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MINIMAL MAPS BETWEEN THE HYPERBOLIC DISCS AND GENERALIZED GAUSS MAPS OF MAXIMAL SURFACES IN THE ANTI-DE SITTER 3-SPACE

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Abstract. Problems related to minimal maps are studied. In particular, we prove an existence result for the Dirichlet problem at infinity for minimal diffeomorphisms between the hyperbolic discs. We also give a representation formula for a minimal diffeomorphism between the hyperbolic discs by means of the generalized Gauss map of a complete maximal surface in the anti-de Sitter 3-space.

Introduction. A smooth map $u : M \to N$ is said to be a *minimal map* from a Riemannian manifold (M, g) to another Riemannian manifold (N, h) if the graph $u(M)$ of u is a minimal submanifold of the product Riemannian manifold $(M \times N, g \times h)$. We will see that, at least in 2-dimension, the theory of minimal maps is closely related to the harmonic map theory. It is well-known that the domain of a harmonic map between surfaces can be considered as a Riemann surface not necessary with a metric. However, the target surface has to be a Riemann surface with a metric and the theory is sensitive to the metric structure. In particular, the theory of harmonic maps between surfaces is not symmetric in the domain and the target. Unlike the harmonic maps, the theory of minimal maps is obviously symmetric in the metrics. Recently minimal maps were used in the studies of Lagrangian minimal submanifolds and the construction of middle points in Teichmüller spaces [16], [9].

For general dimensions, we will prove a version of Ruh-Vilms theorem for minimal Gauss maps (Theorems 1.3 and 1.4). As a corollary, we show that the Gauss map of an immersed surface in the Euclidean 3-space $\boldsymbol{E}^3$ is minimal if and only if

$$\frac{H}{1-K} \equiv \text{constant} \quad \text{or} \quad K \equiv 1\,,$$

where H and K are the mean and Gauss curvatures of the surface, respectively.

In the main part of this paper, we will concentrate on the study of minimal maps between the hyperbolic discs. First, we will prove an existence result for the Dirichlet problem at the ideal boundary with quasi-symmetric boundary data of small dilatation. Next, we will study a representation formula for such minimal maps.

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In order to give the representation formula for minimal maps between the hyperbolic discs, we need a Kenmotsu type representation formula in the anti-de Sitter 3-space $\boldsymbol{H}_1^3$ of constant negative curvature -1. One observes that the "generalized" Gauss map $\mathcal{G}$ of a spacelike surface M in $\boldsymbol{H}_1^3$ can be decomposed into two maps $\mathcal{G}_1$ and $\mathcal{G}_2$ from M into the hyperbolic disc $\boldsymbol{D}$. Each component $\mathcal{G}_i$ satisfies a nonlinear partial differential equation of second order, which is determined by the mean curvature H of the spacelike surface. In particular, when H is constant, each component $\mathcal{G}_i$ is a (non-holomorphic) harmonic map into $\boldsymbol{D}$. Then, following a similar argument as in [2], we will give a representation formula for spacelike surfaces in $\boldsymbol{H}_1^3$ in terms of the mean curvature H and a single component of the generalized Gauss map, say $\mathcal{G}_i$. This is what we call a Kenmotsu type representation formula in $\boldsymbol{H}_1^3$ (see Section 3, Remark 3.6). In this formula, a spacelike surface in $\boldsymbol{H}_1^3$ is represented as a solution of an integrable first order differential equation. Moreover, the components of the generalized Gauss map are related to each other via this representation of the surface.

Having this Kenmotsu type representation formula, we describe a minimal map between the hyperbolic discs by means of the generalized Gauss map of a maximal surface in $\boldsymbol{H}_1^3$. This can be considered as an analogue to representing a harmonic map from a surface into the hyperbolic disc as the Gauss map of a constant mean curvature surface in the Minkowski 3-space.

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1. The equations for minimal maps. In this section, we will first derive a system of equations for the minimal map and then use it to prove a version of Ruh-Vilms theorem.

THEOREM 1.1. *A smooth map $u : (M, g) \to (N, h)$ is minimal if and only if*

$$\mathrm{tr}_{\tilde{g}(u)} \nabla du = 0\,,$$

where $\tilde{g}(u) = g + u^*h$ and $\mathrm{tr}_{\tilde{g}(u)}$ stands for the trace with respect to $\tilde{g}(u)$. In local coordinates x^i on M and u^α on N, it can be written as

$$\sum_{i,j} \tilde{g}^{ij}(u) \left(\frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - {}^M\Gamma_{ij}^k \frac{\partial u^\alpha}{\partial x^k} + {}^N\Gamma_{\beta\gamma}^\alpha (u) \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} \right) = 0\,, \tag{1.1}$$

where ${}^M\Gamma_{ij}^k$ and ${}^N\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols for the Levi-Civita connections of g and h, respectively.

PROOF. We first observe that the graph of u in $M \times N$ is isometric to $(M, \tilde{g}(u))$. Or equivalently, the map from $(M, \tilde{g}(u))$ into $(M \times N, g \times h)$ defined by $x \mapsto (x, u(x))$ is an isometric embedding. It is well-known that an isometric immersion is a minimal immersion if and only if it is harmonic (cf. [6]). Therefore, we conclude that u is a minimal map if and only if $x \mapsto (x, u(x))$ is a harmonic map with respect to the corresponding metrics. That is, u must satisfy

$$
(1.2) \qquad \left\{ \begin{aligned} &\Delta_{\tilde g(u)} x^i + \sum_{j,k} \tilde g^{jk}(u)^M\Gamma^i_{jk} = 0\,, \\ &\Delta_{\tilde g(u)} u^\alpha + \sum_{i,j} \tilde g^{ij}(u)^N\Gamma^\alpha_{\beta\gamma}(u) u^\beta_i u^\gamma_j = 0\,, \end{aligned} \right.
$$

where $u^\alpha_i = \partial u^\alpha / \partial x^i$. It is easy to see that the first equation in (1.2) is equivalent to

$$
(1.3) \qquad \sum_{i,j} \tilde g^{ij}(u)^M\tilde\Gamma^k_{ij}(u) = \sum_{i,j} \tilde g^{ij}(u)^M\Gamma^k_{ij} \quad \text{for all} \quad k\,,
$$

where ${}^M\tilde\Gamma^k_{ij}$ are the Christoffel symbols for the Levi-Civita connection of $\tilde g(u)$. Putting this into the second equation in (1.2), we see that the equation (1.1) is satisfied by a minimal map.

Conversely, we need to show that (1.1) implies both equations in (1.2). In fact, we only need to show that (1.1) implies (1.3) which is equivalent to the first equation of (1.2). Then (1.1) and (1.3) together imply the second equation in (1.2).

To simplify the calculation, we choose normal coordinates x^i and u^α centered at $p \in M$ and $u(p) \in N$ respectively, and use u^α_{ij} to denote the covariant derivatives of u^α_i. Then we need to show that at the center p, $\tilde g^{ij}(u)^M\tilde\Gamma^k_{ij}(u) = 0$. It is easy to find, at p,

$$
\frac{\partial \tilde g_{ij}(u)}{\partial x^k} = \tilde g_{ij,k}(u) = (g_{ij} + u^\alpha_i u^\alpha_j)_{,k} = \sum_\alpha (u^\alpha_{ik} u^\alpha_j + u^\alpha_i u^\alpha_{jk})\,.
$$

Therefore, at p,

$$
\begin{aligned}
\sum_{i,j} \tilde g^{ij}(u)^M\tilde\Gamma^k_{ij}(u) &= \sum_{i,j,l} \frac{1}{2} \tilde g^{ij}(u) \tilde g^{kl}(u) (\tilde g_{lj,i}(u) + \tilde g_{il,j}(u) - \tilde g_{ij,l}(u)) \\
&= \sum_{i,j,l} \frac{1}{2} \tilde g^{ij}(u) \tilde g^{kl}(u) \sum_\alpha ((u^\alpha_{li} u^\alpha_j + u^\alpha_l u^\alpha_{ji}) + (u^\alpha_{ij} u^\alpha_l + u^\alpha_i u^\alpha_{lj}) - (u^\alpha_{il} u^\alpha_j + u^\alpha_i u^\alpha_{jl})) \\
&= \sum_\alpha \left(\sum_l \tilde g^{kl}(u) u^\alpha_l \right) \left(\sum_{i,j} \tilde g^{ij}(u) u^\alpha_{ij} \right)\,.
\end{aligned}
$$

The equation (1.1) says exactly that $\sum_{i,j} \tilde g^{ij}(u) u^\alpha_{ij} = 0$ at p for all α. Hence, we have proved $\sum_{i,j} \tilde g^{ij}(u)^M\tilde\Gamma^k_{ij}(u) = 0$ at any point $p \in M$. This completes the proof of the theorem. □

REMARK 1.2. From Theorem 1.1, it is easy to see that totally geodesic maps are minimal maps. Also, a conformal map is minimal if and only if it is harmonic. In particular, a minimal isometric immersion in the sense that its mean curvature vector vanishes coincides with a minimal map in our sense. So there will be no confusion when we refer to a minimal isometric immersion.

We will now prove a Ruh-Vilms type theorem concerning hypersurfaces with minimal Gauss maps. The Ruh-Vilms theorem [15] says that the Gauss map of a submanifold in the Euclidean N-space $\boldsymbol{E}^N$ is harmonic if and only if its mean curvature vector is parallel.

This theorem and its counterpart for spacelike submanifolds in Minkowski spaces [13], [7], [5] are useful in constructing harmonic maps [5], [19], [20]. In the following, we will find the necessary and sufficient condition for the Gauss map of an isometric immersion into a Euclidean or Minkowski space to be minimal.

Let us demonstrate the Euclidean case. We will follow the calculation as in the notes by Eells-Lemaire [6]. Let $f : (M^n, g) \to \boldsymbol{E}^N$ be an isometric immersion and $\mathcal{G} : (M, g) \to G(N, n)$ be the Gauss map associated to f, where $G(N, n)$ denotes the Grassmann manifold of n-planes in $\boldsymbol{E}^N$ with the standard Riemannian metric. For any point $x_0 \in M$, we consider normal coordinates x^i centered at x_0. Then $\mathcal{G}$ can be identified as the map

$$x \mapsto \frac{\partial f}{\partial x^1}(x) \wedge \cdots \wedge \frac{\partial f}{\partial x^n}(x),$$

and we have

$$d\mathcal{G}\left(\frac{\partial}{\partial x^i}\right) = \sum_{j=1}^{n} \frac{\partial f}{\partial x^1} \wedge \cdots \wedge \frac{\partial f}{\partial x^{j-1}} \wedge \frac{\partial^2 f}{\partial x^i \partial x^j} \wedge \frac{\partial f}{\partial x^{j+1}} \wedge \cdots \wedge \frac{\partial f}{\partial x^n}.$$

Since f is an isometric immersion, we can choose an orthonormal basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_N\}$ of $\boldsymbol{E}^N$ such that

$$e_i = df\left(\frac{\partial}{\partial x^i}\right)\Bigg|_{x_0} = \frac{\partial f}{\partial x^i}(x_0) \quad \text{for all} \quad i = 1, \dots, n.$$

Then, at x_0,

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \nabla df\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_{r=n+1}^{N} h^r_{ij} e_r,$$

where h^r_{ij} are the components of the second fundamental form $\Pi_f = \nabla df$ of f. Putting it into the formula for $d\mathcal{G}$, we see that at x_0,

$$d\mathcal{G}\left(\frac{\partial}{\partial x^i}\right) = \sum_{r=n+1}^{N} \sum_{j=1}^{n} h^r_{ij} e_1 \wedge \cdots \wedge e_{j-1} \wedge e_r \wedge e_{j+1} \wedge \cdots \wedge e_n. \tag{1.4}$$

On the other hand, for any vector field X,

$$\nabla df\left(X, \frac{\partial}{\partial x^i}\right) = \nabla_X df\left(\frac{\partial}{\partial x^i}\right) = e^*_j(X) \nabla_{\frac{\partial}{\partial x^j}} df\left(\frac{\partial}{\partial x^i}\right)$$

and hence at x_0,

$$\nabla df\left(\frac{\partial}{\partial x^i}\right) = \sum_{r=n+1}^{N} \sum_{j=1}^{n} h^r_{ij} e^*_j \otimes e_r.$$

Note that we can identify sections in $\Gamma(T^*M \otimes \mathcal{G}^{-1}TG(N, n))$ with sections in $\Gamma(\otimes^2 T^*M \otimes TM^\perp)$ via the identification

$$e_1 \wedge \cdots \wedge e_{j-1} \wedge e_r \wedge e_{j+1} \wedge \cdots \wedge e_n \leftrightarrow e^*_j \otimes e_r,$$

where $TM^\perp$ is the normal bundle on M. Under this identification, we have

$$d\mathcal{G} = \nabla df$$

and hence the second fundamental form of $\mathcal{G}$ is given by

$$\nabla d\mathcal{G} = \nabla^{\perp}\nabla df\,,$$

where $\nabla^{\perp}$ is the connection on $\otimes^2 T^*M \otimes TM^{\perp}$. Furthermore, if we regard ∇df as a section of $\otimes^2 T^*M \otimes f^{-1}T\boldsymbol{E}^N$, we have

$$\nabla d\mathcal{G} = (\nabla\nabla df)^{\perp}\,,$$

where $(\nabla\nabla df)^{\perp}$ stands for the projection on $TM^{\perp}$ of $\nabla\nabla df$.

By the last formula, we conclude that $\mathcal{G}$ is minimal if and only if

$$\mathrm{tr}_{\tilde{g}(\mathcal{G})}\nabla d\mathcal{G} = (\mathrm{tr}_{\tilde{g}(\mathcal{G})}\nabla_*\nabla_* df)^{\perp} = 0\,,$$

where $\mathrm{tr}_{\tilde{g}(\mathcal{G})}\nabla_*\nabla_*$ indicates that the trace is taken on the two marked vectors with respect to the metric $\tilde{g}(\mathcal{G})$. In terms of local normal coordinates, this is

$$\tilde{g}^{ij}(\mathcal{G})h^r_{ijk} = 0 \quad \text{for all} \quad k = 1, \dots, n \text{ and } r = n+1, \dots, N\,, \tag{1.5}$$

where h^r_{ijk} are the components of the covariant derivative of Π_f. Hence, the above result can be written as

THEOREM 1.3. *Let $f : (M^n, g) \to \boldsymbol{E}^N$ be an isometric immersion. Then the Gauss map $\mathcal{G}$ is minimal if and only if*

$$\mathrm{tr}_{\tilde{g}(\mathcal{G})}\nabla\Pi_f = 0\,.$$

The difficulty in reading off information from Theorem 1.3 is due to the fact that there is no good formula for the inverse of the metric $\tilde{g}(\mathcal{G})$. With respect to the same local coordinates, the metric $\tilde{g}(\mathcal{G}) = (\tilde{g}_{ij}(\mathcal{G}))$ is given by, at x_0,

$$\tilde{g}_{ij}(\mathcal{G}) = \delta_{ij} + \sum_{r=n+1}^{N}\sum_{k=1}^{n} h^r_{ik}h^r_{jk}\,. \tag{1.6}$$

To see this, we note that the standard Riemannian metric on $G(N, n)$ is defined by requiring $\{e_1 \wedge \cdots \wedge e_{j-1} \wedge e_r \wedge e_{j+1} \wedge \cdots \wedge e_n\}_{j=1,\dots,n;r=n+1,\dots,N}$ be orthonormal vectors. Hence by (1.4), the (i, j)-component of the pull-back metric at x_0 is given by

$$\left\langle d\mathcal{G}\left(\frac{\partial}{\partial x^i}\right), d\mathcal{G}\left(\frac{\partial}{\partial x^j}\right)\right\rangle = \sum_{r=n+1}^{N}\sum_{k=1}^{n} h^r_{ik}h^r_{jk}\,.$$

By adding the metric on M, we get the formula (1.6). The formula (1.6) for $\tilde{g}^{ij}(\mathcal{G})$ is complicated due to the noncommutativity of the matrices $\{(h^r_{ij})\}_{n+1\leq r\leq N}$.

However, in codimension 1, this difficulty does not occur and we have the following

THEOREM 1.4. *Let $f : (M^n, g) \to \boldsymbol{E}^{n+1}$ be an isometric immersion. Then the Gauss map $\mathcal{G} : (M^n, g) \to \boldsymbol{S}^n$ is minimal if and only if*

$$\mathrm{tr}_g \tan^{-1}\Pi_f \equiv \text{constant.}$$

PROOF. Note that at the center of a normal coordinate system, $\mathrm{tr}_{\tilde{g}(\mathcal{G})}\nabla\Pi_f$ can be represented as

$$\mathrm{tr}((I+\Pi^2)^{-1}\nabla\Pi)\,,$$

where $\Pi = \Pi_f = (h_{ij})$. We would like to formally integrate the above expression.

To do so, consider the matrix $\tan^{-1}\Pi$ obtained by formally replacing x by Π in the Taylor expansion of $\tan^{-1}x$. That is,

$$\tan^{-1}\Pi = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\Pi^{2k+1}\,,$$

for those Π with all eigenvalues having absolute value smaller than 1. It is easy to see that we can analytically extend the domain of definition of $\tan^{-1}\Pi$ to all real symmetric matrices. In general, $\nabla\tan^{-1}\Pi \neq (I+\Pi^2)^{-1}\nabla\Pi$. In fact, for those Π with absolute values of all eigenvalues smaller than 1, we have

$$\nabla\tan^{-1}\Pi = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}((\nabla\Pi)\Pi^{2k} + \Pi(\nabla\Pi)\Pi^{2k-1} + \cdots + \Pi^{2k}(\nabla\Pi))\,.$$

However, using the fact that $\mathrm{tr}(A_1A_2\cdots A_k) = \mathrm{tr}(A_kA_1\cdots A_{k-1})$, we see that

$$\begin{aligned}\mathrm{tr}\,\nabla\tan^{-1}\Pi &= \sum_{k=0}^{\infty}(-1)^k\mathrm{tr}(\Pi^{2k}(\nabla\Pi))\\ &= \mathrm{tr}\left(\left(\sum_{k=0}^{\infty}(-1)^k\Pi^{2k}\right)(\nabla\Pi)\right)\\ &= \mathrm{tr}((I+\Pi^2)^{-1}\nabla\Pi)\,.\end{aligned}$$

Therefore,

$$\nabla\,\mathrm{tr}\tan^{-1}\Pi = \mathrm{tr}((I+\Pi^2)^{-1}\nabla\Pi)\,.$$

It can be shown that the above formula is true for all real symmetric Π. Therefore, we have proved that the Gauss map $\mathcal{G}$ is minimal if and only if

$$\mathrm{tr}\tan^{-1}\Pi_f \equiv \text{constant.}$$

□

COROLLARY 1.5. *If M^2 is an immersed surface in $\boldsymbol{E}^3$, then the Gauss map of M is minimal if and only if*

$$\frac{H}{1-K} \equiv \text{constant} \quad \textit{or} \quad K \equiv 1\,,$$

where H and K are the mean and Gauss curvatures of M in $\boldsymbol{E}^3$, respectively.

PROOF. By Theorem 1.4, we see that the Gauss map is minimal if and only if

$$\mathrm{tr}\tan^{-1}\Pi_M \equiv \text{constant,}$$

where Π_M is the second fundamental form of M. Let λ_i, $i = 1, 2$, be the principal curvatures of M. Then $\tan^{-1}\Pi_M$ has eigenvalues equal to $\tan^{-1}\lambda_1$ and $\tan^{-1}\lambda_2$, and hence

$$\operatorname{tr}\tan^{-1}\Pi_M = \tan^{-1}\lambda_1 + \tan^{-1}\lambda_2 = \tan^{-1}\frac{\lambda_1+\lambda_2}{1-\lambda_1\lambda_2} = \tan^{-1}\frac{2H}{1-K}.$$

This completes the proof of the corollary. □

Similar results to Theorems 1.3 and 1.4 also hold for isometric immersions into Minkowski spaces. In particular, we have

COROLLARY 1.6. *If M^2 is a spacelike immersed surface in the Minkowski 3-space $\boldsymbol{L}^3$, then the Gauss map of M is minimal if and only if*

$$\frac{H}{1+K} \equiv \text{constant} \quad or \quad K \equiv -1,$$

where H and K are the mean and Gauss curvatures of M in $\boldsymbol{L}^3$, respectively.

PROOF. Note that $K = -\lambda_1\lambda_2$ for a spacelike surface in $\boldsymbol{L}^3$. □

2. Existence for minimal diffeomorphisms between the hyperbolic discs. In this section, we prove an existence result for minimal diffeomorphisms between the hyperbolic discs. Let Σ and $\hat{\Sigma}$ be two Riemann surfaces, and z and w be their isothermal coordinates. Fix a metric $\sigma^2(w)|dw|^2$ on $\hat{\Sigma}$. For any harmonic map $u : \Sigma \to \hat{\Sigma}$, the Hopf differential $\Phi(u) = \phi(u)dz^2$ of u on Σ is defined by

$$\Phi(u) = \sigma^2(u)u_z\bar{u}_z dz^2.$$

We need the following lemma which is due to R. Schoen in [16] implicitly.

LEMMA 2.1. *Let Σ_i, $i = 1, 2$, be two Riemannian 2-manifolds. Then a map $u : \Sigma_1 \to \Sigma_2$ is a minimal map if and only if there exist a Riemann surface Σ homeomorphic to Σ_1 and two harmonic maps u_i from Σ to Σ_i such that u_1 is a diffeomorphism, $u = u_2 \circ u_1^{-1}$, and*

$$\Phi(u_1) + \Phi(u_2) \equiv 0.$$

PROOF. If $u : \Sigma_1 \to \Sigma_2$ is a minimal map, then the induced metric determines a conformal structure on the graph of u and hence gives rise to a Riemann surface Σ. By the definition of minimal maps, the inclusion of Σ into the product manifold $\Sigma_1 \times \Sigma_2$ with product metric is a minimal immersion and hence is a conformal harmonic map. Hence, the projections u_i are harmonic maps from Σ to Σ_i, respectively, such that $u = u_2 \circ u_1^{-1}$. Simple calculation shows that the (2, 0)-part of the pullback metric on Σ of the inclusion is given by $\Phi(u_1) + \Phi(u_2)$. Therefore, one concludes that

$$\Phi(u_1) + \Phi(u_2) \equiv 0.$$

Conversely, if there exist Σ and u_i as in the statement, then we can consider the map $F : \Sigma \to \Sigma_1 \times \Sigma_2$ given by $z \mapsto (u_1(z), u_2(z))$. It is easy to verify that F is a conformal harmonic map and hence the image of F is a minimal submanifold in $\Sigma_1 \times \Sigma_2$. Since u_1

is a diffeomorphism, the graph of $u = u_2 \circ u_1^{-1}$ is exactly identical to the image of F as submanifold. Therefore u is a minimal map. □

Now we consider the Dirichlet problem at infinity of the hyperbolic disc for minimal diffeomorphisms analogous to that for harmonic maps. From the corresponding results for harmonic maps [18], [19], it seems reasonable to consider a special class of boundary data, namely quasi-symmetric functions. That is, for any quasi-symmetric function $\varphi : \boldsymbol{S}^1 \to \boldsymbol{S}^1$, we would like to find a quasi-conformal map $u : \boldsymbol{D} \to \boldsymbol{D}$ such that u have the boundary data φ and u is a minimal map with respect to the Poincaré metric $ds_p^2 = 4|dz|^2/(1 - |z|^2)^2$ on the unit disc $\boldsymbol{D} = \{z \in \boldsymbol{C} \mid |z| < 1\}$. We would like to point out that the existence of the corresponding problem for harmonic maps has not yet been solved completely, although there are many partial results. We will use the method in [18] to obtain the following similar result for minimal diffeomorphisms.

THEOREM 2.2. *Any quasi-symmetric function with sufficiently small dilatation has a quasi-conformal minimal diffeomorphic extension to the hyperbolic disc.*

PROOF. As in [18], we consider a map F from $\mathbf{BQD}(\boldsymbol{D})$ the space of bounded (with respect to the Poincaré metric) holomorphic quadratic differentials on $\boldsymbol{D}$ to $\mathbf{BQD}(\boldsymbol{D}^*)$ the space of bounded (with respect to the Poincaré metric) holomorphic quadratic differentials on $\boldsymbol{D}^* = \boldsymbol{C} \setminus \bar{\boldsymbol{D}}$. The map F is constructed as follows: Given any $\Phi \in \mathbf{BQD}(\boldsymbol{D})$, we solve uniquely two quasi-conformal harmonic diffeomorphisms u_1 and u_2 from $\boldsymbol{D}$ onto $\boldsymbol{D}$ fixing 1, i, and -1 such that

$$\Phi(u_1) = \Phi \quad \text{and} \quad \Phi(u_2) = -\Phi\,.$$

The existence and uniqueness of u_1 and u_2 are ensured by the result of Tam-Wan [17]. Then $u = u_2 \circ u_1^{-1}$ is a quasi-conformal map from $\boldsymbol{D}$ onto $\boldsymbol{D}$ fixing 1, i, and -1. Hence, the complex characteristic of u determines a class in the universal Teichmüller space. Or it is equivalent to say that $u|_{\partial \boldsymbol{D}}$ is a normalized quasi-symmetric function, hence is an element in the universal Teichmüller space. Therefore, it corresponds to an element in $\Psi \in \mathbf{BQD}(\boldsymbol{D}^*)$ via the Bers embedding. Then we define $F(\Phi) = \Psi$. As in [18], we find that the complex characteristics of the harmonic maps u_1 and u_2 are given by

$$\mu_1(z) = \frac{(1 - |z|^2)^2}{4}\overline{\phi(z)}e^{-2w(z)} \quad \text{and} \quad \mu_2(z) = -\frac{(1 - |z|^2)^2}{4}\overline{\phi(z)}e^{-2w(z)}\,, \tag{2.1}$$

where ϕ is the coefficient of Φ, i.e., $\Phi = \phi(z)dz^2$, and w is the unique solution of the equation

$$\Delta_{ds_p^2} w = e^{2w} - \|\phi\|^2 e^{-2w} - 1\,,$$

where $\|\phi\|(z) = (1 - |z|^2)^2|\phi(z)|/4$, such that $4e^{2w}(1 - |z|^2)^{-2}|dz|^2$ defines a complete metric on the unit disc. Then, from (2.1), the complex characteristic of u, denoted by $\mu(\Phi)$, is given by

$$\mu(\Phi)(\zeta) = \frac{\mu_2(z)-\mu_1(z)}{1-\mu_2(z)\overline{\mu_1(z)}}\frac{u_{1,z}(z)}{|u_{1,z}(z)|} \tag{2.2}$$
$$= -2\frac{\rho(z)^{-2}\overline{\phi(z)}e^{-2w(z)}}{1+\rho(z)^{-4}|\phi(z)|^2e^{-4w(z)}}\frac{u_{1,z}(z)}{|u_{1,z}(z)|},$$

where $z = u_1^{-1}(\zeta)$ and $\rho(z) = 2/(1-|z|^2)$.

By [18], w and hence u_1 depend real-analytically on Φ. Therefore, $\mu(\Phi)$ is also real analytic in Φ. It is then easy to see from (2.2) that the differential of $\mu(\Phi)$ in the direction of $\Psi = \psi(\zeta)d\zeta^2$at $\Phi = 0$ is given by

$$D\mu_0(\Psi)(\zeta) = -2\rho^{-2}(\zeta)\overline{\psi(\zeta)}.$$

Here we have used the fact that $w = 0$, $u_1(z) = z$ and hence $u_{1,z} = 1$ for $\Phi = 0$. Then we conclude, as in [18], that F is real-analytic and the differential of F at 0 is

$$DF_0(\Psi) = \left(-\frac{6}{\pi}\int_{\boldsymbol{D}}\frac{\rho^{-2}(\zeta)\overline{(-2\psi(\zeta))}}{(\zeta-\xi)^4}|d\zeta|^2\right)d\xi^2 \quad \text{for any} \quad \xi \in \boldsymbol{D}^*$$
$$= \xi^{-4}\overline{\psi(1/\bar{\xi})}d\xi^2 \in \mathbf{BQD}(\boldsymbol{D}^*).$$

So DF_0 is invertible. Note that $F(0) = 0$, since both u_1 and u_2 are identity maps. Therefore, the inverse function theorem implies that for any element $\Psi \in \mathbf{BQD}(\boldsymbol{D}^*)$ with sufficiently small norm, there exists a unique $\Phi \in \mathbf{BQD}(\boldsymbol{D})$ close to zero such that $F(\Phi) = \Psi$. By the definition of F, this is equivalent to the assertion of the theorem. □

3. Generalized Gauss maps of spacelike surfaces in H_1^3 and Kenmotsu type representation formula. Let $\boldsymbol{E}_2^4$ denote the pseudo-Euclidean 4-space endowed with linear coordinates (x_1, x_2, x_3, x_4) and the scalar product $\langle\cdot,\cdot\rangle$ given by $x_1^2+x_2^2-x_3^2-x_4^2$. The *anti-de Sitter* 3-*space* $\boldsymbol{H}_1^3$, which is a 3-dimensional Lorentzian manifold of constant curvature -1, is defined as the following hyperquadric in $\boldsymbol{E}_2^4$:

$$\boldsymbol{H}_1^3 = \{\mathbf{x} \in \boldsymbol{E}_2^4 | \langle\mathbf{x},\mathbf{x}\rangle = -1\}.$$

Let $SU(1,1)$ be the linear group defined by

$$SU(1,1) = \{h \in \mathfrak{gl}(2;\boldsymbol{C}) \mid h\boldsymbol{\varepsilon}h^* = \boldsymbol{\varepsilon}, \det h = 1\}, \quad \text{where} \quad \boldsymbol{\varepsilon} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Identify $\boldsymbol{E}_2^4$ with the linear hull $\boldsymbol{R}\cdot SU(1,1)$ of $SU(1,1)$ by the map

$$\mathbf{x} = (x_1,x_2,x_3,x_4) \mapsto \underline{\mathbf{x}} = x_1\underline{\mathbf{e}_1} + x_2\underline{\mathbf{e}_2} + x_3\underline{\mathbf{e}_3} + x_4\underline{\mathbf{e}_4},$$

where

$$\underline{\mathbf{e}_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \underline{\mathbf{e}_2} = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}, \quad \underline{\mathbf{e}_3} = \sqrt{-1}\boldsymbol{\varepsilon}, \quad \underline{\mathbf{e}_4} = \mathbf{e} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $\langle\mathbf{x},\mathbf{x}\rangle = -\det\underline{\mathbf{x}}$. Then

$$\boldsymbol{H}_1^3 = SU(1,1).$$

The linear Lie group $SU(1,1)\times SU(1,1)$ acts isometrically on $\boldsymbol{E}_2^4$ by

$$\mathrm{g}\cdot\mathbf{x}=\mathrm{g}_1\mathbf{x}\mathrm{g}_2^*\,,\quad \mathrm{g}=(\mathrm{g}_1,\mathrm{g}_2)\in SU(1,1)\times SU(1,1)\,,\quad \mathbf{x}\in\boldsymbol{E}_2^4=\boldsymbol{R}\cdot SU(1,1)\,.$$

Moreover, $SU(1,1)\times SU(1,1)$ acts on $\boldsymbol{H}_1^3$ isometrically and transitively, and $(SU(1,1)\times SU(1,1))/\boldsymbol{Z}_2$ can be regarded as the identity component of the isometry group $O(2,2)$ of $\boldsymbol{H}_1^3$. The isotropy group $\varDelta$ at a point $\mathbf{e}\in\boldsymbol{H}_1^3$ is given by $\{(\mathrm{h},\boldsymbol{\varepsilon}\mathrm{h}\boldsymbol{\varepsilon})\mid \mathrm{h}\in SU(1,1)\}$, and hence

$$\begin{aligned}\boldsymbol{H}_1^3&=(SU(1,1)\times SU(1,1))/\varDelta\\&=\{\mathrm{g}_1\mathrm{g}_2^*\mid \mathrm{g}=(\mathrm{g}_1,\mathrm{g}_2)\in SU(1,1)\times SU(1,1)\}\,.\end{aligned}$$

Let $\boldsymbol{H}^2$ be the hyperbolic 2-space realized in the subspace $\boldsymbol{L}^3=\mathrm{span}\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_4\}$ of $\boldsymbol{E}_2^4=\boldsymbol{R}\cdot SU(1,1)$ by

$$\boldsymbol{H}^2=\{[\mathrm{h}]:=\mathrm{h}\mathrm{h}^*\mid \mathrm{h}\in SU(1,1)\}\cong SU(1,1)/U(1)\,.$$

We can regard $\boldsymbol{H}^2$ as the hyperbolic disc $\boldsymbol{D}$ by the stereographic projection

$$\boldsymbol{H}^2\ni\left[\begin{pmatrix} q & \bar{p}\\ p & \bar{q}\end{pmatrix}\right]\mapsto\frac{p}{q}\in\boldsymbol{D}\,.$$

Now, let M be a Riemann surface and $f:M\to\boldsymbol{H}_1^3$ a conformal immersion. Since f gives rise to a spacelike surface in $\boldsymbol{H}_1^3$, M has to be noncompact. (We may assume that the orientation of M is compatible with the one induced by the canonical time-orientation of $\boldsymbol{H}_1^3\subset\boldsymbol{E}_2^4$.) Take an isothermal coordinate $z=x+\sqrt{-1}y$ on M, and denote the induced metric f^*ds^2 by $e^{2\lambda}|dz|^2$. The *generalized Gauss map* of f is given by (cf. [14])

$$\mathcal{G}=[f_z]:M\to G_{2,2}^*\,,$$

where $G_{2,2}^*$ is the Grassmann manifold of spacelike oriented 2-planes in $\boldsymbol{E}_2^4$, and the oriented complex null line $[f_z]$ in $\boldsymbol{C}_2^4=\boldsymbol{E}_2^4\otimes\boldsymbol{C}$ is considered as the spacelike oriented 2-plane spanned by f_x and f_y in $\boldsymbol{E}_2^4$. Since $SU(1,1)\times SU(1,1)$ acts transitively on $G_{2,2}^*$, $G_{2,2}^*$ can also be realized as the homogeneous space (cf. [1]):

$$\begin{aligned}G_{2,2}^*&=(SU(1,1)\times SU(1,1))/(U(1)\times U(1))\\&=\{[\mathrm{g}_1(\underline{\mathbf{e}_1}-\sqrt{-1}\underline{\mathbf{e}_2})\mathrm{g}_2^*]\mid \mathrm{g}=(\mathrm{g}_1,\mathrm{g}_2)\in SU(1,1)\times SU(1,1)\}\,,\end{aligned}$$

where $U(1)=\{\cos\theta\mathbf{e}+\sqrt{-1}\sin\theta\boldsymbol{\varepsilon}\mid\theta\in[-\pi,\pi)\}$. Moreover $G_{2,2}^*$ can be identified with $\boldsymbol{D}\times\boldsymbol{D}$ by the map

$$\boldsymbol{D}\times\boldsymbol{D}\cong\boldsymbol{H}^2\times\boldsymbol{H}^2\ni([\mathrm{g}_1],[\mathrm{g}_2])\mapsto[\mathrm{g}_1(\underline{\mathbf{e}_1}-\sqrt{-1}\mathbf{e}_2)\mathrm{g}_2^*]\in G_{2,2}^*\,.$$

Therefore, the generalized Gauss map $\mathcal{G}:M\to G_{2,2}^*$ of f is decomposed into two components $\mathcal{G}=(\mathcal{G}_1,\mathcal{G}_2):M\to\boldsymbol{D}\times\boldsymbol{D}$.

A map $\mathcal{F}=(\mathcal{F}_1,\mathcal{F}_2):M\to SU(1,1)\times SU(1,1)$ is called a *framing* of f if $f=\mathcal{F}_1\mathcal{F}_2^*$. Let $\mathcal{E}=(\mathcal{E}_1,\mathcal{E}_2):M\to SU(1,1)\times SU(1,1)$ be an *adapted framing* of f, that is, $\mathcal{E}$ is a framing of f such that $[f_z]=[\mathcal{E}_1(\underline{\mathbf{e}_1}-\sqrt{-1}\underline{\mathbf{e}_2})\mathcal{E}_2^*]$. (Remark that the adapted framing of f is determined uniquely up to the right action of a $U(1)$-valued function. We can choose an adapted framing globally on a contractible Riemann surface.) Then the components $\mathcal{G}_1, \mathcal{G}_2 : M \to \boldsymbol{H}^2(\cong \boldsymbol{D})$ of the generalized Gauss map $\mathcal{G}$ of f can be given by

$$\mathcal{G}_1 = [\mathcal{E}_1], \quad \mathcal{G}_2 = [\mathcal{E}_2],$$

and these are related as follows:

$$\mathcal{G}_1 = (f\boldsymbol{\varepsilon})\mathcal{G}_2(f\boldsymbol{\varepsilon})^*(= \bar{f}[-\mathcal{G}_2]), \tag{3.1}$$

where $\bar{f}[-\mathcal{G}_2]$ (at $z \in M$) stands for the linear fractional transformation of $-\mathcal{G}_2(z) \in \boldsymbol{D}$ by $\overline{f(z)} \in SU(1,1)$.

Using a similar method to the proof of [2, Proposition 1.1] (cf. [1]), we can describe the induced metric and the Hopf differential in terms of the generalized Gauss map and the mean curvature as follows.

PROPOSITION 3.1. *Let $f : M \to \boldsymbol{H}_1^3$ be a conformal immersion with mean curvature H and the generalized Gauss map $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2) : M \to \boldsymbol{D} \times \boldsymbol{D}$. Then the induced metric f^*ds^2 on M and the Hopf differential Φ_f of f are given by*

$$f^*ds^2 = \frac{4|(\mathcal{G}_1)_{\bar{z}}|^2}{(1+H^2)(1-|\mathcal{G}_1|^2)^2}|dz|^2 = \frac{4|(\mathcal{G}_2)_{\bar{z}}|^2}{(1+H^2)(1-|\mathcal{G}_2|^2)^2}|dz|^2, \tag{3.2}$$

$$\Phi_f = \frac{4(\mathcal{G}_1)_z(\overline{\mathcal{G}_1})_z}{(H-\sqrt{-1})(1-|\mathcal{G}_1|^2)^2}dz^2 = \frac{4(\mathcal{G}_2)_z(\overline{\mathcal{G}_2})_z}{(H+\sqrt{-1})(1-|\mathcal{G}_2|^2)^2}dz^2. \tag{3.3}$$

The Gauss curvature $K(= -1 + H^2 + \|\Phi_f\|^2_{f^*ds^2})$ of f is given by

$$K = -(1+H^2)\left(1 - \left|\frac{(\mathcal{G}_1)_z}{(\mathcal{G}_1)_{\bar{z}}}\right|^2\right) = -(1+H^2)\left(1 - \left|\frac{(\mathcal{G}_2)_z}{(\mathcal{G}_2)_{\bar{z}}}\right|^2\right).$$

REMARK 3.2. Since f^*ds^2 is positive definite on M, $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$ is nowhere-holomorphic. From (3.3), anti-holomorphic points of $\mathcal{G}$ correspond to umbilic points of f.

Furthermore, using a similar method to the proof of [2, Theorem 2.1], we can obtain the following

THEOREM 3.3. *Let f be a conformal immersion from a Riemann surface M to $\boldsymbol{H}_1^3$ with mean curvature H. Then the generalized Gauss map $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2) : M \to \boldsymbol{D} \times \boldsymbol{D}$ satisfies the following equations*:

$$(\mathcal{G}_1)_{z\bar{z}} + \frac{2\overline{\mathcal{G}_1}}{1-|\mathcal{G}_1|^2}(\mathcal{G}_1)_z(\mathcal{G}_1)_{\bar{z}} = \frac{1}{H+\sqrt{-1}}H_z(\mathcal{G}_1)_{\bar{z}}, \tag{3.4}$$

$$(\mathcal{G}_2)_{z\bar{z}} + \frac{2\overline{\mathcal{G}_2}}{1-|\mathcal{G}_2|^2}(\mathcal{G}_2)_z(\mathcal{G}_2)_{\bar{z}} = \frac{1}{H-\sqrt{-1}}H_z(\mathcal{G}_2)_{\bar{z}}, \tag{3.5}$$

REMARK 3.4. When the mean curvature H is constant, the above equations (3.4) and (3.5) imply that each component $\mathcal{G}_i$, $i = 1, 2$, of $\mathcal{G}$ is a harmonic map to the hyperbolic disc $\boldsymbol{D}$ (cf. [14]). For a harmonic map $\nu : M \to \boldsymbol{D}$, the Hopf differential of ν is given by $\Phi(\nu) = 4\nu_z(\bar{\nu})_z(1-|\nu|^2)^{-2}dz^2$. Then the Hopf differentials of $\mathcal{G}_1$ and $\mathcal{G}_2$ are given by

$$\Phi(\mathcal{G}_1) = (H - \sqrt{-1})\Phi_f\,, \quad \Phi(\mathcal{G}_2) = (H + \sqrt{-1})\Phi_f\,.$$

If f is a maximal surface in $\boldsymbol{H}^3_1$, that is, $H \equiv 0$, then we obtain that $\Phi(\mathcal{G}_1) + \Phi(\mathcal{G}_2) \equiv 0$.

Conversely, we can obtain the following representation formula for spacelike surfaces with prescribed mean curvature in $\boldsymbol{H}^3_1$ by means of a single component of the generalized Gauss map (similarly in [2, Theorem 2.2]).

THEOREM 3.5 (Kenmotsu type representation formula in $\boldsymbol{H}^3_1$). *Let M be a simply connected Riemann surface with a reference point z_0 and H a real-valued smooth function H on M. For a non-holomorphic smooth map $\nu : M \to \boldsymbol{D}$ satisfying*

$$\nu_{z\bar{z}} + \frac{2\bar{\nu}}{1-|\nu|^2}\nu_z\nu_{\bar{z}} = \frac{1}{H-\sqrt{-1}}H_z\nu_{\bar{z}}$$

$$\left(\textit{resp. } \nu_{z\bar{z}} + \frac{2\bar{\nu}}{1-|\nu|^2}\nu_z\nu_{\bar{z}} = \frac{1}{H+\sqrt{-1}}H_z\nu_{\bar{z}}\right),$$

define a smooth 1-form ω on M by

$$\omega = \frac{2(\bar{\nu})_z}{(1-\sqrt{-1}H)(1-|\nu|^2)^2}dz \quad \left(\textit{resp. } \omega = \frac{2(\bar{\nu})_z}{(1+\sqrt{-1}H)(1-|\nu|^2)^2}dz\right).$$

Also, define a $\mathfrak{gl}(2;\boldsymbol{C})$-valued 1-form α and an $\mathfrak{su}(1,1)$-valued 1-form μ by

$$\alpha = \begin{pmatrix} \nu & 1 \\ \nu^2 & \nu \end{pmatrix}\omega\,, \quad \mu = \varepsilon(\alpha - \alpha^*)\,.$$

Then there exists uniquely a smooth map $f : M \to SU(1,1)$ such that $f(z_0) = \mathbf{e}$ and $f^{-1}df = \mu$. f (resp. f^*) $: M \to \boldsymbol{H}^3_1$ is a conformal immersion outside $\{w \in M \,|\, \nu_{\bar{z}}(w) = 0\}$ with prescribed mean curvature H, and the generalized Gauss map $\mathcal{G} = (\bar{f}[-\nu], \nu)$ (resp. $\mathcal{G} = (\nu, {}^t f[-\nu])$). Moreover, the induced metric $f^*ds^2 = (1-|\nu|^2)^2\omega\cdot\bar{\omega}$ and the Hopf differential $\Phi_f = (H+\sqrt{-1})^{-1}\Phi(\nu)$ (resp. $= (H-\sqrt{-1})^{-1}\Phi(\nu)$).

REMARK 3.6. In Theorems 3.3 and 3.5, if we replace the ambient space $\boldsymbol{H}^3_1$ by $\boldsymbol{H}^3_1(-c^2)$ (the anti-de Sitter 3-space of constant curvature $-c^2$ ($c > 0$)), then the equations satisfied by $\mathcal{G}_1$ and $\mathcal{G}_2$ change from (3.4), (3.5) to the following:

$$(\mathcal{G}_1)_{z\bar{z}} + \frac{2\overline{\mathcal{G}_1}}{1-|\mathcal{G}_1|^2}(\mathcal{G}_1)_z(\mathcal{G}_1)_{\bar{z}} = \frac{1}{H+\sqrt{-1}c}H_z(\mathcal{G}_1)_{\bar{z}}\,,$$

$$(\mathcal{G}_2)_{z\bar{z}} + \frac{2\overline{\mathcal{G}_2}}{1-|\mathcal{G}_2|^2}(\mathcal{G}_2)_z(\mathcal{G}_2)_{\bar{z}} = \frac{1}{H-\sqrt{-1}c}H_z(\mathcal{G}_2)_{\bar{z}}\,.$$

If we put $c = 0$, we can then obtain the generalized harmonic map equation satisfied by the Gauss map of a spacelike surface in $\boldsymbol{L}^3$ [4]. Spacelike surfaces in $\boldsymbol{L}^3$ can also be represented in terms of the mean curvature and the Gauss map [4], which is a Lorentzian version of the Kenmotsu representation formula in $\boldsymbol{E}^3$ [8].

4. Representation of minimal maps between the hyperbolic discs. Finally, we apply the Kenmotsu type representation formula, Theorem 3.5, to representing a minimal map between the hyperbolic discs $\boldsymbol{D}$ as the composition $\mathcal{G}_1 \circ \mathcal{G}_2^{-1}$ of the generalized Gauss map $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$ of a maximal surface in $\boldsymbol{H}_1^3$.

One easily observes that a composition of the generalized Gauss map of a maximal surface in $\boldsymbol{H}_1^3$ can be regarded as a minimal map, at least locally, between $\boldsymbol{D}$, by Lemma 2.1 and Remark 3.4. More precisely, let $f : \Sigma \to \boldsymbol{H}_1^3 = SU(1,1)$ be a conformal maximal immersion, where Σ is an arbitrary Riemann surface. Let $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2) : \Sigma \to \boldsymbol{D} \times \boldsymbol{D}$ be its generalized Gauss map. Then, both components are harmonic maps from Σ into $\boldsymbol{D}$ and the sum of the Hopf differentials is equal to zero (see Remark 3.4). Therefore, if one of the components is a diffeomorphism, then according to Lemma 2.1, the generalized Gauss map represents a minimal map from $\boldsymbol{D}$ into itself. In this case, if $\mathcal{G}_2 : \Sigma \to \boldsymbol{D}$ is a harmonic diffeomorphism, then we can use the relation (3.1) between $\mathcal{G}_1$ and $\mathcal{G}_2$ to write the corresponding minimal map as

$$(\mathcal{G}_1 \circ \mathcal{G}_2^{-1})(\zeta) = \overline{(f \circ \mathcal{G}_2^{-1})(\zeta)}[-\zeta] \quad (\zeta \in \boldsymbol{D}).$$

We would like to point out that, by Proposition 3.1, the Gauss curvature K of the conformal maximal immersion f has a fixed sign if and only if one of the components of the generalized Gauss map is a local diffeomorphism, and hence, if and only if both components are local diffeomorphisms. Suppose further that the curvature K of f is bounded away from zero and the surface is complete, then both components of the generalized Gauss map are simultaneously orientation preserving or reversing local harmonic diffeomorphisms with bounded dilatation. Therefore, by a result in [19, Theorem 13], both components are harmonic diffeomorphisms onto the hyperbolic disc. Hence, we obtain a minimal diffeomorphism between the hyperbolic disc in this case. Moreover, this minimal map is in fact a quasi-conformal map of the unit disc.

Conversely, we have the following

THEOREM 4.1. *Any orientation preserving minimal diffeomorphism between the hyperbolic discs can be represented as the composition $\mathcal{G}_1 \circ \mathcal{G}_2^{-1}$ of the generalized Gauss map $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$ of a complete maximal surface in $\boldsymbol{H}_1^3$.*

PROOF. Given a minimal map u between the hyperbolic discs, Lemma 2.1 implies that there exist a Riemann surface Σ homeomorphic to $\boldsymbol{D}$ and two harmonic maps u_i, $i = 1, 2$, from Σ to $\boldsymbol{D}$ such that u_2 is a diffeomorphism, $u = u_1 \circ u_2^{-1}$, and

$$\Phi(u_1) + \Phi(u_2) \equiv 0.$$

(Note that we have interchanged the notation for the harmonic maps u_i.) Since u_2 is a diffeomorphism, either $|u_{2,z}|$ or $|u_{2,\bar{z}}|$ is non-vanishing. We may assume that $|u_{2,\bar{z}}| > |u_{2,z}| \geqq 0$ (i.e., u_2 is orientation reversing), otherwise one can consider $\overline{u_2}$. Then by Theorem 3.5, there is a conformal maximal immersion $f : \boldsymbol{D} \to \boldsymbol{H}_1^3$ such that the second component of the generalized Gauss map is given by u_2 (considered as a map from $\boldsymbol{D}$). Therefore, we only need to show that the first component of the generalized Gauss map is equal to u_1.

By assumption, u is an orientation preserving minimal diffeomorphism onto $\boldsymbol{D}$. Then u_1 is an orientation reversing diffeomorphism onto $\boldsymbol{D}$ as u_2 is orientation reversing. Using Proposition 3.1, the first component $\mathcal{G}_1 : \boldsymbol{D} \to \boldsymbol{D}$ of the generalized Gauss map satisfies

$$\|\bar{\partial} u_2\| = \|\bar{\partial}\mathcal{G}_1\|\,.$$

Since u_2 is a diffeomorphism and the pullback metric $u_2^* ds_p^2$ is dominated, up to a constant multiple, by $\|\bar{\partial} u_2\|^2 ds_p^2$, $\|\bar{\partial} u_2\|^2 ds_p^2$ and hence $\|\bar{\partial}\mathcal{G}_1\|^2 ds_p^2$ defines a complete metric on $\boldsymbol{D}$. By the same reason, $\|\bar{\partial} u_1\|^2 ds_p^2$ defines a complete metric on $\boldsymbol{D}$. From the equation $\Phi(u_1) = \Phi(\mathcal{G}_1)$ $(= \phi dz^2)$ and the uniqueness result [17], [19] for complete solutions of

$$\Delta_{ds_p^2} w = e^{2w} - \|\phi\| e^{-2w} - 1\,,$$

we obtain

$$\|\bar{\partial} u_1\| = \|\bar{\partial}\mathcal{G}_1\| \ \ (= e^w)\,. \tag{4.1}$$

Now, (4.1) together with the equation $\Phi(u_1) = \Phi(\mathcal{G}_1)$ implies that u_1 is identically equal to the first component $\mathcal{G}_1$ of the generalized Gauss map up to an isometry. Finally, we note that the induce metric $f^* ds^2$ is complete, since $f^* ds^2 = \|\bar{\partial} u_2\|^2 ds_p^2$. Therefore we have proved the theorem. □

As a final remark, we would like to claim that Theorem 3.5 (Kenmotsu type representation formula) can be used, in principle, to construct minimal maps between the hyperbolic discs as follow. By the existence results of Li and Tam [10], [11], [12], and the second author [3] for the Dirichlet problem at infinity of harmonic maps between $\boldsymbol{D}$, one can get a nonconformal harmonic diffeomorphism of $\boldsymbol{D}$ by assigning suitable boundary data. Or, one can use the result of Tam-Wan [17] to obtain a nonconformal harmonic diffeomorphism by prescribing the Hopf differential. From such a harmonic diffeomorphism, one can use the Kenmotsu type representation formula in $\boldsymbol{H}_1^3$ to construct a complete maximal surface in $\boldsymbol{H}_1^3$. Then the composition $\mathcal{G}_1 \circ \mathcal{G}_2^{-1}$ of the generalized Gauss map $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$ is a minimal map between $\boldsymbol{D}$.

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