzcd} X' \arrow[r, "\Phi"] \arrow[dr, "p"'] & \boldsymbol{P}_C(V) = \boldsymbol{P} \arrow[d, "\pi"] \\ & C \end{tikzcd}$$

We also have $\Phi(X') \in |\mathcal{O}_{\boldsymbol{P}_C(V)}(3) + \pi^*(M)|$ for some line bundle M on C.

PROOF. By (2.1) all the fibres of the morphism p are irreducible. Let $V = p_*(L')$. As in (1.6), it follows that rank $V = n+1$. By (1.2) L' is locally very ample with respect to p. Hence we have a morphism $\Phi: X' \to \boldsymbol{P}_C(V)$ such that the above diagram commutes, (see [10, (4.4.4)]). Moreover, Φ is an embedding. Note that Φ_F is the map associated to the linear system $|L'_F|$, where $F = p^{-1}(c)$ is a general fibre of p. We also have $\Phi(X') \cap \pi^{-1}(c) = \Phi(F)$. Hence it follows that

(3.1.1) $$N_{\Phi(X')/\boldsymbol{P}_C(V), \Phi(F)} = N_{\Phi(X')/\boldsymbol{P}^n} = \mathcal{O}_{\boldsymbol{P}^n}(3)\,.$$

Since $\mathrm{Pic}(\boldsymbol{P}_C(V)) \cong \mathrm{Pic}(C) \times \boldsymbol{Z}$, we have $[\Phi(X')] = \mathcal{O}_{\boldsymbol{P}_C(V)}(\alpha) + \pi^*(M)$ for some $\alpha \in Z$, and $M \in \mathrm{Pic}(C)$. By (3.1.1) we conclude that $\alpha = 3$, whence

$$\Phi(X') \in |\mathcal{O}_{\boldsymbol{P}_C(V)}(3) + \pi^*(M)|\,.$$

■

4. The case $d=4$.

(4.1) THEOREM. *Let X, L, X', L' and $p: X' \to C$ be as in* (1.0). *Then there exists a morphism $\Phi: X' \to \boldsymbol{P}_C(V)$ with $V = p_*(L')$ such that the following diagram commutes*:

$$\begin{tikzcd} X' \arrow[r, "\Phi"] \arrow[dr, "p"'] & \boldsymbol{P}_C(V) = \boldsymbol{P} \arrow[d, "\pi"] \\ & C \end{tikzcd}$$

and that $\Phi(X')$ restricted to each fibre $\boldsymbol{P}^{n+1}$ of π is a complete intersection of type (2,2) *in $\boldsymbol{P}^{n+1}$.*

PROOF. By (2.2) p has irreducible fibres. By (1.2) L' is locally very ample with respect to p. Hence there exists a morphism $\Phi: X' \to \boldsymbol{P}_C(V)$ such that the above diagram commutes. Also by [10, (4.4.4)] Φ is an embedding. Let F denote a general fibre of p. By noting that Φ_F is the morphism associated to the linear system $|L'_F|$, and since F is a del Pezzo manifold with $d=4$, F is isomorphic via Φ_F to a complete intersection of type (2,2) in $\boldsymbol{P}^{n+1}$, see [8, (2.2)]. We will show that this is also true for the possible singular fibre Σ of p. After slicing Σ with $n-3$ general members $A_i \in |L'|$, we obtain $\Sigma \cap X'^3 = \Gamma$, which is a possible singular fibre of p_3. But Γ is irreducible are reduced, hence Σ itself is irreducible and reduced. Note that Σ is a del Pezzo variety, whence, (a) $\Delta(\Sigma, L')=1=g(\Sigma, L')$; (b) Σ is locally Gorenstein, $\omega_\Sigma=\mathcal{O}_\Sigma(1-\dim \Sigma)$; and (c) $H^i(\Sigma, tL')=0$ for any $0<i<\dim \Sigma$, $t\in \boldsymbol{Z}$. (a) is trivially checked. (b) is obtained from $(K_{X'}+L')_\Sigma=\mathcal{O}_\Sigma$; (c) follows from [7, (5.7.6)]. Hence $h^0(\Sigma, 2L')=\chi(\Sigma, 2L')=(n^2+5n+2)/2=h^0(\boldsymbol{P}^{n+1}, \mathcal{O}_{\boldsymbol{P}^{n+1}}(2))-2$. Thus there are two distinct hyperquadrics Q_1 and Q_2 in $\boldsymbol{P}^{n+1}$ which contain $\Phi_\Sigma(\Sigma)$. ∎

5. The case $d=8$. In this section we will deal with the cases $\dim X'\geq 3$. From (1.5) it follows that the only cases to look at are $\dim X'=3$ and $\dim X'=4$. Let F be a general fibre of $p\colon X'\to C$. Note that if $\dim X'=3$, then F is either isomorphic to $Q_b\boldsymbol{P}^2$, the blow up of $\boldsymbol{P}^2$ at one point $b\in \boldsymbol{P}^2$ or is isomorphic to $\boldsymbol{P}^1\times \boldsymbol{P}^1$.

(5.0) $\dim X'=3$ and $F\cong Q_b\boldsymbol{P}^2$.

(5.1) $\dim X'=3$ and $F\cong \boldsymbol{P}^1\times \boldsymbol{P}^1$.

(5.2) $\dim X'=4$.

(5.3) THEOREM. *Let, X, L, X', L' and $p\colon X'\to C$ be as in* (1.0) *with the additional assumption in* (5.0). *Then there exists a birational morphism $\Phi\colon X'\to Y$ with Y a $\boldsymbol{P}^2$-bundle over C such that the following diagram commutes*:

$$\begin{tikzcd} X' \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & Y \arrow[dl, "p''"] \\ & C & \end{tikzcd}$$

PROOF. A general fibre of $p\colon X'\to C$ is isomorphic to $Q_b\boldsymbol{P}^2$. By (2.3), the possible reducible fibre Γ of p has at most three components. Moreover either Γ is as in Table 1, or $\Gamma=F_0+F_1+F_1$.

(5.3.1) Let Γ be as in Table 1, with $\Gamma=\tilde{F}_2+F_2$. Since $N_{F_2/X',f}=\mathcal{O}_F(-1)$ then F_2 can be smoothly blown down to a curve $\bar{C}$ to give a manifold X''. Let $\pi\colon X'\to X''$ be the blow down morphism. Let L'' be the line bundle on X'' such that $\pi^*L''=L'+[F_2]$. Note that $\pi(F_2)=\bar{C}\cong E\cong \boldsymbol{P}^1$, where E is as in (0.7).

(5.3.1.a) CLAIM. *$L''_{|\bar{C}}$ is ample.*

PROOF OF CLAIM. It is enough to show that $\deg L''_{|\bar{C}}>0$. Note that $\deg L''_{|\bar{C}}=\deg(\pi^*L'')_{|E}=\deg(L'+[F_2])_{|E}=\deg(E+4f-E)_{|E}=4f\cdot E=4$. This proves our claim.

Since $L''_{|\bar{C}}$ is ample, by [6, (5.7)] it follows that L'' is ample on X''.

Let Γ be as in the remaining cases in Table 1. A reasoning similar to that in (5.3.1) shows that one of the two components Γ^α of Γ can be smoothly blown down to a curve to give a manifold X''. Moreover the line bundle L'' on X'' with $\pi^*L''=L'+[\Gamma^\alpha]$ is ample ($\pi\colon X'\to X''$ is the blow-down morphism).

(5.3.2) Let $\Gamma=F_0+F_1+F_1$. Note that a reasoning similar to that in (5.3.1) holds true also in this case. Of course in this case we need two steps as in (5.3.1) in order to get a manifold which we still call it X''. Moreover the line bundle L'' on X'' with

$\pi^*L'' = L' + [\Gamma^\alpha] + [\Gamma^\beta]$ is ample (Γ^α, Γ^β are two components of Γ and $\pi: X' \to X''$ is the blow-down morphism).

We repeat this process for all reducible fibres Γ_i of p. Since p has only finitely many reducible fibres, after a finite number of steps we get a manifold which, for simplicity, we still call X'', a morphism which again, for simplicity, we still call $\pi: X' \to X''$, and an ample line bundle L'' on X'' such that $\pi^*L'' = L' + \sum_i \Gamma_i^\alpha$, where Γ_i^α stand for the components of Γ_i which are blown down by π. Hence we have a morphism $p': X'' \to C$ such that $p' \circ \pi = p$, and all of the fibres of p' are *irreducible and reduced, and a general* fibre F' of p' is isomorphic to $Q_b\boldsymbol{P}^2$.

Let E be the exceptional curve in F'. It is easy to see that there are no obstructions to deformations of E in X''. Let $\mathscr{H}$ be the irreducible component of the Hilbert scheme of X'' parametrizing flat deformations of E in X''. Let $\mathscr{U}$ be the universal family in $\mathscr{H} \times X''$ and denote $q_2(\mathscr{U})$ by D, where q_2 is the projection of $\mathscr{H} \times X''$ onto the second factor. From the natural identification of the tangent space $T_{\mathscr{H},\alpha}$ of $\mathscr{H}$ at the point α which corresponds to E with $\Gamma(E, N_{E/X''})$, we see that $\dim D = 2$. Also $\dim p'(D) = 1$. Indeed, $\dim p'(D) \leq 1$. On the other hand since E is contained in a general fibre of p', it follows that $\dim p'(D) \geq 1$. Hence $\dim p'(D) = 1$. Moreover, the general fibre of $p'_{|D}$ is isomorphic to $E\,(\cong \boldsymbol{P}^1)$, and each fibre of $p'_{|D}$ is irreducible and reduced, since L'' is ample and $L''_E = \mathcal{O}_E(1)$. Applying [5, (5.4)] to $(D, L''_D, p'_{|D}, C)$ we conclude that D is a $\boldsymbol{P}^1$-bundle over C. Let f be a general fibre of $p'_{|D}: D \to C$. From $N_{f/D} = \mathcal{O}_f$, $\det N_{f/X''} = \mathcal{O}_f(-1)$ and the exact sequence

$$0 \to N_{f/D} \to N_{f/X''} \to N_{D/X'',f} \to 0\,,$$

it follows that $N_{D/X'',f} = \mathcal{O}_f(-1)$. Hence there exists a manifold Y and a birational morphism $\pi': X'' \to Y$ expressing X'' as Y with $C' = \pi'(D)$ blown up, see [16]. Let $\mathscr{L} = \pi'_*(L'' + [D])$ and let $p'': Y \to C$ be the morphism such that $p'' \circ \pi' = p'$. The morphism p'' is such that all of its fibres are *irreducible and reduced*, *a general* fibre F'' of p'' is isomorphic to $\boldsymbol{P}^2$, and $\mathscr{L}_{F''} = \mathcal{O}_{\boldsymbol{P}^2}(3)$.

(5.3.3) CLAIM. *The line bundle $\mathscr{L} = \pi'_*(L'' + [D])$ is relatively ample with respect to p''.*

PROOF OF CLAIM. It suffices to show that $\mathscr{L} + p''^*\mathscr{M}$ is ample for some $\mathscr{M} \in \operatorname{Pic}(C)$. Let $C' = \pi'(D)$. By [6, (5.7)], it suffices to show that $(\mathscr{L} + p''^*\mathscr{M})_{|C'}$ is ample and that $\pi'^*(\mathscr{L} + p''^*\mathscr{M}) - D$ is ample.

Take $\mathscr{M}$, such that $\deg \mathscr{M} = m \gg 0$. Since $\pi'^*(\mathscr{L} + p''^*\mathscr{M}) - D = L'' + p'^*\mathscr{M}$ is ample, we need only to show that $(\mathscr{L} + p''^*\mathscr{M}) \cdot C' > 0$.

To see this, let $A = L'' \cdot D$. Note that A is isomorphic to C' and so $(\mathscr{L} + p''^*\mathscr{M}) \cdot C' = \pi'^*(\mathscr{L} + p''^*\mathscr{M}) \cdot A = (L'' + [D] + p'^*\mathscr{M}) \cdot L'' \cdot D = L''^2 \cdot D + L'' \cdot D^2 + m > 0$ for $m \gg 0$.

Let $Z = \bigcup_{1 \leq i \leq k} \bar{F}_i$, where $\bar{F}_i$ denotes the possible singular fibre of p''. Note that $\bar{F}_i$ is irreducible and reduced. Let $U = Y - Z$. We note that $\mathscr{L}_{F''} = \mathcal{O}_{F''}(3)$, where F'' is a general fibre of p''. Hence over U, $\mathscr{L}_U = 3H$ (modulo line bundles coming from $\operatorname{Pic}(C)$), where $H \in \mathrm{Pic}(U)$, or equivalently, $\mathscr{L}_U = 3H + p''^*(\mathscr{M})$ for some $\mathscr{M} \in \mathrm{Pic}\,(C)$.

(5.3.4) CLAIM. *There exists a line bundle $\bar{H}$ on Y such that $\mathscr{L} = 3\bar{H} + p''^*(\mathscr{N})$ for some $\mathscr{N} \in \mathrm{Pic}(C)$.*

PROOF OF CLAIM. Let $A^1(Y)$ denote the group of cycles of codimension 1 on Y modulo rational equivalence. Let $i\colon U \to Y$ be the inclusion. Note that Y and U are smooth. Hence the morphism $i^*\colon A^1(Y) \to A^1(U)$ is surjective. Moreover $A^1(Y) \cong \mathrm{Pic}(Y)$ and $A^1(U) \cong \mathrm{Pic}(U)$, (e.g. [11, p. 428]). Thus $i^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(U)$ is surjective. Hence $H \in \mathrm{Pic}(U)$ lifts to $\bar{H}$ on Y. Now consider $\mathscr{L} - 3\bar{H} - p''^*(\mathscr{M})$, which is trivial when restricted to U. Since the possible singular fibres of p'' in Z are irreducible we have $\mathscr{L} - 3\bar{H} - p''^*(\mathscr{M}) = p''^*(\mathscr{A})$ for some $\mathscr{A} \in \mathrm{Pic}(C)$. And so $\mathscr{L} = 3\bar{H} + p''^*(\mathscr{N})$, where $\mathscr{N} = \mathscr{M} + \mathscr{A}$.

(5.3.5) REMARK. From (5.3.4) it follows that $\bar{H}$ is relatively ample with respect to p''. Hence the quadruple $(Y, \bar{H}, p'', C)$ is a family of polarized varieties, [5, (5.1)].

Note that $\bar{H}_{F''} = \mathcal{O}_{F''}(1)$ and $\deg \bar{H}_{F''} = 1$, hence $\Delta(F'', \bar{H}) = 0$. Now we use [5, (5.3) and (5.4)] to conclude that Y is isomorphic to $\boldsymbol{P}_C(p''_*(\bar{H}))$. ■

(5.4) THEOREM. *Let X, L, X', L', and $p\colon X' \to C$ be as in* (1.0) *with the additional assumption in* (5.1). *Then there exists a birational morphism $\pi\colon X' \to X''$ to a projective manifold X'' such that the following diagram commutes*:

$$\begin{tikzcd} X' \arrow[rr, "\pi'"] \arrow[dr, "p"'] & & X'' \arrow[dl, "p'"] \\ & C & \end{tikzcd}$$

and that X'' is embedded in a $\boldsymbol{P}^3$-bundle $\boldsymbol{P}$ over C with its restriction to each fibre of $\boldsymbol{P}$ being an irreducible reduced quadric in $\boldsymbol{P}^3$.

PROOF. Here a general fibre of $p\colon X' \to C$ is isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. By (2.3), the possible reducible fibre Γ of p has at most three components. Moreover either Γ is as in Table 1, or $\Gamma = F_0 + F_1 + F_1$. Note also that the morphism p has only *finitely many* reducible fibres Γ_i. As in the proof of (5.3) we see that there exists a birational morphism $\pi\colon X' \to X''$ and an ample line bundle L'' on X'' such that $\pi^* L'' = L' + \sum_i \Gamma_i^\alpha$, where Γ_i^α stand for the components of Γ_i which are blown down by π. Moreover all the fibres of the morphism $p'\colon X'' \to C$ with $p' \circ \pi = p$ are *irreducible and reduced, a general* fibre F' of p' is isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, and $L''_{F'} = \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(2,2) = \mathcal{O}_{F'}(2)$ say.

Let $Z = \bigcup_{1 \le i \le k} \bar{F}_i$, where $\bar{F}_i$ denotes the possible singular fibre of p'. Note that $\bar{F}_i$ are irreducible and reduced. Let $U = X'' - Z$, then over U, $L''_U = 2H$ (modulo line bundles coming from $\mathrm{Pic}(C)$), where $H \in \mathrm{Pic}(U)$, or equivalently, $L''_U = 2H + p'^*(\mathscr{M})$ for some $\mathscr{M} \in \mathrm{Pic}(C)$.

(5.4.1) CLAIM. *There exists a line bundle $\bar{H}$ on X'' such that $L'' = 2\bar{H} + p'^*(\mathscr{N})$ for some $\mathscr{N} \in \mathrm{Pic}(C)$.*

The proof is as in (5.3.4), and so we omit it.

(5.4.2) REMARK. From (5.4.1) it follows that $\bar{H}$ is relatively ample with respect to p'. Hence the quadruple $(X'', \bar{H}, p', C)$ is a family of polarized varieties, see [5, (5.1)].

Note that $\bar{H}_{F'} = \mathcal{O}_{F'}(1)$ and $\deg \bar{H}_{F'} = 2$, hence $\Delta(F', \bar{H}) = 0$. By [5, (5.5)] it follows that we can map X'' into a $\boldsymbol{P}^3$-bundle $\boldsymbol{P}$ over C such that its restriction to each fibre of $\boldsymbol{P}$ is a hyperquadric. ■

(5.5) THEOREM. *Let X, L, X', L', and $p\colon X' \to C$ be as in* (1.0) *with the additional assumption in* (5.2). *Then there exists a birational morphism $\pi''\colon X' \to X''$ to a projective manifold X'' such that the following diagram commutes*:

$$\begin{tikzcd} X' \arrow[rr, "\pi''"] \arrow[dr, "p"'] & & X'' \arrow[dl, "p'"] \\ & C & \end{tikzcd}$$

and that X'' is isomorphic to a $\boldsymbol{P}^3$-bundle $\boldsymbol{P}$ over C.

PROOF. Let F' and F denote the general fibres of p_4 and p_3, respectively. Note that (F', L') is a del Pezzo manifold with $\Delta(F', L') = 1$ and $d = 8$. From [8, (5.6)] it follows that $(F', L') = (\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(2))$. Let Γ be a possible reducible fibre of p_3, let $c = p_3(\Gamma)$ and let $\Sigma = p_4^{-1}(c)$. Choose $X'^3 \in |L'|$ general enough so that $\Sigma \cap X'^3 = \Gamma$ is transverse. It follows from (2.3) that either Γ is as in Table 1 or $\Gamma = F_0 + F_1 + F_1$.

If $\Gamma = F_0 + F_1 + F_1$, then $\Sigma = \Sigma^0 + \Sigma^1 + \Sigma^2$. Say $\Gamma^0 \cong F_0$, and $\Gamma^\alpha \cong F_1$ for $\alpha = 1, 2$. Note that Γ^α is ample on Σ^α. If $\Gamma^\alpha \cong F_1$, then from [1] it follows that Σ^1 is a $\boldsymbol{P}^2$-bundle $\pi'\colon \Sigma^1 \to \boldsymbol{P}^1$ over $\boldsymbol{P}^1$. Here π' denotes the extension of π, where π is as in (0.7). As for Σ^0 we claim that Σ^0 is a hyperquadric. Indeed, $N_{\Gamma^0/X'^3} = -L'_{\Gamma^0}$. Since $\Sigma^0 \cap X'^3 = \Gamma^0$ is transverse in X', it follows that $N_{\Sigma^0/X', \Gamma^0} = N_{\Gamma^0/X'^3} = -L'_{\Gamma^0}$. Since $\mathrm{Pic}(\Sigma^0)$ injects into $\mathrm{Pic}(\Gamma^0)$, $N_{\Sigma^0/X'} = -L'_{\Sigma^0}$. We also know that $(K_{X'} + 2L')_{\Sigma^0} = \mathcal{O}_{\Sigma^0}$. These last two facts together give $K_{\Sigma^0} + L'_{\Sigma^0} = \mathcal{O}_{\Sigma^0}$. Hence Σ^0 is a hyperquadric. Note that Σ^0 and Σ^1 meet on a surface S. Such S must contain $B = F_0 \cap F_1$, a linear $\boldsymbol{P}^1$ (see (2.3)) as ample divisor. Hence $S \cong \boldsymbol{P}^2$. Since S is in Σ^1, it follows that S must be a fibre of $\Sigma^1 \to \boldsymbol{P}^1$. Also S contains the exceptional curve B of F_1, and B cannot be contracted by $\pi\colon F_1 \to \boldsymbol{P}^1$. Hence there are no such Σ's.

If $\Gamma = \Gamma^0 + \Gamma^1$, then $\Sigma = \Sigma^0 + \Sigma^1$. Say $\Sigma^0 \supset \Gamma^0$ and $\Sigma^1 \supset \Gamma^1$, where Γ^0 and Γ^1 are as in Table 1. By [1], note that Σ^1 is a $\boldsymbol{P}^2$-bundle $\pi'\colon \Sigma^1 \to \boldsymbol{P}^1$ over $\boldsymbol{P}^1$. Since a general fibre f of Γ^1 is ample in $\boldsymbol{P}^2$ and $N_{\Gamma^1/X'^4, f} = \mathcal{O}_f(-1)$, we see that $N_{\Sigma^1/X'^4, \boldsymbol{P}^2} = \mathcal{O}_{\boldsymbol{P}^2}(-1)$. Hence Σ^1 can be smoothly blown down on X' to give a manifold X''. Let π'' be the blow-down morphism and let L'' be the line bundle on X'' such that $\pi^* L'' = L' + [\Sigma^1]$. We will show that L'' is ample. Let $\bar{C} = \pi''(\Sigma^1)$.

We will examine only the case $\Sigma \supset \Gamma$, with $\Gamma = \tilde{F}_2 + F_2$, since all the remaining cases use a similar reasoning. Note that in this case $\Sigma^1 \supset F_2$, and $\bar{C} = \pi''(\Sigma^1) = \pi''(F_2) \cong E$, where E is as in (0.7). By noting that $\deg(L' + [\Sigma^1])_{|\Sigma^1} = \deg(L' + [F_2])_{|F_2}$, the same proof of

(5.3.1.a) gives that $L''_{|\bar{C}}$ is ample. Thus L'' is ample on X'', [6, (5.7)].

We repeat this process for all reducible fibres Σ_i of p. Since p has only finitely many reducible fibres, after a finite number of steps we get a manifold which, for simplicity, *we still call* X'', a morphism which again, for simplicity, *we still call* $\pi''\colon X' \to X''$ and an ample line bundle L'' on X'' such that $(\pi'')^*L'' = L' + \sum_i \Sigma_i^{\alpha}$. Hence we have a morphism $p'\colon X'' \to C$ *with* $p' \circ \pi'' = p$. Moreover all the fibres of the morphism p' are *irreducible and reduced*, a *general* fibre F' of p' is isomorphic to $\boldsymbol{P}^3$, and $L''_{F'} = \mathcal{O}_{\boldsymbol{P}^3}(2)$.

Let $\bar{H}$ be the line bundle on X''^3 as in (5.4.1). By the Lefschetz theorem there is a unique extension $\bar{H}'$ of $\bar{H}$ to X''. Since $\bar{H}'_{F'} = \mathcal{O}_{\boldsymbol{P}^3}(\alpha)$ for some $\alpha \in \boldsymbol{Z}$, and $2 = \deg(\bar{H}_F) = \deg((\bar{H}'_{F'})_F) = \mathcal{O}_{\boldsymbol{P}^3}(\alpha) \cdot \mathcal{O}_{\boldsymbol{P}^3}(\alpha) \cdot \mathcal{O}_{\boldsymbol{P}^3}([F]) = \mathcal{O}_{\boldsymbol{P}^3}(\alpha) \cdot \mathcal{O}_{\boldsymbol{P}_3}(\alpha) \cdot \mathcal{O}_{\boldsymbol{P}^3}(2)$, we get $\alpha = 1$. Hence $\bar{H}'_{F'} = \mathcal{O}_{\boldsymbol{P}^3}(1)$. Since $\mathrm{Pic}(F')$ injects into $\mathrm{Pic}(F)$ and $L''_F = 2\bar{H}_F$, we conclude that $L''_{F'} = 2\bar{H}'_{F'}$. Hence $\bar{H}'$ is relatively ample with respect to p'. Now by [5, (5.3) and (5.4)] we conclude that X'' is isomorphic to $\boldsymbol{P}_C(p'_*(\bar{H}'))$. ∎

6. The case $d = 7$.

(6.1) THEOREM. *Let X, L, X', L', and $p\colon X' \to C$ be as in* (1.0). *Then there exists a birational morphism $\pi\colon X' \to X''$, where X'' is a $\boldsymbol{P}^3$-bundle over C.*

PROOF. From (1.5) it follows that $\dim X' = 4$. Let F be a general fibre of $p\colon X' \to C$. Note that $F \cong Q_b\boldsymbol{P}^3$, the blow up of $\boldsymbol{P}^3$ at a point $b \in \boldsymbol{P}^3$. A reasoning similar to that in (5.5) gives that up to smooth blow down of a component of Σ_i (the possible reducible fibres of p), we can always assume that all the fibres of p are irreducible and reduced.

Let E be the exceptional divisor on F over $b \in \boldsymbol{P}^3$. It is easy to see that there are no obstructions to deformations of E in X'. Let $\mathscr{H}$ be the irreducible component of the Hilbert scheme of X' parametrizing flat deformations of E in X'. Let $\mathscr{U}$ be the universal family in $\mathscr{H} \times X'$ and denote $q_2(\mathscr{U})$ by D, where q_2 is the projection of $\mathscr{H} \times X'$ onto the second factor. We claim that D is a $\boldsymbol{P}^2$-bundle over C. From the natural identification of the tangent space $T_{\mathscr{H},\alpha}$ of $\mathscr{H}$ at the point α, which corresponds to E with $\Gamma(E, N_{E/X'})$, we see that $\dim D = 3$. Clearly $\dim p(D) \leq 1$. On the other hand since E is contained in a general fibre of p, it follows that $\dim p(D) \geq 1$. Hence $\dim p(D) = 1$. Moreover, the general fibre of $p_{|D}$ is isomorphic to E ($\cong \boldsymbol{P}^2$), and each fibre of $p_{|D}$ is irreducible and reduced, since L' is ample and $L'_E = \mathcal{O}_E(1)$. Now consider the quadruple (D, L'_D, p_D, C). Such a quadruple is a family of polarized varieties [5, (5.1)]. By [5, (5.4)] we conclude that D is a $\boldsymbol{P}^2$-bundle over C. Let f be a general fibre of $p_{|D}\colon D \to C$. From $N_{f/D} = \mathcal{O}_f$, $\det N_{f/X'} = \mathcal{O}_f(-1)$ and the exact sequence $0 \to N_{f/D} \to N_{f/X'} \to N_{D/X',f} \to 0$, it follows that $N_{D/X',f} = \mathcal{O}_f(-1)$. Hence there exists a manifold X'' and a birational morphism $\pi\colon X' \to X''$ expressing X' as X'' with $C' = \pi(D)$ blown up, see [16]. Let $L'' = \pi_*(L' + [D])$ and let p'$\colon X'' \to C$ be the morphism such that $p' \circ \pi = p$. The morphism p' is such that all of its fibres are *irreducible and reduced*, a *general* fibre F' of p' is isomorphic to $\boldsymbol{P}^3$, and $L''_{F'} = \mathcal{O}_{\boldsymbol{P}^3}(2)$.

The same proof as that in (5.3.3) gives that L'' is relatively ample with respect to

p'. Moreover as in (5.4.1) we can show that there exists a line bundle $\bar{H}$ on X'' such that $L''=2\bar{H}+p'^*(\mathcal{N})$ for some $\mathcal{N}\in\mathrm{Pic}(C)$. Hence (X'', L'', p', C) is a family of polarized varieties. Now use [5, (5.3) and (5.4)] to conclude that $(X'', \bar{H})=(\boldsymbol{P}_C(\mathscr{E}), \xi_{\mathscr{E}})$ where $\mathscr{E}=p'_*(\bar{H})$ and $\xi_{\mathscr{E}}$ is the tautological line bundle on $\boldsymbol{P}_C(\mathscr{E})$. ■

REFERENCES

[1] L. BADESCU, On ample divisors, II, Proceedings of the Week of Algebraic Geometry, Bucarest, 1980, Teubner, Leipzig, 1981, 12–32.

[2] H. D'SOUZA, Threefolds whose hyperplane sections are elliptic surfaces, Pacific J. Math. 134 (1988), 57–78.

[3] M. L. FANIA, Extension of modifications of ample divisors on fourfolds, J. Math. Soc. Japan 36 (1984), 107–120.

[4] M. L. FANIA, Extension of modifications of ample divisors on fourfolds: II, J. Math. Soc. Japan 38 (1986), 285–294.

[5] T. FUJITA, On the structure of polarized varieties with Δ-genera zero, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (1975), 103–115.

[6] T. FUJITA, On the hyperplane section principle of Lefschetz, J. Math. Soc. Japan 32 (1980), 153–169.

[7] T. FUJITA, On polarized varieties of small Δ-genera, Tohoku Math. J. 34 (1982), 319–341.

[8] T. FUJITA, On the structure of polarized manifolds with total deficiency one, I, II, III, J. Math. Soc. Japan 32 (1980), 709–725; 33 (1981), 415–434; 36 (1984), 75–89.

[9] T. FUJITA, On polarized manifolds whose adjoint bundles are not semipositive, Advanced Study in Pure Math. 10 (1987), 167–178.

[10] A. GROTHENDIECK, Elements de géométrie algébrique, Publ. I.H.E.S. 4 (1960), 8 (1961).

[11] R. HARTSHORNE, Algebraic Geometry, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[12] H. HIRONAKA, Smoothing of algebraic cycles of small dimensions, Amer. J. Math. 90 (1968), 1–54.

[13] P. IONESCU, Generalized adjunction and applications, Math. Proc. Cambridge Philos. Soc. 99 (1986), 457–472.

[14] K. KODAIRA, On compact analytic surfaces II, Ann. of Math. 77 (1963), 563–623.

[15] S. MORI, Threefolds whose canonical bundle is not numerically effective, Ann. of Math. 116 (1982), 133–176.

[16] S. NAKANO, On the inverse of monoidal transformations, Publ. Res. Inst. Math. Sci. 6 (1971), 483–501.

[17] A. J. SOMMESE, On the minimality of hyperplane sections of projective threefolds, J. Reine Angew. Math. 329 (1981), 16–41.

[18] A. J. SOMMESE, On the configurations of -2 rational curves on hyperplane sections of projective threefolds. In: Classifications of algebraic and analytic manifolds. Progr. Math. 39 (1980), 465–497.

[19] A. J. SOMMESE, On the adjunction theoretic structure of projective varieties. Proc. of Complex Analysis and Algebraic Geometry Conference, ed. by H. Grauert, Göttingen, 1985 Lecture Notes in Math. 1194, Springer-Verlag, Berlin-Heidelberg-New York, (1986), 175–213.

[20] A. J. SOMMESE AND A. VAN DE VEN, On the adjunction mapping, Math. Ann. 278 (1987), 593–603.

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