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SMOOTH $Sp(2, \boldsymbol{R})$-ACTIONS ON THE 4-SPHERE

Dedicated to Professor Tsuyoshi Watabe on his sixtieth birthday

KAZUO MUKŌYAMA

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Abstract. We construct a one-to-one correspondence between the equivariant diffeomorphism classes of smooth $Sp(2, \boldsymbol{R})$-actions on the standard 4-sphere without fixed points and the equivalence classes of certain pairs of $\boldsymbol{R}$-actions and maps defined on the circle subject to five conditions. Consequently, we show that there are infinitely many smooth $Sp(2, \boldsymbol{R})$-actions on the space without fixed points up to equivariant diffeomorphisms.

Introduction. Asoh [2] classified smooth $SL(2, \boldsymbol{C})$-actions on S^3 topologically, and Uchida [7] classified $SO_0(p, q)$-actions on S^{p+q-1} for $p, q \geq 3$ such that the restricted $SO(p) \times SO(q)$-actions are standard. Each of their actions is characterized by a pair (φ, f) satisfying certain conditions, where φ is a one-parameter transformation group on S^1 and $f: S^1 \to P_1(\boldsymbol{R})$ is a smooth function. The pair, introduced by Asoh and improved by Uchida, is constructed by using the following two facts: first, the restricted maximal compact subgroup action has codimension one principal orbits and secondly, the fixed point set of the action restricted to the principal isotropy subgroup is diffeomorphic to S^1.

In this paper, we shall study smooth $Sp(2, \boldsymbol{R})$-actions on S^4 without fixed points. Since $Sp(2, \boldsymbol{R})$ is simple and contains $U(2)$ as a maximal compact subgroup, it follows that the principal isotropy subgroup of the restricted $U(2)$-action is conjugate to a circle T. Hence the $U(2)$-action has codimension one principal orbits, but the fixed point set of the restricted T-action is diffeomorphic to S^2. Thus we are in a situation slightly different from [2] and [7]. Instead of the pair, we shall construct a triple (S, φ, f) satisfying the conditions defined in §4, where S is diffeomorphic to S^1 in S^2, φ is a one-parameter transformation group on S and $f: S \to P_1(\boldsymbol{R})$ is a smooth map, and show that the triple is finally represented by a pair (φ', f') defined in §6.

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1. Preliminaries.

In this section, we give relevant known facts and basic properties for later convenience.

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1.1. $Sp(2, \boldsymbol{R})$ and $\mathfrak{sp}(2, \boldsymbol{R})$. Let $Sp(2, \boldsymbol{R})$ be the real symplectic group of order 2 defined by

$$Sp(2, \boldsymbol{R})=\{g\in M(4, \boldsymbol{R})\,|\,gJ{}^tg=J\} \qquad \text{for} \quad J=\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

where $M(4, \boldsymbol{R})$ denotes the set of real 4×4 matrices, ${}^tg$ the transposed matrix of g and I_2 the identity matrix of order 2. $Sp(2, \boldsymbol{R})$ contains $U(2)$ as a maximal compact subgroup, which is naturally embedded in $SO(4)$ by

$$U(2)\ni k=k_1+ik_2\longmapsto\begin{pmatrix} k_1 & k_2 \\ -k_2 & k_1 \end{pmatrix}\in SO(4)\,.$$

The Lie algebra $\mathfrak{sp}(2, \boldsymbol{R})$ of $Sp(2, \boldsymbol{R})$ is

$$(1.1)\quad \mathfrak{sp}(2, \boldsymbol{R})=\{A\in M(4, \boldsymbol{R})\,|\,AJ+J{}^tA=O\}$$
$$=\left\{\begin{pmatrix} A_1 & A_2 \\ A_3 & -{}^tA_1 \end{pmatrix}\,\middle|\, A_i \text{ are } 2\times 2 \text{ matrices with } A_2 \text{ and } A_3 \text{ symmetric}\right\}.$$

We can take a basis of $\mathfrak{sp}(2, \boldsymbol{R})$ as follows:

$$E_1=\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},\quad E_2=\begin{pmatrix} 0 & P \\ -P & 0 \end{pmatrix},\quad E_3=\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix},\quad E_4=\begin{pmatrix} 0 & -Q \\ Q & 0 \end{pmatrix},$$
$$E_5=\begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix},\quad E_6=\begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix},\quad E_7=\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},\quad E_8=\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},$$
$$E_9=\begin{pmatrix} P & 0 \\ 0 & -P \end{pmatrix},\quad E_{10}=\begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix},$$

where P, Q, R are 2×2 matrices defined by

$$P=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad Q=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad R=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

respectively. The Lie algebra $\mathfrak{u}(2)$ of $U(2)$ is given by

$$\mathfrak{u}(2)=\langle E_1, E_2, E_3, E_4\rangle\,,$$

where $\langle\ \ \rangle$ denotes the linear subspace generated by the elements in the angle bracket.

1.2. The 5-dimensional standard representation of $Sp(2, \boldsymbol{R})$. We denote the inner product on $M(4, \boldsymbol{R})$ by

$$(X, Y)=\mathrm{trace}(X{}^tY) \qquad \text{for} \quad X, Y\in M(4, \boldsymbol{R})\,,$$

and define an action of $Sp(2, \boldsymbol{R})$ on $M(4, \boldsymbol{R})$ by

$$(1.2)\qquad g\cdot X=gX{}^tg \qquad \text{for} \quad g\in Sp(2, \boldsymbol{R})\,,\quad X\in M(4, \boldsymbol{R})\,.$$

Then $M_{\mathrm{alt}}=\{X\in M(4, \boldsymbol{R}) \mid {}^tX=-X\}$ is an $Sp(2, \boldsymbol{R})$-invariant subspace of $M(4, \boldsymbol{R})$ and has an orthonormal basis

$$e_1=\frac{1}{2}\begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix}, \quad e_2=\frac{1}{2}\begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix},$$

$$e_3=\frac{1}{2}E_2\,, \quad e_4=\frac{1}{2}E_3\,, \quad e_5=\frac{1}{2}E_4\,, \quad e_6=\frac{1}{2}E_1\,.$$

Since $e_6=(1/2)J$, the space $\boldsymbol{R}^5=\langle e_1, e_2, e_3, e_4, e_5\rangle$ is $Sp(2, \boldsymbol{R})$-invariant. We call this space $\boldsymbol{R}^5$ the standard representation space of $Sp(2, \boldsymbol{R})$ and the action (1.2) the standard action of $Sp(2, \boldsymbol{R})$ on $\boldsymbol{R}^5$.

Let $R_1=\{X\in\boldsymbol{R}^5 \mid J\cdot X=-X\}=\langle e_1, e_2\rangle$ and $R_2=\{X\in\boldsymbol{R}^5 \mid J\cdot X=X\}=\langle e_3, e_4, e_5\rangle$. Then $\boldsymbol{R}^5=R_1\oplus R_2$ and we have the following properties:

(1.3) The standard $Sp(2, \boldsymbol{R})$-action on $\boldsymbol{R}^5$ leaves invariant the quadratic form

$$-v_1^2-v_2^2+w_1^2+w_2^2+w_3^2=(J\cdot X, X)\,,$$

for any $X=v_1e_1+v_2e_2+w_1e_3+w_2e_4+w_3e_5$ of $\boldsymbol{R}^5$.

(1.4) R_1 and R_2 are $U(2)$-invariant subspaces. Moreover, $U(2)$ acts on $S(R_i)$ $(i=1, 2)$ transitively and

$$S(R_1)=U(2)/SU(2)\,, \quad S(R_2)=U(2)/T^2\,,$$

where $S(R_i)=\{X\in R_i \mid \|X\|=1\}$. The normal subgroup $U(1)$ of $U(2)$ acts trivially on R_2 and so does $SU(2)$ on R_1.

REMARK 1.5. The above 5-dimensional representation of $Sp(2, \boldsymbol{R})$ is a homomorphism from $Sp(2, \boldsymbol{R})$ onto $SO_0(2, 3)$ and sends J to

$$\begin{pmatrix} -I_2 & 0 \\ 0 & I_3 \end{pmatrix}.$$

1.3. Subgroups and subalgebras. Put $\boldsymbol{R}^3=\langle e_1, e_2, e_3\rangle\subset\boldsymbol{R}^5$. Let $H(a, b, c)$ (resp. $\mathfrak{h}(a, b, c)$) denote the isotropy subgroup (resp. the isotropy subalgebra) of the standard action of $Sp(2, \boldsymbol{R})$ at $ae_1+be_2+ce_3$ for $(a, b, c)\neq(0, 0, 0)$.

LEMMA 1.6. *$\mathfrak{h}(a, b, c)=\langle B_1, B_2, B_3, B_4, B_5, B_6\rangle$, where in the case $c\neq 0$,*

$$B_1=bE_3+aE_4+c(E_7+E_9)\,, \quad B_2=-aE_3+bE_4+c(E_8+E_{10})\,,$$
$$B_3=-bE_3+aE_4+c(E_7-E_9)\,, \quad B_4=aE_3+bE_4+c(E_8-E_{10})\,,$$
$$B_5=E_2\,, \quad B_6=-cE_1+aE_5+bE_6\,,$$

and in the case $c=0$,

$$B_1=b^2E_7+a^2E_9-ab(E_8-E_{10})\,, \quad B_2=a^2E_8+b^2E_{10}-ab(E_7-E_9)\,,$$
$$B_3=E_3\,, \quad B_4=E_4, \quad B_5=E_2\,, \quad B_6=aE_5+bE_6\,.$$

PROOF. Note that $A\in\mathfrak{h}(a, b, c)$ if and only if $AX+X^tA=O$ for $X=ae_1+be_2+ce_3$. Then the result follows by routine calculations. q.e.d.

We define $m(\theta)\in Sp(2, \boldsymbol{R})$ $(\theta\in\boldsymbol{R})$ by

$$m(\theta)=\exp\left(-\frac{\theta}{2}E_5\right)=\left(\cosh\frac{\theta}{2}\right)I-\left(\sinh\frac{\theta}{2}\right)E_5\,, \tag{1.7}$$

and put $M=\{m(\theta)\mid\theta\in\boldsymbol{R}\}$. Then we have

$$m(\theta)\cdot(ae_1+be_2+ce_3)=ae_1+b'e_2+c'e_3\,, \tag{1.8}$$

where $b'=b\cosh\theta+c\sinh\theta$, $c'=b\sinh\theta+c\cosh\theta$. Let T be the maximal torus of $SU(2)$ defined by

$$\left\{\begin{pmatrix} t & 0\\ 0 & \bar{t}\end{pmatrix}\in SU(2)\,\middle|\,|t|=1\right\}.$$

Then we have

$$\mathfrak{t}=\langle E_2\rangle\,,\qquad \mathrm{Lie}(N(T, Sp(2, \boldsymbol{R})))=\langle E_1, E_2, E_5, E_6\rangle\,, \tag{1.9}$$

where $\mathfrak{t}$ and $\mathrm{Lie}(N(T, Sp(2, \boldsymbol{R})))$ denote the Lie algebras of T and $N(T, Sp(2, \boldsymbol{R}))$, respectively, and $N(T, Sp(2, \boldsymbol{R}))$ the normalizer of T in $Sp(2, \boldsymbol{R})$.

LEMMA 1.10. $Sp(2, \boldsymbol{R})=U(2)MH(0, b, c)$.

PROOF. Let $g\in Sp(2, \boldsymbol{R})$ and $g\cdot(be_2+ce_3)=\boldsymbol{v}\oplus\boldsymbol{w}\in R_1\oplus R_2$. By (1.4) there exist $k\in U(2)$ and $\varepsilon_i=\pm1$ $(i=1, 2)$ such that

$$k^{-1}g\cdot(be_2+ce_3)=\varepsilon_1\|\boldsymbol{v}\|e_2+\varepsilon_2\|\boldsymbol{w}\|e_3\,.$$

Since $-\|\boldsymbol{v}\|^2+\|\boldsymbol{w}\|^2=-b^2+c^2$ by (1.3), there exists $\theta\in\boldsymbol{R}$ such that

$$m(\theta)\cdot(be_2+ce_3)=\varepsilon_1\|\boldsymbol{v}\|e_2+\varepsilon_2\|\boldsymbol{w}\|e_3\,.$$

Hence $m(-\theta)k^{-1}g\in H(0, b, c)$. q.e.d.

It should be noted that $\bigcap_{(a,b,c)}\mathfrak{h}(a, b, c)=\mathfrak{t}$ by Lemma 1.6.

LEMMA 1.11. *Let* $\mathfrak{g}$ *be a proper subalgebra of* $\mathfrak{sp}(2, \boldsymbol{R})$ *which contains* $\mathfrak{t}$. *If* $\dim\mathfrak{g}\geq 6$, *then* $\mathfrak{g}=\mathfrak{h}(a, b, c)$ *for some* (a, b, c) *or* $\mathfrak{g}=\mathfrak{h}(a, b, c)\oplus\langle bE_5-aE_6\rangle$ *for* $a^2+b^2=c^2$.

PROOF. By the $\mathrm{Ad}(T)$-action on $\mathfrak{sp}(2, \boldsymbol{R})$, we can first decompose $\mathfrak{sp}(2, \boldsymbol{R})$ into $\mathrm{Ad}(T)$-invariant subspaces as vector spaces:

$$\mathfrak{sp}(2, \boldsymbol{R})=V_1\oplus V_2\oplus V_3\oplus W\,,$$

where $V_1=\langle E_3, E_4\rangle$, $V_2=\langle E_7, E_{10}\rangle$, $V_3=\langle E_8, E_9\rangle$, $W=\langle E_1, E_2, E_5, E_6\rangle$, and $\mathrm{Ad}(T)$ acts trivially on W. Hence we see that

$$\mathfrak{g}=(\mathfrak{g}\cap(V_1\oplus V_2\oplus V_3))\oplus(\mathfrak{g}\cap W)\,.$$

Then the result follows by the Lie algebra structure of $\mathfrak{sp}(2, \boldsymbol{R})$ and the bracket operations on these $\mathrm{Ad}(T)$-invariant subspaces (cf. Uchida [6, §2]). q.e.d.

By Lemma 1.6, we see that $\mathfrak{h}(a, b, c)=\mathfrak{h}(a', b', c')$ if and only if $(a, b, c)=r(a', b', c')$ for $0\neq r\in\boldsymbol{R}$. Hence from now on we rewrite $H(a, b, c)$ (resp. $\mathfrak{h}(a, b, c)$) as $H(a:b:c)$ (resp. $\mathfrak{h}(a:b:c)$), where $(a:b:c)$ is an element of the real projective space $P_2(\boldsymbol{R})$.

Next we denote the element $t(\tau)\in U(2)$ by

$$t(\tau)=\exp\left(-\frac{\tau}{2}E_1\right)=\left(\cos\frac{\tau}{2}\right)I-\left(\sin\frac{\tau}{2}\right)E_1 \qquad \text{for} \quad \tau\in\boldsymbol{R}\,.$$

Then $\{t(\tau)\,|\,\tau\in\boldsymbol{R}\}=U(1)$ is a normal subgroup of $U(2)$ and acts on $\boldsymbol{R}^3$ by

$$t(\tau)\cdot(ae_1+be_2+ce_3)=a'e_1+b'e_2+ce_3\,, \tag{1.12}$$

where $a'=a\cos\tau-b\sin\tau$, $b'=a\sin\tau+b\cos\tau$. The M- and $U(1)$-actions on $\boldsymbol{R}^3$ derive M- and $U(1)$-actions on $P_2(\boldsymbol{R})$, respectively. We call these derived actions on $P_2(\boldsymbol{R})$ the standard actions on $P_2(\boldsymbol{R})$ and use the same notation as for the actions on $\boldsymbol{R}^3$.

2. Standard $Sp(2, \boldsymbol{R})$-action on S^4. We set $S^4=\{X\in\boldsymbol{R}^5\,|\,\|X\|=1\}$. Let Φ_0: $Sp(2, \boldsymbol{R})\times S^4\to S^4$ denote the smooth $Sp(2, \boldsymbol{R})$-action on S^4 defined by

$$\Phi_0(g, X)=\|g\cdot X\|^{-1}g\cdot X \qquad \text{for} \quad g\in Sp(2, \boldsymbol{R}) \text{ and } X\in S^4\,. \tag{2.1}$$

We call Φ_0 the standard action of $Sp(2, \boldsymbol{R})$ on S^4. By (1.4) and (1.7), this action has the following properties:

(2.2) The restricted $U(2)$-action ψ has the principal orbit $U(2)/T$ of codimension one and two singular orbits $U(2)/T^2$ and $U(2)/SU(2)$. Let $F(T)$ be fixed point set of the restricted T-action on S^4. Then $F(T)=\{ue_1+ve_2+we_3\,|\,u^2+v^2+w^2=1\}\subset\boldsymbol{R}^3$ and

$$F(T)/(N(T, U(2))/T)=S^4/U(2)\,,$$

where $N(T, U(2))=T^2\cup E_3T^2$ (cf. Bredon [3, p. 191]).

(2.3) $\qquad S^1=\{ve_2+we_3\,|\,v^2+w^2=1\}$ is an M-invariant subspace of $F(T)$.

By (1.8), Lemma 1.10, (1.12), (2.2) and (2.3), we see that the standard $Sp(2, \boldsymbol{R})$-action on S^4 has three orbits.

REMARK 2.4. By the classification theorem due to Asoh [1], any almost effective smooth $U(2)$-action on S^4 is equivariantly diffeomorphic to one of the following:

(1) the $U(2)$-action ψ defined above.

(2) ψ': $U(2)\times S^4\to S^4$ defined by

$$\psi'(g, (x, y))=(gx, y) \qquad \text{for} \quad (x, y)\in S^4\subset\boldsymbol{C}^2\times\boldsymbol{R}^1.$$

We notice that the action ψ' has two fixed points as singular orbits.

3. Smooth $Sp(2, \boldsymbol{R})$-actions on S^4.

LEMMA 3.1. *Let $\Phi: Sp(2, \boldsymbol{R}) \times N \to N$ be a smooth $Sp(2, \boldsymbol{R})$-action on a smooth 4-manifold N. Then the action Φ has a fixed point if and only if its restricted $U(2)$-action has a fixed point.*

PROOF. Suppose the restricted $U(2)$-action has a fixed point X_0. Let $\mathfrak{g}$ be the isotropy subalgebra at X_0 with respect to the $Sp(2, \boldsymbol{R})$-action. Then $\mathfrak{t} \subset \mathfrak{u}(2) \subset \mathfrak{g}$. On the other hand $\mathfrak{g} = \mathfrak{h}(a:b:c)$, $\mathfrak{h}(a:b:c) \oplus \langle bE_5 - aE_6 \rangle$ or $\mathfrak{sp}(2, \boldsymbol{R})$ by Lemma 1.11. Hence $\mathfrak{g} = \mathfrak{sp}(2, \boldsymbol{R})$. Thus X_0 is a fixed point of the $Sp(2, \boldsymbol{R})$-action. q.e.d

By this lemma and Remark 2.4, we have:

LEMMA 3.2. *Let $\Phi: Sp(2, \boldsymbol{R}) \times S^4 \to S^4$ be a smooth $Sp(2, \boldsymbol{R})$-action on S^4. Then the action Φ has no fixed point if and only if its restricted $U(2)$-action is equivariantly diffemorphic to the action ψ in (2.2).*

In the rest of this paper, we shall study smooth $Sp(2, \boldsymbol{R})$-actions on S^4 without fixed points. By Lemma 3.2, we assume that the restricted $U(2)$-action coincides with ψ. We put

$$
(3.3) \quad \begin{aligned} &G = Sp(2, \boldsymbol{R}), \quad K = U(2), \quad T = \left\{ \begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix} \middle| \, |t| = 1 \right\}, \\ &\psi = \Phi_0 \,|\, K \times S^4, \quad F(T) = \{(u, v, w) = ue_1 + ve_2 + we_3 \,|\, u^2 + v^2 + w^2 = 1\}. \end{aligned}
$$

Let $\Phi: G \times S^4 \to S^4$ be a smooth G-action on S^4 satisfying $\Phi \,|\, (K \times S^4) = \psi$ We shall construct a smooth map $f: F(T) \to P_2(\boldsymbol{R})$ uniquely determined by the condition

$$
(3.4) \qquad \mathfrak{h}(f(X)) \subset \mathfrak{g}_X \quad \text{for} \quad X \in F(T),
$$

where $\mathfrak{g}_X$ is the isotropy subalgebra at X with respect to the given G-action Φ and $\mathfrak{h}(f(X))$ is a subalgebra of $\mathfrak{sp}(2, \boldsymbol{R})$ in Lemma 1.6. Because $\mathfrak{g}_X$ is a proper subalgebra of $\mathfrak{sp}(2, \boldsymbol{R})$ containing $\mathfrak{t}$, there exists a unique $(a:b:c) \in P_2(\boldsymbol{R})$ such that $\mathfrak{h}(a:b:c) \subset \mathfrak{g}_X$ by Lemma 1.11.

Comparing $\mathfrak{h}(a:b:c)$ with the isotropy subalgebra of the restricted K-action, we have

$$
(3.5) \qquad f(X) = (0:0:1) \Leftrightarrow X = (0, 0, \pm 1),
$$

and

$$
(3.6) \qquad f(X) = (a:b:0) \Leftrightarrow X = (u, v, 0).
$$

Let $m(\theta)$ be the matrix defined by (1.7). The set $F(T)$ is invariant under the M-action $\Phi \,|\, (M \times S^4)$, because $m(\theta)$ commutes with each element of T. Let $\varphi: \boldsymbol{R} \times F(T) \to F(T)$ denote the smooth $\boldsymbol{R}$-action on $F(T)$ defined by $\varphi(\theta, X) = \Phi(m(\theta), X)$. Then we see that f is $U(1)$- and M-equivariant by the definitions of f and $\mathfrak{h}(a:b:c)$. Hence we have

(3.7) $$f(\varphi(\theta, X))=m(\theta)\cdot(a:b:c)=(a:b':c') \quad \text{for} \quad f(X)=(a:b:c),$$

where $b'=b\cosh\theta+c\sinh\theta$, $c'=b\sinh\theta+c\cosh\theta$, and also

(3.8) $$f(t(\tau)\cdot X)=t(\tau)\cdot(a:b:c)=(a':b':c) \quad \text{for} \quad f(X)=(a:b:c),$$

where $a'=a\cos\tau-b\sin\tau$, $b'=a\sin\tau+b\cos\tau$. By (3.6), (3.8) and (1.12), we see that the restriction $f\,|\,\{X=(u, v, 0)\in F(T)\}$ is a double covering.

LEMMA 3.9. *The map $f: F(T)\to P_2(\boldsymbol{R})$ is smooth.*

PROOF. Put $f(X)=(a:b:c)$ for $X=(u, v, w)$. Then $\mathfrak{h}(a:b:c)\subset\mathfrak{g}_X$. First assume that $w\neq 0$. Then $c\neq 0$ and we have

$$aE_3-cE_{10}, \quad bE_3+cE_9\in\mathfrak{g}_X,$$

by Lemma 1.6. Hence

$$a\|E_3\|_X^2-c\langle\!\langle E_3, E_{10}\rangle\!\rangle_X=0, \qquad b\|E_3\|_X^2+c\langle\!\langle E_3, E_9\rangle\!\rangle_X=0,$$

where $\langle\!\langle\ ,\ \rangle\!\rangle$ denotes the standard Riemannian metric on S^4 and each element of $\mathfrak{sp}(2, \boldsymbol{R})$ can be considered naturally as a smooth vector field on S^4 (cf. Palais [5, ch. II, Th. II]). Hence $f(X)=(a:b:c)=(a/c:b/c:1)$ is smooth, since $E_3\notin\mathfrak{g}_X$ by Lemma 1.11.

Next assume that $w=0$. Then $c=0$. If $b\neq 0$, then $f(\varphi(\theta, X))$ has a non-vanishing third coordinate for some $\theta\in\boldsymbol{R}$ by (3.7). Hence f is smooth, since $f(X)=m(-\theta)\cdot f(\varphi(\theta, X))$ by (3.7). In the same way we see that f is smooth in a neighborhood of the points X_i $(i=1, 2)$ satisfying $f(X_i)=(1:0:0)$ by (3.8).

Thus f is smooth on $F(T)$. q.e.d.

By (3.5), (3.6), (3.7) and (3.8), the image of f contains $P_2(\boldsymbol{R})-C$, where C is the standard $U(1)$-orbits of the set $\{(0:1:\pm 1)\}$. Hence we see that f is surjective by the continuity of f.

Let $J_i: F(T)\to F(T)$ $(i=1, 2)$ denote the involutions defined by $J_1(u, v, w)=(-u, -v, w)$ and $J_2(u, v, w)=(u, v, -w)$. Then $J_1J_2(X)=-X$ and we have

(3.10) $$f(J_1(X))=f(J_2(X))=(a:b:-c) \quad \text{for} \quad f(X)=(a:b:c),$$

which follows from $J_i(X)=\psi(j_i, X)$ $(i=1, 2)$, where

(3.11) $$j_1=E_1=J\in U(1), \qquad j_2=E_3\in N(T, U(2)).$$

Since $j_i m(\theta)=m(-\theta)j_i$, we have

(3.12) $$J_i(\varphi(\theta, X))=\varphi(-\theta, J_i(X)) \quad (i=1, 2).$$

Put $P_1(\boldsymbol{R})=\{(b:c)=(0:b:c)\in P_2(\boldsymbol{R})\}$ and $S=f^{-1}(P_1(\boldsymbol{R}))$.

LEMMA 3.13. *S is a one-dimensional submanifold of $F(T)$ which is diffeomorphic to a great circle in $F(T)$.*

PROOF. Let f_0 be the restriction of f on $F(T)-\{\pm e_3\}$. Then f_0 maps $F(T)-\{\pm e_3\}$ onto $P_2(\boldsymbol{R})-\{(0:1)\}$. Since f_0 is t-regular on $P_1(\boldsymbol{R})-\{(0:1)\}$ by (3.8), $f_0^{-1}(P_1(\boldsymbol{R})-\{(0:1)\})$ is a one-dimensional submanifold of $F(T)-\{\pm e_3\}$. By (3.5), $S=f_0^{-1}(P_1(\boldsymbol{R})-\{(0:1)\})\cup\{\pm e_3\}$. On the other hand, $\varphi(\theta, \pm e_3)\in S$ by (3.7) and $\varphi(-, \pm e_3)$ gives a local diffeomorphism from a neighborhood of 0 in $\boldsymbol{R}$ to a neighborhood of $\pm e_3$. q.e.d.

Let us denote the restriction of f and φ to S also by f and φ, respectively. By the definition of S, S is J_i-invariant for $i=1, 2$, and f and φ also satisfy the conditions (3.5), (3.6), (3.7), (3.10) and (3.12). Moreover $S-\{\pm e_3\}$ intersects transversely $U(1)$-orbits on $F(T)-\{\pm e_3\}$.

4. Properties of (S, φ, f). Let $S^2=\{X=(u, v, w)\in\boldsymbol{R}^3 \mid u^2+v^2+w^2=1\}$ and $P_1(\boldsymbol{R})=\{(b:c)=(0:b:c)\}\subset P_2(\boldsymbol{R})$. Let (S, φ, f) be a triple of a one-dimensional closed submanifold S of S^2, a smooth $\boldsymbol{R}$-action $\varphi: \boldsymbol{R}\times S\to S$ and a smooth map $f: S\to P_1(\boldsymbol{R})$ satisfying the following conditions:

(i) S is J_i-invariant and diffeomorphic to a great circle containing $\{(0, 0, \pm 1)\}$, where J_i $(i=1, 2)$ are involutions on S^2 defined in §3. $S-\{(0, 0, \pm 1)\}$ intersects each circles $\{(u, v, w)\in S^2 \mid w=c\}$ $(-1<c<1)$ transversely.

(ii) $J_i(\varphi(\theta, X))=\varphi(-\theta, J_i(X))$ $(i=1, 2)$,

(iii) $f(J_1(X))=f(J_2(X))=(b:-c)$ for $f(X)=(b:c)$,

(iv) $f(\varphi(\theta, X))=(b':c')$ for $f(X)=(b:c)$,

where $b'=b\cosh\theta+c\sinh\theta$, $c'=b\sinh\theta+c\cosh\theta$,

(v) $f(X)=(0:1)\Leftrightarrow X=(0, 0, \pm 1)\in S$,

(vi) $f(X)=(1:0)\Leftrightarrow X=(u, v, 0)\in S$.

Let (S, φ, f) be a triple defined above. Let W_{bc} and $P(X)$ denote matrices defined by

$$W_{bc}=(b^2+c^2)^{-1/2}(be_2+ce_3)\,, \qquad P(X)=W_{bc}{}^tW_{bc} \tag{4.1}$$

for $f(X)=(b:c)$, respectively. Let $U(X)$ denote the subset of G defined by

$$U(X)=\{g\in G \mid (g\cdot W_{bc})^t(g\cdot W_{bc})=W_{bc}{}^tW_{bc}\}\,.$$

Then trace $P(X)=1$ and $H(0:b:c)\subset U(X)$. We have

$$(m(\theta)\cdot W_{bc})^t(m(\theta)\cdot W_{bc})=\lambda(\theta, X)P(\varphi(\theta, X))\,, \tag{4.2}$$

where

$$\lambda(\theta, X)=\cosh 2\theta+2bc(b^2+c^2)^{-1}\sinh 2\theta$$

for $f(X)=(b:c)$. By the conditions (v) and (vi), we have

$$K\cap H(0:b:c)=K_X\,, \tag{4.3}$$

where K_X denotes the isotropy subgroup at $X\in S$ for the K-action ψ.

5. Construction of $Sp(2, \boldsymbol{R})$-actions.

5.1. Let (S, φ, f) be a triple of a one-dimensional closed smooth submanifold S of S^2, a smooth $\boldsymbol{R}$-action φ on S and a smooth map $f: S \to P_1(\boldsymbol{R})$ satisfying the six conditions in §4. We construct a smooth G-action on S^4 from the triple (S, φ, f). We use the notation in (3.3) and (3.11).

Let $X \in S$. Then by Lemma 1.10,

$$G = KMH(0:b:c) \tag{5.1}$$

for $f(X) = (b:c)$. Take $(g, p) \in G \times S^4$. Let us choose

$$\begin{aligned} &k \in K, \quad X \in S : \psi(k, X) = p\,, \\ &k' \in K, \quad \theta \in \boldsymbol{R}, \quad u \in H(0:b:c) : gk = k'm(\theta)u\,, \end{aligned} \tag{5.2}$$

and put

$$\Phi(g, p) = \psi(k', \varphi(\theta, X)) \in S^4\,. \tag{5.3}$$

Then we have the following:

PROPOSITION 5.4. *$\Phi: G \times S^4 \to S^4$ of (5.3) is a smooth G-action on S^4 such that $\Phi \mid (K \times S^4) = \psi$.*

In the rest of this section, we shall prove this proposition. The proof is divided into two parts.

5.2. First we shall show that Φ of (5.3) is well-defined and defines a G-action on S^4 such that $\Phi \mid (K \times S^4) = \psi$.

LEMMA 5.5. *Let $f(X) = (b:c)$ and*

$$(*) \qquad km(\theta)u = k'm(\theta')u' \quad \textit{for} \quad k, k' \in K \quad \textit{and} \quad u, u' \in H(0:b:c)\,.$$

Then $\psi(k, \varphi(\theta, X)) = \psi(k', \varphi(\theta', X))$.

To show this, we need the following lemma.

LEMMA 5.6. *In Lemma 5.5, the following hold.*

(1) *If $f(X) = (\varepsilon : 1)$ $(\varepsilon = \pm 1)$, then $(*)$ implies $\theta = \theta'$ and $k^{-1}k' \in T$.*

(2) *If $f(X) = (1:0)$, then $(*)$ implies one of the following:*

(a) *$\theta = \theta' = 0$ and $k^{-1}k' \in SU(2)$, (b) $\theta = \theta' \neq 0$ and $k^{-1}k' \in T$,* (c) *$\theta = -\theta' \neq 0$ and $k^{-1}k' \in j_2 T$.*

(3) *If $f(X) = (0:1)$, then $(*)$ implies one of the following:*

(a) *$\theta = \theta' = 0$ and $k^{-1}k' \in T^2$,* (b) *$\theta = \theta' \neq 0$ and $k^{-1}k' \in T$,* (c) *$\theta = -\theta' \neq 0$ and $k^{-1}k' \in j_1 T$.*

PROOF. We only prove the case (2). Since $km(\theta) \cdot \boldsymbol{e}_2 = k'm(\theta') \cdot \boldsymbol{e}_2$, $\|m(\theta) \cdot \boldsymbol{e}_2\| = \|m(\theta') \cdot \boldsymbol{e}_2\|$. Hence $\theta = \pm \theta'$. If $\theta = \theta'$, then (a) or (b) holds. If $\theta = -\theta' \neq 0$, then

$$j_2k^{-1}k'm(\theta')\cdot e_2=j_2m(\theta)\cdot e_2=m(-\theta)\cdot e_2=m(\theta')\cdot e_2\,.$$

Hence (c) holds. q.e.d.

PROOF OF LEMMA 5.5. In the case $f(X)=(\varepsilon:1)$, we have $\theta=\theta'$ and $k^{-1}k'\in T$ by Lemma 5.6. Put $k^{-1}k'=u$. Then

$$\psi(k', \varphi(\theta', X))=\psi(ku, \varphi(\theta, X))=\psi(k, \varphi(\theta, X))\,.$$

In the case $f(X)=(1:0)$, if the case (c) of Lemma 5.6 (2) holds, put $k^{-1}k'=j_2u$. Then

$$\psi(k', \varphi(\theta', X))=\psi(kj_2u, \varphi(-\theta, X))=\psi(k, \varphi(\theta, J_2(X)))=\psi(k, \varphi(\theta, X))\,,$$

by the condition (ii). The other cases of Lemma 5.6 (2) are clear. In the case $f(X)=(0:1)$, we can also show the result by Lemma 5.6, (3). Now we shall show the equality in the other case. Let $f(X)=(b:c)$, where $bc\neq 0$ and $|b|\neq|c|$. If $|b|<|c|$ (resp. $|b|>|c|$), then by the condition (iv), there exists $\theta_0\in\boldsymbol{R}$ such that $f(\varphi(\theta_0, X))=(0:1)$ (resp. $(1:0)$). Put $X_0=\varphi(\theta_0, X)$. Since $km(\theta-\theta_0)\cdot e_3=k'm(\theta'-\theta_0)\cdot e_3$ (resp. $km(\theta-\theta_0)\cdot e_2=k'm(\theta'-\theta_0)\cdot e_2$), we have

$$\psi(k', \varphi(\theta', X))=\psi(k', \varphi(\theta'-\theta_0, X_0))=\psi(k, \varphi(\theta-\theta_0, X_0))=\psi(k, \varphi(\theta, X))\,.$$

q.e.d.

By Lemma 5.5 and the definition of Φ, we can show that Φ of (5.3) is a well-defined G-action satisfying $\Phi\,|\,(K\times S^4)=\psi$. Since the proof is same as [7, §4], we omit it.

5.3. Next we shall show the smoothness of Φ of (5.3). For $i=1, 2$, define

$$S_i(\Phi)=\{\Phi(g, e_{i+1})\,|\,g\in G\}\,,\quad S_i(\Phi_0)=\{\Phi_0(g, e_{i+1})\,|\,g\in G\}$$

for the G-action Φ of (5.3) and the standard G-action Φ_0, respectively. Then clearly

$$S_1(\Phi_0)=\{\boldsymbol{v}\oplus\boldsymbol{w}\in S(R_1\oplus R_2)\,|\,\|\boldsymbol{v}\|>\|\boldsymbol{w}\|\}\,,$$
$$S_2(\Phi_0)=\{\boldsymbol{v}\oplus\boldsymbol{w}\in S(R_1\oplus R_2)\,|\,\|\boldsymbol{v}\|<\|\boldsymbol{w}\|\}\,.$$

Put $X_0=(u, v, 0)\in S$. Let r_1 (resp. r) be the supremum (resp. the infimum) of the third coordinate of $\{\varphi(\theta, X_0)\,|\,\theta\in\boldsymbol{R}\}$ (resp. $\{\varphi(\theta, e_3)\,|\,\theta\in\boldsymbol{R}\}$) and set $r_2=(1-r^2)^{1/2}$. Then $0<r_i<1$ $(i=1, 2)$ and we see that

$$S_1(\Phi)=\{\boldsymbol{v}\oplus\boldsymbol{w}\in S(R_1\oplus R_2)\,|\,\|\boldsymbol{w}\|<r_1\}\,,$$
$$S_2(\Phi)=\{\boldsymbol{v}\oplus\boldsymbol{w}\in S(R_1\oplus R_2)\,|\,\|\boldsymbol{v}\|<r_2\}\,,$$

by (5.3) and the conditions of (S, φ, f).

LEMMA 5.7. *Φ is smooth on $G\times S_i(\Phi)$ $(i=1, 2)$.*

To show Lemma 5.7, we define diffeomorphisms F_i $(i=1, 2)$. Let $D^3(\delta)=\{\boldsymbol{w}\in R_2\,|\,\|\boldsymbol{w}\|<\delta\}$ and $D^2(\delta)=\{\boldsymbol{v}\in R_1\,|\,\|\boldsymbol{v}\|<\delta\}$ for $\delta>0$. The subset $S_1(\Phi)\cap S$ of S has two components. We denote one of them by S_1. Then there is a smooth real-valued function

h_1 on $(-r_1, r_1)$ such that $f(u, v, w)=(1: h_1(w))$ for $(u, v, w) \in S_1$ by the condition (vi). By the conditions (iv), (vi), h_1 is a diffeomorphism from $(-r_1, r_1)$ onto $(-1, 1)$. Moreover, we have $h_1(-w)=-h_1(w)$, because

$$(1: h_1(-w))=f(u, v, -w)=f(J_2(u, v, w))=(1: -h_1(w)) .$$

Since $w \mapsto w^{-1}h_1(w)$ is a smooth even function, $F_1(\boldsymbol{w})=\|\boldsymbol{w}\|^{-1}(h_1(\|\boldsymbol{w}\|))\ \boldsymbol{w}$ is a diffeomorphism from $D^3(r_1)$ onto $D^3(1)$ (cf. [4, ch. VIII, § 14, Problem 6-c]).

The subset $S_2(\Phi) \cap S$ of S also has two components. We denote by S_2 the one containing the point $\boldsymbol{e}_3$. Then $S_2=\{\varphi(\theta, \boldsymbol{e}_3) \mid \theta \in \boldsymbol{R}\}$. Let $p: S_2 \to D^2(r_2)$ be the map defined by $p(u, v, w)=(u, v)$ and let $L=p(S_2)$. Then there is a smooth real-valued function h_2 on L such that $f(u, v, w)=(h_2(u, v): 1)$ for $(u, v, w) \in S_2$ by the condition (v). We see that h_2 is a diffeomorphism from L onto $(-1, 1)$ satisfying $h_2(-u, -v)=-h_2(u, v)$ and $h_2(p(\varphi(\theta, \boldsymbol{e}_3)))=\tanh \theta$. We put $L_0=h_2^{-1}([0, 1))$. By using the standard $U(1)$-action on $D^2(\delta)$, we define a map $F_2: D^2(r_2) \to D^2(1)$ by

$$F_2(t \cdot \boldsymbol{v})=h_2(\boldsymbol{v})(t \cdot \boldsymbol{e}_2) \qquad \text{for} \quad t \in U(1),\ \boldsymbol{v} \in L_0 .$$

Then F_2 is a diffeomorphism from $D^2(r_2)$ onto $D^2(1)$, because we see that F_2 is regular on $D^2(r_2)$ by the definition of (S, φ, f).

PROOF OF LEMMA 5.7. Let $\alpha: D^2(1) \times S(R_2) \to S_2(\Phi_0)$ be the diffeomorphism defined by

$$\alpha(\boldsymbol{v}, \boldsymbol{w})=(\|\boldsymbol{v}\|^2+1)^{-1/2}(\boldsymbol{v} \oplus \boldsymbol{w}) ,$$

and let $F_2': S_2(\Phi) \to S_2(\Phi_0)$ be the diffeomorphism defined by

$$F_2'(\boldsymbol{v} \oplus \boldsymbol{w})=\alpha(F_2(\boldsymbol{v}), \|\boldsymbol{w}\|^{-1}\boldsymbol{w}) .$$

Since $SU(2)$ acts trivially on R_1 by (1.4), we see that F_2' is K-equivariant. By the definitions of F_2 and h_2, we have

$$F_2'(\varphi(\theta, \boldsymbol{e}_3))=\Phi_0(m(\theta), \boldsymbol{e}_3) \qquad \text{for} \quad \theta \in \boldsymbol{R} .$$

Take $g \in G$ and put $g=km(\theta)u$ for $k \in K$, $u \in H(0:0:1)$. Then

$$\begin{aligned} F_2'(\Phi(g, \boldsymbol{e}_3)) &= F_2'(\psi(k, \varphi(\theta, \boldsymbol{e}_3)))=\Phi_0(k, F_2'(\varphi(\theta, \boldsymbol{e}_3))) \\ &= \Phi_0(k, \Phi_0(m(\theta), \boldsymbol{e}_3))=\Phi_0(g, \boldsymbol{e}_3) . \end{aligned}$$

Hence the diffeomorphism F_2' is G-equivariant. Thus we see that the restriction $\Phi \mid (G \times S_2(\Phi))$ is smooth.

Let $\boldsymbol{v}_0$ be the element of S_1 satisfying $f(\boldsymbol{v}_0)=(1:0)$. Then $S_1=\{\varphi(\theta, \boldsymbol{v}_0) \mid \theta \in \boldsymbol{R}\}$. Let $\eta: S_1(\Phi) \to S(R_1) \times D^3(r_1)$ be the map defined by

$$\eta(\boldsymbol{v} \oplus \boldsymbol{w})=(\|\boldsymbol{v}\|^{-1}\boldsymbol{v}, \boldsymbol{w}) .$$

Then η is a K-equivariant diffeomorphism by (1.4). We denote $D(S_1)=S_1(\Phi) \cap S^2$ and denote by S' the intersection of $D(S_1)$ with the great circle in S^2 through $\boldsymbol{v}_0$ and $\boldsymbol{e}_3$.

Then $\eta(D(S_1))=S(R_1)\times\{we_3\in D^3(r_1)\}$ and $\eta(S')=\{(v_0, we_3)\,|\,|w|<r_1\}\subset S(R_1)\times D^3(r_1)$. Moreover $\eta(S_1)$ is a smooth curve in $\eta(D(S_1))$ such that

$$(*)\qquad (v, we_3)\in\eta(S_1)\Leftrightarrow(v, -we_3)\in\eta(S_1)\,,$$

since $J_2\eta(\varphi(\theta, v_0))=\eta\varphi(-\theta, v_0)$. It follows from the conditions (i), (ii) in §4 and $(*)$ that there exists a smooth map $\sigma:(-r_1, r_1)\to U(1)$ such that $\sigma(w)=\sigma(-w)$ and that the map $\delta:\eta(S')\to\eta(S_1)$, defined by $\delta(X)=(\sigma(w)\cdot v_0, we_3)$ for $X=(v_0, we_3)\in\eta(S')$, is a diffeomorphism. Let $\Delta_1: S(R_1)\times D^3(r_1)\to S(R_1)\times D^3(r_1)$ be the K-equivariant diffeomorphism defined by

$$\Delta_1(v, w)=(t_0\cdot\sigma(\|w\|)^{-1}\cdot v, w)\,,$$

where $t_0\in U(1)$ is the element satisfying $t_0\cdot v_0=e_2$, $\sigma(\|\cdot\|)$ being smooth since σ is an even function. Let $\Delta_2: S_1(\Phi)\to S_1(\Phi)$ be the map defined by

$$\Delta_2(v\oplus w)=(t_0\cdot\sigma(\|w\|)^{-1}\cdot v)\oplus w\,.$$

Since $\Delta_1\eta=\eta\Delta_2$, Δ_2 is a K-equivariant diffeomorphism. Let $\alpha': S(R_1)\times D^3(1)\to S_1(\Phi_0)$ be the diffeomorphism defined by

$$\alpha'(v, w)=(1+\|w\|^2)^{-1/2}(v\oplus w)\,.$$

Put $F_1'=\alpha'\circ(1\times F_1)\circ\eta\circ\Delta_2$. Then $F_1': S_1(\Phi)\to S_1(\Phi_0)$ is K-equivariant and we have

$$F_1'(\varphi(\theta, v_0))=\Phi_0(m(\theta), e_2)\qquad\text{for}\quad\theta\in\boldsymbol{R}\,,$$

by the definitions of F_1 and σ. Hence we see that F_1' is a G-equivariant diffeomorphism in the same way as above and that the restriction $\Phi\,|\,(G\times S_1(\Phi))$ is also smooth.

q.e.d.

Put $X=(u, v, w)\in S$ and $f(X)=(b:c)$. If $w>0$, then $c\neq 0$ and there is a smooth function β on $\{(u, v, w)\in S\,|\,w>0\}$ such that $f(X)=(\beta(X):1)$. We define the subsets S_+ and S_- of S by

$$S_+(\text{resp. } S_-)=\{X=(u, v, w)\in S\,|\,w>0, \beta(X)>0\ (\text{resp. } \beta(X)<0)\}\,.$$

Then each of S_+ and S_- is connected and $J_1(S_+)=S_-$ and $J_1(S_-)=S_+$ by (5.8) and the definition of β.

LEMMA 5.8. *Let* $(\theta, X)\in\boldsymbol{R}\times S_+$ (resp. $\boldsymbol{R}\times S_-$) *be given. Then* $\varphi(\theta, X)\in S_+$ *(resp.* S_-*) if and only if*

$$(5.9)\qquad \{2\beta(X)\cosh 2\theta+(1+\beta(X)^2)\sinh 2\theta\}>0\quad(\textit{resp.}\ <0)\,.$$

PROOF. $f(\varphi(\theta, X))=(\beta(X)\cosh\theta+\sinh\theta:\beta(X)\sinh\theta+\cosh\theta)$ by the condition (iv). Hence if $\varphi(\theta, X)\in S_+$ (resp. S_-), then $(\beta(X)\cosh\theta+\sinh\theta)(\beta(X)\sinh\theta+\cosh\theta)>0$ (resp. <0). Thus we have (5.9). Conversely, if (5.9) holds, then $\varphi(\theta, X)\in S_+\cup J_1J_2(S_+)$ (resp. $S_-\cup J_1J_2(S_-)$). Hence we see that $\varphi(\theta, X)\in S_+$ (resp. S_-) by (5.8) and the connectivity of the orbit of X under the $\boldsymbol{R}$-action φ. q.e.d.

We define

$$D_+ = \{(\theta, X) \in \boldsymbol{R} \times S_+ \mid \varphi(\theta, X) \in S_+\},$$
$$W_+ = \{(km(\theta)u, X) \in G \times S_+ \mid k \in K, (\theta, X) \in D_+, u \in H(0:\beta(X):1)\}.$$

Then D_+ is an open set of $\boldsymbol{R} \times S_+$ and we have the following.

LEMMA 5.10. *For $(g, X) \in G \times S_+$, we have $(g, X) \in W_+$ if and only if*

(5.11) $$\operatorname{trace}(g \cdot W_{\beta(X)1})^t(g \cdot W_{\beta(X)1}) \neq |(1-\beta(X)^2)(1+\beta(X)^2)^{-1}|,$$

where $W_{\beta(X)1}$ is the matrix in (4.1).

PROOF. By Lemma 1.10, for any $g \in G$ we always have a decomposition $g = km(\theta)u$, where $k \in K$, $\theta \in \boldsymbol{R}$ and $u \in H(0:\beta(X):1)$. Hence we see that

(*) $$\operatorname{trace}(g \cdot W_{\beta(X)1})^t(g \cdot W_{\beta(X)1}) = \cosh 2\theta + 2\beta(X)(\beta(X)^2+1)^{-1} \sinh 2\theta$$

by (4.2). We denote the right hand side of this equation by $\alpha(\theta)$.

First suppose $(g, X) \in W_+$. We may assume that $\varphi(\theta, X) \in S_+$. If $\beta(X) = 1$, then $\alpha(\theta) > 0$. Hence (5.11) holds. If $\beta(X) \neq 1$, then $\alpha(\theta)$ has the minimum $|(1-\beta(X)^2)(1+\beta(X)^2)^{-1}|$ if and only if $\tanh 2\theta = -2\beta(X)(1+\beta(X)^2)^{-1}$. Hence (5.11) follows from (5.9).

Next suppose (5.11) holds. Then $\tanh 2\theta \neq -2\beta(X)(1+\beta(X)^2)^{-1}$. Hence $\varphi(\theta, X) \in S_+ \cup S_-$ by Lemma 5.8. If $\varphi(\theta, X) \in S_-$, then we can take a decomposition of g satisfying $\varphi(\theta', X) \in S_+$. We shall show this as follows: By considering the $\boldsymbol{R}$-action φ, $\beta(X) \neq 1$. First suppose $0 < \beta(X) < 1$. Then $f(\varphi(\theta_0, X)) = (0:1)$ for $\theta_0 \in \boldsymbol{R}$ with $\beta(X) + \tanh \theta_0 = 0$. Put $k' = kj_1$, $u' = m(-\theta_0)j_1m(\theta_0)u$ and $\theta' = 2\theta_0 - \theta$. Then we have

$$g = k'm(\theta')u'; \; u' \in H(0:\beta(X):1).$$

Moreover $\varphi(\theta', X) \in S_+$, because

$$\varphi(\theta_0, X) = J_1\varphi(\theta_0, X) = \varphi(-\theta_0, J_1(X))$$

by conditions (ii), (v) and then

$$J_1(\varphi(\theta', X)) = \varphi(\theta-\theta_0, \varphi(-\theta_0, J_1(X))) = \varphi(\theta-\theta_0, \varphi(\theta_0, X)) = \varphi(\theta, X).$$

Next suppose $1 < \beta(X)$. Then $f(\varphi(\theta_0, X)) = (1:0)$ for $\theta_0 \in \boldsymbol{R}$ with $\beta(X) \tanh \theta_0 + 1 = 0$. Now we put $k' = kj_2$, $u' = m(-\theta_0)j_2m(\theta_0)u$ and $\theta' = 2\theta_0 - \theta$. Then we see that $g = k'm(\theta')u'$, $u' \in H(0:\beta(X):1)$ and $\varphi(\theta', X) \in S_+$ in the same way as above. q.e.d.

LEMMA 5.12. *For any $(g, X) \in W_+$, there exist unique $kT \in K/T$ and $\theta \in \boldsymbol{R}$ such that*

(5.13) $$g = km(\theta)u; \; u \in H(0:\beta(X):1), \; (\theta, X) \in D_+.$$

Furthermore, the correspondence $\Delta\colon W_+ \to (K/T) \times D_+$ defined by $\Delta(g, X) = (kT, \theta, X)$ is smooth.

PROOF. First we shall show the uniqueness of the decomposition. If $g=km(\theta)u=k'm(\theta')u'$, then $\|m(\theta)\cdot(0,\beta(X),1)\|=\|m(\theta')\cdot(0,\beta(X),1)\|$. Hence we have $\theta=\theta'$ by Lemma 5.8. This implies $k^{-1}k'\in T$. Next we shall show that Δ is smooth. Let $\theta=\theta(g,X)$ and $\delta(g,X)=kT$ for $(g,X)\in W_+$. We consider the smooth function γ on $W_+\times \boldsymbol{R}$ defined by

$$\gamma(g,X,\theta)=\cosh 2\theta+2\beta(X)(1+\beta(X)^2)^{-1}\sinh 2\theta-\mathrm{trace}((g\cdot W_{\beta(X)1})^t(g\cdot W_{\beta(X)1}))\,.$$

Then $\gamma(g,X,\theta(g,X))=0$ by (5.13) and $(*)$ in the proof of Lemma 5.10. By Lemma 5.8

$$\partial\gamma/\partial\theta=2(\sinh 2\theta+2\beta(X)(1+\beta(X)^2)^{-1}\cosh 2\theta)>0$$

at (g,X,θ) satisfying $\gamma(g,X,\theta)=0$. Thus we see that the function $\theta(g,X)$ is smooth by the implicit function theorem.

Next consider the smooth maps $\delta_1: W_+\to\boldsymbol{R}^5$, $\delta_3: K/T\to\boldsymbol{R}^5$ and the smooth map δ_2 on $(R_1-\{\boldsymbol{0}\})\oplus(R_2-\{\boldsymbol{0}\})$ defined by

$$\delta_1(g,X)=(1+\beta(X)^2)^{-1/2}g\cdot(\beta(X)\boldsymbol{e}_2+\boldsymbol{e}_3)\,,$$
$$\delta_3(kT)=k\cdot(\boldsymbol{e}_2+\boldsymbol{e}_3)\,,$$
$$\delta_2(\boldsymbol{v}\oplus\boldsymbol{w})=\|\boldsymbol{v}\|^{-1}\boldsymbol{v}\oplus\|\boldsymbol{w}\|^{-1}\boldsymbol{w}\,,$$

respectively. Since $\delta_3\delta=\delta_2\delta_1$ and δ_3 is an embedding, δ is smooth. q.e.d.

Now we show that Φ of (5.3) is smooth. Define $W(\Phi)=\{(g,\psi(k,X))\in G\times S^4\mid k\in K, (gk,X)\in W_+\}$. Since W_+ is an open set of $G\times S_+$ by Lemma 5.10, we see that $W(\Phi)$ is an open set of $G\times S^4$. Moreover, we see that $\Phi\mid W(\Phi)$ is smooth, because Δ is smooth by Lemma 5.12. Therefore, Φ is smooth on $G\times S^4$, since $G\times S^4$ is covered by the open sets $G\times\{\Phi(g,\boldsymbol{e}_2)\mid g\in G\}$, $G\times\{\Phi(g,\boldsymbol{e}_3)\mid g\in G\}$ and $W(\Phi)$, and Φ is smooth on each open set.

6. Equivalences and the theorem. Let Φ_i $(i=1,2)$ be smooth G-actions on S^4 without fixed points. Φ_1 and Φ_2 are said to be equivalent if Φ_1 is equivariantly diffeomorphic to Φ_2, i.e., there exists a diffeomorphism $\Psi: S^4\to S^4$ satisfying $\Psi(\Phi_1(g,X))=\Phi_2(g,\Psi(X))$ for any $(g,X)\in G\times S^4$.

Triples (S_i,φ_i,f_i) $(i=1,2)$ satisfying the conditions (i) to (vi) in §4 are said to be equivalent if there exists a diffeomorphism ξ from S_1 onto S_2 such that $\xi J_j=J_j\xi$ for $j=1,2$ and if the following diagram is commutative:

(6.1)
$$\begin{array}{ccccc} \boldsymbol{R}\times S_1 & \xrightarrow{\varphi_1} & S_1 & \overset{f_1}{\searrow} & \\ {\scriptstyle 1\times\xi}\downarrow & & \downarrow{\scriptstyle\xi} & & P_1(\boldsymbol{R}) \\ \boldsymbol{R}\times S_2 & \xrightarrow{\varphi_2} & S_2 & \underset{f_2}{\nearrow} & \end{array}$$

If $S=S^1=\{(0,v,w)\}\subset S^2$, then we simply write the triple (S^1,φ,f) as (φ,f). The pair (φ,f) is characterized by the conditions (ii) to (vi) in §4. The pairs (φ_i,f_i) $(i=1,2)$

are said to be equivalent if the triples (S^1, φ_i, f_i) are equivalent.

THEOREM. *There is a one-to-one correspondence between the equivalence classes of smooth $Sp(2, \boldsymbol{R})$-actions on S^4 without fixed points and the equivalence classes of pairs (φ, f) satisfying the conditions* (ii) *to* (vi) *in* §4.

To prove this theorem we need the following lemmas.

LEMMA 6.2. *Let Φ_i $(i=1, 2)$ be smooth G-actions on S^4 satisfying $\Phi_i \mid (K\times S^4)=\psi$. Then the corresponding triples (S_i, φ_i, f_i) defined in* §3 *are equivalent if Φ_i are equivalent.*

PROOF. *Let $\Psi: S^4 \to S^4$* be a diffeomorphism satisfying $\Psi\circ\Phi_1(g, X)=\Phi_2(g, \Psi(X))$. Then $G_{\Psi(X)}=G_X$ for any $X\in S^4$. Hence $\Psi(S_1)=S_2$ and $f_1=f_2\circ\Psi$. Let $\xi=\Psi \mid S_1$. Then $\xi J_j=J_j\xi$ $(j=1, 2)$ and $\xi(\varphi_1(\theta, X))=\varphi_2(\theta, \xi(X))$. Hence (S_1, φ_1, f_1) and (S_2, φ_2, f_2) are equivalent. q.e.d.

LEMMA 6.3. *Let (S_i, φ_i, f_i) $(i=1, 2)$ be triples satisfying the conditions* (i) *to* (vi) *in* §4. *Then the corresponding G-actions Φ_i $(i=1, 2)$ constructed by* (5.3) *are equivalent if (S_i, φ_i, f_i) are equivalent.*

PROOF. If (S_i, φ_i, f_i) $(i=1, 2)$ are equivalent, then there exists a diffeomorphism $\xi: S_1\to S_2$ such that $\xi J_j=J_j\xi$ $(j=1, 2)$ and the diagram (6.1) is commutative. Since $\psi \mid (K\times S_i): K\times S_i\to S^4$ are smooth, closed and surjective, there exists a K-equivariant homeomorphism Ψ of S^4 satisfying $\Psi(\psi(k, X))=\psi(k, \xi(X))$ for $k\in K$, $X\in S_1$. Now for any $(g, p)\in G\times S^4$, let us choose $\Phi_1(g, p)=\psi(k', \varphi_1(\theta, X))$ as in (5.3), where $p=\psi(k, X)$, $gk=k'm(\theta)u$, $u\in H(0:b:c)$ for $f_1(X)=(b:c)$. Then we have

$$\begin{aligned}\Psi(\Phi_1(g, p))&=\Psi(\psi(k', \varphi_1(\theta, X)))=\psi(k', \xi\varphi_1(\theta, X))\\&=\psi(k', \varphi_2(\theta, \xi(X)))=\Phi_2(g, \Psi(p)).\end{aligned}$$

Thus Ψ is G-equivariant.

Let $S_i(T)=\{X\in S_i \mid f_i(X)\neq(1:0), f_i(X)\neq(0:1)\}$. Since $\psi \mid (K\times S_i(T))$ are open maps and have smooth local sections, Ψ is a diffeomorphism on $S^4-\{B(T^2)\cup B(SU(2))\}$, where $B(T^2)=\{\psi(k, e_3) \mid k\in K\}$ and $B(SU(2))=\{\psi(k, e_2) \mid k\in K\}$ are two singular orbits of the K-action ψ on S^4. On the other hand, open orbits $\{\Phi_i(g, e_3) \mid g\in G\}$ and $\{\Phi_i(g, e_2) \mid g\in G\}$ of the G-actions Φ_i are equivariantly diffeomorphic to $G/H(0:0:1)$ and $G/H(0:1:0)$, respectively. Hence the G-equivariant homeomorphisms $\Psi \mid \{\Phi_1(g, e_i) \mid g\in G\}: \{\Phi_1(g, e_i) \mid g\in G\}\to\{\Phi_2(g, e_i) \mid g\in G\}$ $(i=2, 3)$ are diffeomorphisms. Thus Ψ is a G-equivariant diffeomorphism and hence Φ_1 and Φ_2 are equivalent.
q.e.d.

LEMMA 6.4. *Let Φ be a smooth G-action on S^4 satisfying $\Phi \mid (K\times S^4)=\psi$, and let (S, φ, f) be the triple defined in* §3. *Then the G-action Φ', constructed from (S, φ, f) by* (5.3), *coincides with the given one.*

PROOF. Let $(g, p)\in G\times S^4$, and set $\Phi'(g, p)=\psi(k', \varphi(\theta, X))$ as in (5.3), where

$p=\psi(k, X)$, $gk=k'm(\theta)u$, $u\in H(0:b:c)$ for $f(X)=(b:c)$. Then we have

$$\Phi(g, p)=\Phi(k'm(\theta)uk^{-1}, \psi(k, X))=\psi(k', \varphi(\theta, X))=\Phi'(g, p) .$$

q.e.d.

LEMMA 6.5. *Let (S, φ, f) be a triple satisfying the conditions* (i) *to* (vi) *in* §4, *and let Φ be the G-action on S^4 constructed from (S, φ, f) by* (5.3). *Then the triple (S', φ', f') constructed from Φ coincides with the given one.*

PROOF. Let $X\in S$ and $f(X)=(b:c)$. Then $H(0:b:c)\subset G_X$ by the definition of Φ. Hence $f'(X)=(b:c)$ and we have $S=S'$ by the condition (i). Therefore $f=f'$ and $\varphi=\varphi'$. q.e.d.

LEMMA 6.6. *Let (S, φ, f) be a triple satisfying the conditions* (i) *to* (vi) *in* §4. *Then the triple is equivalent to a pair (φ', f') satisfying the conditions* (ii) *to* (vi) *in* §4.

PROOF. By the condition (i), there exists a J_i-equivariant diffeomorphism $h: S^1\to S$ for $i=1, 2$. We define a smooth $\boldsymbol{R}$-action $\varphi': \boldsymbol{R}\times S^1\to S^1$ and a smooth map $f': S^1\to P_1(\boldsymbol{R})$ by

$$\varphi'(\theta, X)=h^{-1}(\varphi(\theta, h(X))) \quad \text{and} \quad f'(X)=f(h(X)) \qquad \text{for} \quad \theta\in\boldsymbol{R}, X\in S^1 ,$$

respectively. Then we see that the pair (φ', f') satisfies the conditions (ii) to (vi) in §4 and is equivalent to the triple (S, φ, f). q.e.d.

PROOF OF THEOREM. Let Φ be a smooth G-action on S^4 without fixed points. Then Φ is equivalent to a smooth G-action Φ' on S^4 satisfying $\Phi'|(K\times S^4)=\psi$ by Lemma 3.2. Hence we are done by the above lemmas. q.e.d.

7. Examples and Corollary. Let (φ, f) be a pair defined in §6. Then we denote

$$F(\varphi, f)=\{X\in S^1 \mid \varphi(\theta, X)=X \text{ for any } \theta\in\boldsymbol{R}\} .$$

We say that $X_1, X_2\in F(\varphi, f)$ are equivalent if $X_2=J_1^r J_2^s(X_1)$ for some $r, s\in\{0, 1\}$ and we denote the set of the equivalence classes by $\{F(\varphi, f)\}$. Then we have the following lemma by the definition of (φ, f).

LEMMA 7.1. *If $\{F(\varphi, f)\}$ consists of m elements, then the G-action on S^4 constructed from (φ, f) by* (5.3) *consists of* $(2m+1)$ *orbits.*

Now we give two examples.

EXAMPLE 1. Let Φ_0 be the standard G-action on S^4 introduced in §2. Then the triple (S_0, φ_0, f_0) is as follows:

$$S_0=S^1 , \quad f_0(0, v, w)=(v:w) \quad \text{and} \quad \varphi_0(\theta, (0, v, w))=(v'^2+w'^2)^{-1/2}(0, v', w') ,$$

where $v'=v\cosh\theta+w\sinh\theta$, $w'=v\sinh\theta+w\cosh\theta$. Moreover $\{F(\varphi_0, f_0)\}$ consists of one element.

EXAMPLE 2. Let m be a positive integer. Now we shall construct a pair (φ, f) defined in §6 such that $\{F(\varphi, f)\}$ consists of $(2m-1)$ elements. Let L be the unit vector field on S^1 defined by $L_X = -w(\partial/\partial v)_X + v(\partial/\partial w)_X$ for $X=(0, v, w)\in S^1$. We put

$$\rho(x)=\begin{cases} \exp(-1/x^2) & \text{if} \quad x>0\,, \\ 0 & \text{if} \quad x\le 0\,, \end{cases}$$

and $\eta(x)=\rho(\rho(x))$. We define smooth functions $\alpha(x)$ and $\beta(x)$ by

$$\alpha(x)=(\eta(x_1)-\eta(x_2))/(\eta(x_1)+\eta(x_2))\,,$$
$$\beta(x)=x_1^3x_2^3\rho(x_1)^2\rho(x_2)^2/(x_1^3\rho(x_1)^2+x_2^3\rho(x_2)^2)\,,$$

where $x_1=(1+x)/2$, $x_2=(1-x)/2$. Put $\gamma(x)=1/\alpha(x)$ for $x\neq 0$ and

$$\begin{aligned} &a(\tau)=\gamma(\omega_0(\tau))\alpha(\omega_{2m-1}(\tau))\gamma(\omega_{4m-2}(\tau)) && (0<\tau<\pi)\,, \\ &b(\tau)=s\textstyle\sum_{j=0}^{4m-2}(-1)^j\beta(\omega_j(\tau)) && (0\le\tau\le\pi)\,, \end{aligned} \tag{7.2}$$

where $s=\pi/(8m-4)$ and $\omega_j(\tau)=(\tau-2js)/s$ $(0\le j\le 4m-2)$. Then we see that

$$b(\tau)(da/d\tau)=1-a(\tau)^2\,, \tag{7.3}$$

and

$$a(\pi-\tau)=-a(\tau)\,, \qquad b(\pi-\tau)=b(\tau)\,, \tag{7.4}$$

by routine calculations (cf. Asoh [2, §10]).

Put $X=(0, \cos\tau, \sin\tau)\in S^1$ $(-\pi<\tau\le\pi)$ and

$$\begin{aligned} h(X)&=\begin{cases} a(\tau) & \text{if} \quad 0<\tau<\pi\,, \\ -a(-\tau) & \text{if} \quad -\pi<\tau<0\,, \end{cases} \\ g(X)&=\begin{cases} b(\tau) & \text{if} \quad 0\le\tau\le\pi\,, \\ b(-\tau) & \text{if} \quad -\pi<\tau<0\,. \end{cases} \end{aligned} \tag{7.5}$$

Then g and h are smooth functions on S^1 and $S^1-\{(0, \pm 1, 0)\}$, respectively. Also there exists a smooth function $h'(X)$ in a neighborhood U of $(0, \pm 1, 0)$ satisfying $h'(X)=1/h(X)$ for any $X\in U-\{(0, \pm 1, 0)\}$. We define a smooth map $f: S^1\to P_1(\boldsymbol{R})$ by

$$f(X)=\begin{cases} (h(X):1) & \text{if} \quad X\neq(0, \pm 1, 0)\,, \\ (1:h'(X)) & \text{if} \quad X\in U\,. \end{cases}$$

Then conditions (iii), (v) and (vi) in §4 hold by (7.4). Moreover we have

$$g(J_i(X))=g(X)\,, \qquad h(J_i(X))=-h(X)\,,$$

by (7.4) and (7.5). We see by (7.3) that

$$(gL)_Xh=1-h(X)^2 \qquad \text{for} \quad X\in S^1-\{(0, \pm 1, 0)\}\,.$$

Hence the vector field gL defines a smooth $\boldsymbol{R}$-action φ on S^1 which satisfies $h(\varphi(\theta, X)) = (h(X) + \tanh\theta)/(1 + h(X)\tanh\theta)$ and conditions (ii), (iv) in §4 (cf. Asoh [2, Lemma 9.3 and (6.8)]). We also see that $\{F(\varphi, f)\}$ consists of $(2m-1)$ elements.

By Example 2, we have the following:

COROLLARY. *There are infinitely many non-equivalent smooth $Sp(2, \boldsymbol{R})$-actions on S^4 without fixed points.*

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TOKYO METROPOLITAN COLLEGE OF AERONAUTICAL ENGINEERING
52–1, MINAMI-SENJU 8-CHOME
ARAKAWA-KU
TOKYO 116
JAPAN