# Small generating sets for the Torelli group

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Proving a conjecture of Dennis Johnson, we show that the Torelli subgroup $\mathcal{I}_g$ of the genus $g$ mapping class group has a finite generating set whose size grows cubically with respect to $g$. Our main tool is a new space called the handle graph on which $\mathcal{I}_g$ acts cocompactly.

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## 1 Introduction

Let $\Sigma_{g,n}$ be a compact connected oriented genus $g$ surface with $n$ boundary components. The *mapping class group* of $\Sigma_{g,n}$, denoted $\mathrm{Mod}_{g,n}$, is the group of orientation-preserving homeomorphisms of $\Sigma_{g,n}$ that fix the boundary pointwise modulo isotopies that fix the boundary pointwise. We will often omit the $n$ if it vanishes. For $n \leq 1$, the *Torelli group*, denoted $\mathcal{I}_{g,n}$, is the kernel of the action of $\mathrm{Mod}_{g,n}$ on $\mathrm{H}_1(\Sigma_{g,n}; \mathbb{Z})$. The Torelli group has been the object of intensive study ever since the seminal work of Dennis Johnson in the early '80's. See [10] for a survey of Johnson's work.

**Finite generation of Torelli** One of Johnson's most celebrated theorems says that $\mathcal{I}_{g,n}$ is finitely generated for $g \geq 3$ and $n \leq 1$ (see [9]). This is a surprising result – though $\mathrm{Mod}_{g,n}$ is finitely presentable, $\mathcal{I}_{g,n}$ is an infinite-index normal subgroup of $\mathrm{Mod}_{g,n}$, so there is no reason to hope that $\mathcal{I}_{g,n}$ has any finiteness properties. Moreover, McCullough and Miller [13] proved that $\mathcal{I}_{2,n}$ is *not* finitely generated for $n \leq 1$, and later Mess [14] proved that $\mathcal{I}_2$ is an infinite rank free group.

**Johnson's generating set** Johnson's generating set for $\mathcal{I}_{g,n}$ when $g \geq 3$ and $n \leq 1$ is enormous. Indeed, for $\mathcal{I}_g$ (resp. $\mathcal{I}_{g,1}$), it contains $9 \cdot 2^{2g-3} - 4g^2 + 2g - 6$ (resp. $9 \cdot 2^{2g-3} - 4g^2 + 4g - 5$) elements. In [11], Johnson proved that the abelianization of $\mathcal{I}_g$ (resp. $\mathcal{I}_{g,1}$) has rank $\frac{1}{3}(4g^3 + 5g + 3)$ (resp. $\frac{1}{3}(4g^3 - g)$). These give large lower bounds on the size of generating sets for $\mathcal{I}_{g,n}$; however, there is a huge gap between this cubic lower bound and Johnson's exponentially growing generating set. At the end of [9] and in [10, page 168], Johnson conjectures that there should be a generating set for $\mathcal{I}_{g,n}$ whose size grows cubically with respect to the genus. Later, in

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[4, Problem 5.7] Farb asked whether there at least exists a generating set whose size grows polynomially.

**Main theorem** In this paper, we prove Johnson's conjecture. Our main theorem is as follows.

**Theorem A** *For $g \geq 3$, the group $\mathcal{I}_g$ has a generating set of size at most $57\binom{g}{3}$ and the group $\mathcal{I}_{g,1}$ has a generating set of size at most $57\binom{g}{3} + 2g + 1$.*

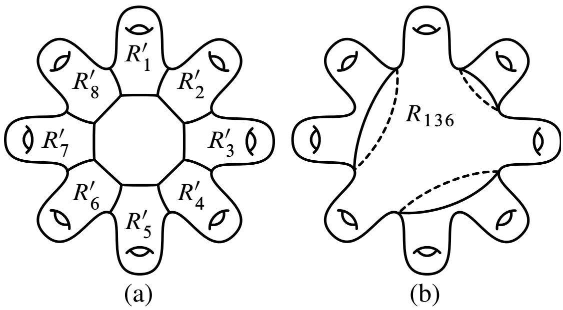


Figure 1: (a) The subsurfaces $R'_i \cong \Sigma_{1,1}$ To avoid cluttering the picture, the portion of the boundaries of the $R'_i$ which lie on the back side the figure are not drawn. (b) A subsurface isotopic to $R_{136}$

The generating set we construct was conjectured to generate $\mathcal{I}_{g,n}$ by Brendle and Farb [2]. To describe it, we must introduce some notation. As in Figure 1(a), let $R'_1, \ldots, R'_g$ be $g$ subsurfaces of $\Sigma_g$ each homeomorphic to $\Sigma_{1,1}$ such that the following hold. Interpret all indices modulo $g$.

- If $1 \leq i < j \leq g$ satisfy $i \notin \{j-1, j+1\}$, then $R'_i \cap R'_j = \varnothing$.
- For all $1 \leq i \leq g$, the intersection $R'_i \cap R'_{i+1}$ is homeomorphic to an interval.

For $1 \leq i < j < k \leq g$, define a subsurface $R_{ijk}$ of $\Sigma_g$ by $R_{ijk} = \overline{\Sigma_g \setminus \bigcup_{l \neq i,j,k} R'_l}$. Thus $R_{ijk}$ is a genus 3 surface with at most 3 boundary components such that $R'_i, R'_j, R'_k \subset R_{i,j,k}$ (see Figure 1(b)).

If $S$ is a subsurface of $\Sigma_g$, define $\text{Mod}(\Sigma_g, S)$ to be the subgroup of $\text{Mod}_g$ consisting of mapping classes that can be realized by homeomorphisms supported on $S$ and $\mathcal{I}(\Sigma_g, S)$ to equal $\mathcal{I}_g \cap \text{Mod}(\Sigma_g, S)$. The key result for the proof of Theorem A is the following theorem.

**Theorem B** *For $g \geq 3$, the group $\mathcal{I}_g$ is generated by the set*

$$\bigcup_{1 \leq i < j < k \leq g} \mathcal{I}(\Sigma_g, R_{ijk}).$$

Using Johnson's work, it is easy to see that $\mathcal{I}(\Sigma_g, R_{ijk})$ is finitely generated by a generating set with at most $57$ generators (see Lemma 2.2). Also, standard techniques (see Lemma 2.1) show that if $\mathcal{I}_g$ has a generating set with $k$ elements, then $\mathcal{I}_{g,1}$ has a generating set with $k+2g+1$ elements. Since there are $\binom{g}{3}$ subsurfaces $R_{ijk}$, Theorem A follows from Theorem B.

**Remark** To illustrate the relative sizes of our generating sets, Johnson's generating set for $\mathcal{I}_{20}$ contains more than one trillion elements while our generating set for $\mathcal{I}_{20}$ has $64980$ elements.

**New proof of Johnson's theorem** Our deduction of Theorem A from Theorem B depends on Johnson's theorem that $\mathcal{I}_3$ is finitely generated. However, Hain [6] has recently announced a direct conceptual proof that $\mathcal{I}_3$ is finitely generated. Hain's proof uses special properties of the moduli space of genus $3$ Riemann surfaces and cannot be easily generalized to $g > 3$. Combining this with our paper, we obtain a new proof that $\mathcal{I}_{g,n}$ is finitely generated for $g \geq 3$ and $n \leq 1$.

Our new proof is more conceptual than Johnson's original one. To illustrate this, we will sketch Johnson's proof. He starts by writing down an enormous finite subset $S \subset \mathcal{I}_{g,n}$ which is known (from work of Powell [15]) to normally generate $\mathcal{I}_{g,n}$ as a subgroup of $\mathrm{Mod}_{g,n}$. Letting $T$ be a standard generating set for $\mathrm{Mod}_{g,n}$, Johnson then proves via a laborious computation that for $t \in T$ and $s \in S$, the element $tst^{-1} \in \mathcal{I}_{g,n}$ can be written as a word in $S$. This implies that the subgroup $\Gamma$ of $\mathcal{I}_{g,n}$ generated by $S$ is a normal subgroup of $\mathrm{Mod}_{g,n}$, and thus that $\Gamma = \mathcal{I}_{g,n}$.

**Remark** Our proof of Theorem B appeals to a theorem of [17] whose proof depends on Johnson's theorem. However, Hatcher and Margalit [12] have recently given a new proof of this result that is independent of Johnson's work.

**Nature of generators** Some basic elements of $\mathcal{I}_{g,n}$ are as follows (see, eg [16]). If $x$ is a simple closed curve on $\Sigma_{g,n}$, then denote by $T_x \in \mathrm{Mod}_{g,n}$ the Dehn twist about $x$. If $x$ is a separating simple closed curve, then $T_x \in \mathcal{I}_{g,n}$; these are called *separating twists*. If $x$ and $y$ are disjoint homologous nonseparating simple closed curves, then $T_x T_y^{-1} \in \mathcal{I}_{g,n}$; these are called *bounding pair maps*. Following work of Birman [1], Powell [15] proved that $\mathcal{I}_{g,n}$ is generated by bounding pair maps and separating twists for $g \geq 1$ and $n \leq 1$ (see [16] and [12] for alternate proofs). Johnson's finite generating set for $\mathcal{I}_{g,n}$ for $g \geq 3$ and $n \leq 1$ consists entirely of bounding pair maps. It follows easily from our proofs of Lemma 2.1 and 2.2 that our generating set consists of bounding pair maps and separating twists; see the remark after Lemma 2.2.

**The handle graph** Our proof of Theorem B is topological. To prove that a group $G$ is finitely generated, it is enough to find a connected simplicial complex upon which $G$ acts cocompactly with finitely generated stabilizers. We use a variant on the curve complex. If $\gamma$ is an oriented simple closed curve on $\Sigma_g$, then denote by $[\gamma] \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ its homology class. Also, if $\gamma_1$ and $\gamma_2$ are isotopy classes of simple closed curves on $\Sigma_g$, then denote by $i_g(\gamma_1,\gamma_2)$ their *geometric intersection number*, ie the minimal possible number of intersections between two curves in the isotopy classes of $\gamma_1$ and $\gamma_2$. Finally, denote by $i_a(\cdot,\cdot)$ the algebraic intersection pairing on $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$.

**Definition** Let $a, b \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ satisfy $i_a(a,b) = 1$. The *handle graph* associated to $a$ and $b$, denoted $\mathcal{H}_{a,b}$, is the graph whose vertices are isotopy classes of oriented simple closed curves on $\Sigma_g$ that are homologous to either $a$ or $b$ and where two vertices $\gamma_1$ and $\gamma_2$ are joined by an edge exactly when $i_g(\gamma_1,\gamma_2) = 1$.

We will show that $\mathcal{H}_{a,b}/\mathcal{I}_g$ consists of a single edge (see Lemma 5.2) and that $\mathcal{H}_{a,b}$ is connected for $g \geq 3$ (see Lemma 3.1).

**A complication** It would appear that we have all the ingredients in place to use the space $\mathcal{H}_{a,b}$ to prove that $\mathcal{I}_g$ is finitely generated. However, there is one remaining complication. Namely, we do not know the answer to the following question.

**Question 1.1** For some $g \geq 4$, let $\gamma$ be the isotopy class of a nonseparating simple closed curve on $\Sigma_g$. Is the stabilizer subgroup $(\mathcal{I}_g)_\gamma$ of $\gamma$ finitely generated?

In other words, we do not know if the vertex stabilizer subgroups of the action of $\mathcal{I}_g$ on $\mathcal{H}_{a,b}$ are finitely generated. Nonetheless, in Section 4 we will prove a weaker statement that suffices to prove Theorem B. The proof of Theorem B is in Section 5.

**Smaller generating sets** A positive answer to Question 1.1 would likely lead to a smaller generating set for $\mathcal{I}_g$, though of course this depends on the nature of the finite generating sets for the stabilizer subgroups. Let us describe one way this could work. For $g \geq 3$, let $\sigma_g$ be the smallest cardinality of a generating set for $\mathcal{I}_g$. Consider

$g \geq 4$, and fix an edge $\{\alpha, \beta\}$ of $\mathcal{H}_{a,b}$. The proof of Theorem B shows that $\mathcal{I}_g$ is generated by $(\mathcal{I}_g)_\alpha \cup (\mathcal{I}_g)_\beta$. Let $S$ be a subsurface of $\Sigma_g$ such that $S \cong \Sigma_{g-1,1}$ and $\alpha \cup \beta \subset \Sigma_g \setminus S$. We have $\mathcal{I}(\Sigma_g, S) \cong \mathcal{I}_{g-1,1}$ (see Section 2) and $\mathcal{I}(\Sigma_g, S) \subset (\mathcal{I}_g)_\alpha$ and $\mathcal{I}(\Sigma_g, S) \subset (\mathcal{I}_g)_\beta$. Assume that there exists a finite set $V_\alpha$ (resp. $V_\beta$) such that $(\mathcal{I}_g)_\alpha$ (resp. $(\mathcal{I}_g)_\beta$) is generated by $\mathcal{I}(\Sigma_g, S) \cup V_\alpha$ (resp. $\mathcal{I}(\Sigma_g, S) \cup V_\beta$). The group $\mathcal{I}_g$ is then generated by $\mathcal{I}(\Sigma_g, S) \cup V_\alpha \cup V_\beta$. Lemma 2.1 says that $\mathcal{I}(\Sigma_g, S) \cong \mathcal{I}_{g-1,1}$ can be generated by $\sigma_{g-1} + 2g + 1$ elements. Moreover, it seems likely that there exists some relatively small $K$ such that $|V_\alpha|, |V_\beta| \leq Kg^2$. This would imply that

$$\sigma_g \leq \sigma_{g-1} + 2g + 1 + 2Kg^2.$$

Iterating this, we would get that

$$\sigma_g \leq \sigma_3 + \sum_{i=4}^{g} (2i + 1 + 2Ki^2)$$

for $g \geq 4$. This bound is cubic in $g$ (as it needs to be), but as long as $K$ is not too large it is much smaller than $57\binom{g}{3}$.

**Finite presentability** Perhaps the most important open question about the combinatorial group theory of $\mathcal{I}_g$ is whether or not it is finitely presentable for $g \geq 3$. One way of proving that a group $G$ is finitely presentable is to construct a simply-connected simplicial complex $X$ upon which $G$ acts cocompactly with finitely presentable stabilizer subgroups (see, eg [3]). For example, Hatcher and Thurston use this technique in [7] to prove that the mapping class group is finitely presentable.

The handle graph $\mathcal{H}_{a,b}$ appears to be the first example of a useful space upon which $\mathcal{I}_g$ acts cocompactly (of course, there are trivial non-useful examples of such spaces; for example, the Cayley graph of $\mathcal{I}_g$ or a 1–point space). Unfortunately, while $\mathcal{H}_{a,b}$ is connected for $g \geq 3$, it is not simply connected. Indeed, it does not even have any 2–cells (and is not a tree). However, one could probably attach 2–cells to $\mathcal{H}_{a,b}$ to obtain a simply connected complex upon which $\mathcal{I}_g$ acts cocompactly. This would not be enough, however – one would also have to prove that the simplex stabilizer subgroups were finitely presentable. In other words, this complex would provide the inductive step in a proof that $\mathcal{I}_g$ was finitely presentable, but one would still need a base case.

**A complex that does not work** We close this introduction by discussing an approach to Theorem B that does not work. One might think of trying to prove Theorem B using the following complex. Let $a \in \mathrm{H}_1(\Sigma_g; \mathbb{Z})$ be a primitive vector. Define $\mathcal{C}_a$ to be the graph whose vertices are isotopy classes of oriented simple closed curves $\gamma$ on $\Sigma_g$ such that $[\gamma] = a$ and where two vertices $\gamma$ and $\gamma'$ are joined by an edge if

$i_g(\gamma, \gamma') = 0$. It is known ([17, Theorem 1.9]; see [12] for an alternate proof) that $\mathcal{C}_a$ is connected for $g \geq 3$. Moreover, $\mathcal{I}_g$ acts transitively on the vertices of $\mathcal{C}_a$. However, it does *not* act cocompactly; indeed, there are infinitely many edge orbits. To see this, consider edges $e_1 = \{\gamma_1, \gamma_1'\}$ and $e_2 = \{\gamma_2, \gamma_2'\}$ of $\mathcal{C}_a$. Assume that there exists some $f \in \mathcal{I}_g$ such that $f(e_1) = e_2$. Since $\gamma_1$ is homologous to $\gamma_1'$, the multicurve $\gamma_1 \cup \gamma_1'$ divides $\Sigma_g$ into two subsurfaces $S_1$ and $S_1'$. Similarly, $\gamma_2 \cup \gamma_2'$ divides $\Sigma_g$ into two subsurfaces $S_2$ and $S_2'$. Relabeling if necessary, we have $f(S_1)$ isotopic to $S_2$ and $f(S_1')$ isotopic to $S_2'$. Since $f \in \mathcal{I}_g$, the images of $\mathrm{H}_1(S_1;\mathbb{Z})$ and $\mathrm{H}_1(S_2;\mathbb{Z})$ in $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$ must be the same, and similarly for $\mathrm{H}_1(S_1';\mathbb{Z})$ and $\mathrm{H}_1(S_2';\mathbb{Z})$. It is easy to see that infinitely many such images occur for different edges of $\mathcal{C}_a$, so there must be infinitely many edges orbits. We remark that Johnson proved in [8, Corollary to Lemma 9 on page 250] that the images of $\mathrm{H}_1(S_1;\mathbb{Z})$ and $\mathrm{H}_1(S_1';\mathbb{Z})$ in $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$ are a complete invariant for the edge orbits.

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# 2 The Torelli group on subsurfaces

We will need to understand how the Torelli group restricts to subsurfaces. For a general discussion of this, see [16]. In this section, we will extract from [16] results on two kinds of subsurfaces. In Section 2.1, we will show how to analyze subsurfaces like the subsurfaces $R_{ijk}$ from Section 1. In Section 2.2, we will show how to analyze stabilizers of nonseparating simple closed curves (which are supported on the subsurface obtained by taking the complement of a regular neighborhood of the curve).

## 2.1 Analyzing the subsurfaces $R_{ijk}$

We begin by defining groups $\mathcal{I}_{g,n}$ for $n \geq 2$. There is a map $\mathrm{Mod}_{g,n} \to \mathrm{Mod}_g$ induced by gluing discs to the boundary components of $\Sigma_{g,n}$ and extending homeomorphisms by the identity. Define $\mathcal{I}_{g,n}$ to be the kernel of the resulting action of $\mathrm{Mod}_{g,n}$ on $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$. For the case $n=1$, the map $\mathrm{H}_1(\Sigma_{g,1};\mathbb{Z}) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ is an isomorphism, so this agrees with our previous definition of $\mathcal{I}_{g,1}$.

**Remark** In [16], the different definitions of the Torelli group on a surface with boundary are parametrized by partitions of the boundary components. The above definition of $\mathcal{I}_{g,n}$ corresponds to the discrete partition $\{\{\beta_1\}, \ldots, \{\beta_n\}\}$ of the set $\{\beta_1, \ldots, \beta_n\}$ of boundary components of $\Sigma_{g,n}$.

In [16, Theorem 1.2], a version of the Birman exact sequence is proven for the Torelli group. For $\mathcal{I}_{g,n}$ with $g \geq 2$, it takes the form

$$1 \longrightarrow \pi_1(U\Sigma_{g,n}) \longrightarrow \mathcal{I}_{g,n+1} \longrightarrow \mathcal{I}_{g,n} \longrightarrow 1. \tag{1}$$

Here $U\Sigma_{g,n}$ is the unit tangent bundle of $\Sigma_{g,n}$. The subgroup $\pi_1(U\Sigma_{g,n})$ of $\mathcal{I}_{g,n+1}$ is often called the "disc-pushing subgroup" – the mapping class associated to $\gamma \in \pi_1(U\Sigma_{g,n})$ "pushes" a fixed boundary component around $\gamma$ while allowing it to rotate. The following is an immediate consequence of (1) and the fact that $\pi_1(U\Sigma_g)$ can be generated by $2g+1$ elements.

**Lemma 2.1** *$\mathcal{I}_{g,1}$ can be generated by $k+2g+1$ elements if $\mathcal{I}_g$ can be generated by $k$ elements.*

Now assume that $S \cong \Sigma_{h,n}$ is an embedded subsurface of $\Sigma_g$ and that all the boundary components of $S$ are non-nullhomotopic separating curves in $\Sigma_g$. For example, $S$ could be one of the surfaces $R_{ijk}$ from Section 1. Letting $\mathrm{Mod}(S)$ be the mapping class group of $S$, the induced map $\mathrm{Mod}(S) \to \mathrm{Mod}_g$ is an injection. This gives a natural identification of $\mathrm{Mod}(S)$ with $\mathrm{Mod}(\Sigma_g, S)$. The group $\mathcal{I}(\Sigma_g, S)$ is thus naturally a subgroup of $\mathrm{Mod}(S) \cong \mathrm{Mod}_{h,n}$, and in [16, Theorem 1.1] it is proven that $\mathcal{I}(\Sigma_g, S) = \mathcal{I}_{h,n}$. Johnson [9] proved that $\mathcal{I}_3$ can be generated by 35 elements. Applying (1) repeatedly, we see that $\mathcal{I}_{3,1}$ can be generated by 42 elements, $\mathcal{I}_{3,2}$ by 49 elements, and $\mathcal{I}_{3,3}$ by 57 elements. Since $R_{ijk} \cong \Sigma_{3,k}$ with $k \leq 3$, we obtain the following.

**Lemma 2.2** *For all $1 \leq i < j < k \leq g$, the group $\mathcal{I}(\Sigma_g, R_{ijk})$ can be generated by 57 elements.*

**Remark** It is well-known (see, eg [16, Section 2.1]) that the mapping classes corresponding to the generators of $\pi_1(U\Sigma_{g,n})$ used to prove Lemmas 2.1 and 2.2 can be chosen to be bounding pair maps and separating twists. Additionally, Johnson's minimal-size generating set for $\mathcal{I}_3$ consists entirely of bounding pair maps, so the generating set for $\mathcal{I}(\Sigma_g, R_{ijk})$ in Lemma 2.2 can be taken to consist of bounding pair maps and separating twists.

## 2.2 Stabilizers of nonseparating simple closed curves

Let $\gamma$ be a nonseparating simple closed curve on $\Sigma_g$. Define $\Sigma_{g,\gamma}$ to be the result of cutting $\Sigma_g$ along $\gamma$, so $\Sigma_{g,\gamma} \cong \Sigma_{g-1,2}$. Letting $\mathrm{Mod}_{g,\gamma}$ be the mapping class group of $\Sigma_{g,\gamma}$, the natural map $\Sigma_{g,\gamma} \to \Sigma_g$ induces a map $i\colon \mathrm{Mod}_{g,\gamma} \to \mathrm{Mod}_g$. Define $\mathcal{I}_{g,\gamma} = i^{-1}(\mathcal{I}_g)$. The map $i$ restricts to a surjection $\mathcal{I}_{g,\gamma} \to (\mathcal{I}_g)_\gamma$, where $(\mathcal{I}_g)_\gamma$ is the stabilizer subgroup of $\gamma$.

**Remark** In the notation of [16], the group $\mathcal{I}_{g,\gamma}$ corresponds to the Torelli group of $\Sigma_{g-1,2}$ with respect to the "indiscrete partition" $\{\{\beta,\beta'\}\}$ of the boundary components $\beta$ and $\beta'$ of $\Sigma_{g,\gamma}$. Also, the kernel of the map $\mathcal{I}_{g,\gamma} \to (\mathcal{I}_g)_\gamma$ is isomorphic to $\mathbb{Z}$ and is generated by $T_\beta T_{\beta'}^{-1}$, where $T_\beta$ and $T_{\beta'}$ are the Dehn twists about $\beta$ and $\beta'$, respectively.

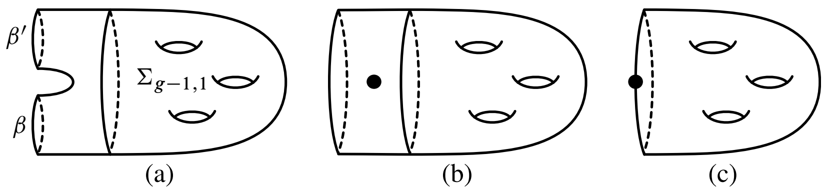


Figure 2: (a) The surface $\Sigma_{g,\gamma}$ and the subsurface $\Sigma_{g-1,1}$ of $\Sigma_{g,\gamma}$ such that the induced map $\mathcal{I}_{g-1,1} \to \mathcal{I}_{g,\gamma}$ splits the exact sequence (2) (b) The basepoint for $\pi_1(\Sigma_{g-1,1})$ is obtained from $\Sigma_{g,\gamma}$ by collapsing the boundary component $\beta$ to a point. (c) The surface in b deformation retracts to $\Sigma_{g-1,1}$ such that the basepoint ends up on the boundary component.

In [16, Theorem 1.2], it is proven that for $g \geq 2$ there is a short exact sequence

(2) $$1 \longrightarrow K_{g,\gamma} \longrightarrow \mathcal{I}_{g,\gamma} \longrightarrow \mathcal{I}_{g-1,1} \longrightarrow 1.$$

Here $K_{g,\gamma} \cong [\pi_1(\Sigma_{g-1,1}), \pi_1(\Sigma_{g-1,1})]$. This exact sequence splits via the inclusion $\mathcal{I}_{g-1,1} \hookrightarrow \mathcal{I}_{g,\gamma}$ induced by the inclusion $\Sigma_{g-1,1} \hookrightarrow \Sigma_{g,\gamma}$ indicated in Figure 2(a). In other words, the following holds.

**Lemma 2.3** $\mathcal{I}_{g,\gamma} = K_{g,\gamma} \ltimes \mathcal{I}_{g-1,1}$ *for* $g \geq 3$ *and* $\gamma$ *a simple closed nonseparating curve on* $\Sigma_g$*.*

The group $\mathcal{I}_{g-1,1}$ acts on $K_{g,\gamma} < \pi_1(\Sigma_{g-1,1})$ as follows. As is clear from [16, Theorem 1.2], the basepoint for $\pi_1(\Sigma_{g-1,1})$ is as indicated in Figure 2(b). As shown in Figure 2(c), the surface $\Sigma_{g-1,1}$ deformation retracts onto the surface $\Sigma_{g-1,1}$ on which $\mathcal{I}_{g-1,1}$ is supported. After this deformation retract, the basepoint ends up on $\partial\Sigma_{g-1,1}$. Summing up, $\mathcal{I}_{g-1,1}$ acts on $K_{g,\gamma} < \pi_1(\Sigma_{g-1,1})$ via the action of $\mathrm{Mod}_{g-1,1}$ on $\pi_1(\Sigma_{g-1,1})$, where the basepoint for $\pi_1(\Sigma_{g-1,1})$ is on $\partial\Sigma_{g-1,1}$.

# 3 The handle graph is connected

In this section, we prove the following.

**Lemma 3.1** *Fix* $g \geq 3$*. Let* $a, b \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ *satisfy* $i_a(a,b) = 1$*. Then* $\mathcal{H}_{a,b}$ *is connected.*

We will need two lemmas. In the first, if $\epsilon$ is an oriented arc in a surface, then $\epsilon^{-1}$ denotes the arc obtained by reversing the orientation of $\epsilon$.

**Lemma 3.2** *Let the boundary components of $\Sigma_{g,2}$ be $\delta_0$ and $\delta_1$. Choose points $v_i \in \delta_i$ for $i = 0, 1$ and let $\epsilon$ be an oriented properly embedded arc in $\Sigma_{g,2}$ whose initial point is $v_0$ and whose terminal point is $v_1$. Then for any $h \in \mathrm{H}_1(\Sigma_{g,2}; \mathbb{Z})$, there exists an oriented properly embedded arc $\epsilon'$ in $\Sigma_{g,2}$ whose initial point is $v_0$ and whose terminal point is $v_1$ such that the homology class of the loop $\epsilon' \cdot \epsilon^{-1}$ is $h$.*

**Proof** Gluing $(\delta_0, v_0)$ to $(\delta_1, v_1)$, we obtain a surface $S \cong \Sigma_{g+1}$. Let $\alpha$ and $*$ be the images of $\delta_0$ and $v_0$ in $S$, respectively. The image of $\epsilon$ in $S$ is an oriented simple closed curve $\beta$ with $i_g(\alpha, \beta) = 1$. There is a natural isomorphism $\mathrm{H}_1(\Sigma_{g,2}; \mathbb{Z}) \cong [\alpha]^{\perp}$, where the orthogonal complement is taken with respect to $i_a(\cdot, \cdot)$. Under this identification, we can apply [16, Lemma A.3] to find an oriented simple closed curve $\beta'$ on $S$ such that $[\beta'] = [\beta] + h$ and such that $\alpha \cap \beta' = \{*\}$. Cutting $S$ open along $\alpha$, the curve $\beta'$ becomes the desired arc $\epsilon'$. □

**Lemma 3.3** *Let $a, b \in \mathrm{H}_1(\Sigma_g; \mathbb{Z})$ satisfy $i_a(a, b) = 1$. Let $\alpha_1$ and $\alpha_2$ be disjoint oriented simple closed curves on $\Sigma_g$ such that $[\alpha_i] = a$ for $i = 1, 2$. There then exists some oriented simple closed curve $\beta$ on $\Sigma_g$ such that $[\beta] = b$ and $i_g(\alpha_i, \beta) = 1$ for $i = 1, 2$.*

**Proof** Let $\beta'$ be any simple closed curve on $\Sigma_g$ such that $i(\alpha_i, \beta') = 1$ for $i = 1, 2$. Orient $\beta'$ so that its intersections with $\alpha_1$ and $\alpha_2$ are positive. Let $X_1$ and $X_2$ be the two subsurfaces of $\Sigma_g$ that result from cutting $\Sigma_g$ along $\alpha_1 \cup \alpha_2$. For $i = 1, 2$, the surface $X_i$ has 2 boundary components and the intersection of $\beta'$ with $X_i$ is an oriented properly embedded arc $\epsilon_i$ running between these boundary components. Also, the induced map $\mathrm{H}_1(X_i; \mathbb{Z}) \to \mathrm{H}_1(\Sigma_g; \mathbb{Z})$ is an injection, and we will identify $\mathrm{H}_1(X_i; \mathbb{Z})$ with its image in $\mathrm{H}_1(\Sigma_g; \mathbb{Z})$. The orthogonal complement to $a$ with respect to the algebraic intersection pairing is spanned by $\mathrm{H}_1(X_1; \mathbb{Z}) \cup \mathrm{H}_1(X_2; \mathbb{Z})$. Since $i_a(a, b) = i_a(a, [\beta'])$, the homology class $b - [\beta']$ is orthogonal to $a$. There thus exist $h_i \in \mathrm{H}_1(X_i; \mathbb{Z})$ for $i = 1, 2$ such that $b = [\beta'] + h_1 + h_2$. Lemma 3.2 says that for $i = 1, 2$ there exists an oriented properly embedded arc $\epsilon_i'$ in $X_i$ with the same endpoints as $\epsilon_i$ such that the homology class of the loop $\epsilon_i' \cdot \epsilon_i^{-1}$ equals $h_i$. Letting $\beta$ be the loop $\epsilon_1' \cdot \epsilon_2'$, it follows that $[\beta] = [\beta'] + h_1 + h_2 = b$, as desired. □

**Proof of Lemma 3.1** Let $\delta$ and $\delta'$ be vertices of $\mathcal{H}_{a,b}$. We will construct a path in $\mathcal{H}_{a,b}$ from $\delta$ to $\delta'$. Without loss of generality, $[\delta] = [\delta'] = a$. By [17, Theorem 1.9] (see [12] for an alternate proof), we can find a sequence

$$\delta = \alpha_1, \alpha_2, \ldots, \alpha_n = \delta'$$

of isotopy classes of oriented simple closed curves on $\Sigma_g$ such that $[\alpha_i]=a$ for $1\le i\le n$ and $i_g(\alpha_i,\alpha_{i+1})=0$ for $1\le i<n$ (this is where we use the condition $g\ge 3$). Lemma 3.3 implies that there exist isotopy classes $\beta_1,\dots,\beta_{n-1}$ of oriented simple closed curves on $\Sigma_g$ such that $[\beta_i]=b$ and $i_g(\alpha_i,\beta_i)=i_g(\alpha_{i+1},\beta_i)=1$ for $1\le i<n$. Since $\beta_i$ is adjacent to both $\alpha_i$ and $\alpha_{i+1}$ in $\mathcal{H}_{a,b}$, the desired path from $\delta$ to $\delta'$ is thus

$$\delta=\alpha_1,\beta_1,\alpha_2,\beta_2,\dots,\beta_{n-1},\alpha_n=\delta'. \qquad \square$$

## 4 Generating the stabilizer of a nonseparating simple closed curve

Let the subsurfaces $R'_i$ of $\Sigma_g$ be as in the introduction. Define $S_i=\overline{\Sigma_g\setminus R'_i}$. The goal of this section is to prove the following lemma.

**Lemma 4.1** *Assume that $g\ge 4$. Let $\gamma$ be the isotopy class of a simple closed nonseparating curve on $\Sigma_g$ that is contained in $R'_1$. Then the subgroup $(\mathcal{I}_g)_\gamma$ of $\mathcal{I}_g$ stabilizing $\gamma$ is contained in the subgroup of $\mathcal{I}_g$ generated by $\cup_{i=1}^{g}\mathcal{I}(\Sigma_g,S_i)$.*

Before proving this, we need a technical lemma. Set $\pi=\pi_1(\Sigma_{g,1},*)$, where $*\in\partial\Sigma_{g,1}$. Let $T'_1,\dots,T'_g$ be disjoint subsurfaces of $\Sigma_{g,1}$ such that $T'_i\cong\Sigma_{1,1}$ and $T'_i\cap\partial\Sigma_{g,1}=\varnothing$ for $1\le i\le g$ (see Figure 3(a)). Define $T_i=\overline{\Sigma_{g,1}\setminus T'_i}$. We have $T_i\cong\Sigma_{g-1,2}$ and $*\in T_i$ for $1\le i\le g$. The maps $\pi_1(T_i,*)\to\pi_1(\Sigma_{g,1},*)$ and $\mathrm{H}_1(T'_i;\mathbb{Z})\to\mathrm{H}_1(\Sigma_{g,1};\mathbb{Z})$ are injective; we will identify $\pi_1(T_i,*)$ and $\mathrm{H}_1(T'_i;\mathbb{Z})$ with their images in $\pi_1(\Sigma_{g,1},*)$ and $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$, respectively. Define $K_i=[\pi,\pi]\cap\pi_1(T_i,*)$. We then have the following.

**Lemma 4.2** *For $g\ge 3$, the group $[\pi,\pi]$ is generated by the $\mathcal{I}_{g,1}$–orbits of the set $\cup_{i=1}^{g}K_i$.*

The proof of this will have two ingredients. The first is the following theorem of Tomaszewski. As notation, if $G$ is a group and $a,b\in G$, then $[a,b]:=a^{-1}b^{-1}ab$ and $a^b:=b^{-1}ab$.

**Theorem 4.3** (Tomaszewski, [20]) *Let $F_n$ be the free group on $\{x_1,\dots,x_n\}$. Then the set*

$$\{[x_i,x_j]^{x_i^{k_i}x_{i+1}^{k_{i+1}}\cdots x_n^{k_n}} \mid 1\le i<j\le n \text{ and } k_m\in\mathbb{Z} \text{ for all } i\le m\le n\}$$

*is a free basis for $[F_n,F_n]$.*

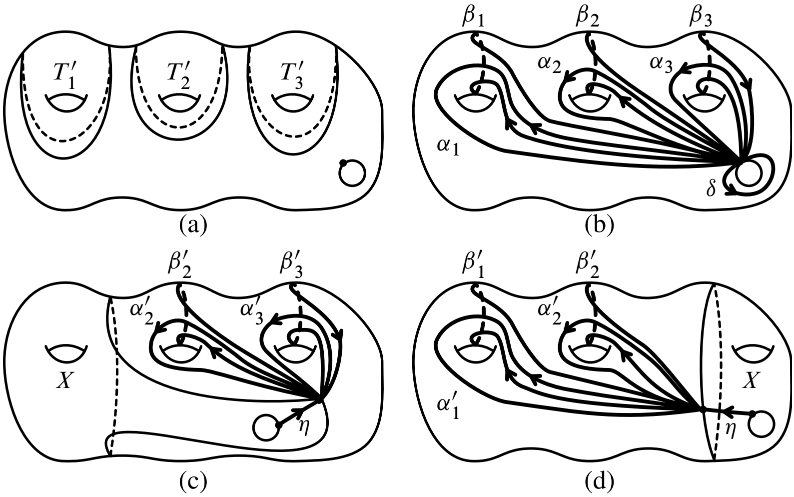


Figure 3: (a) The subsurfaces $T_i'$ (b) The standard basis for $\pi$ (c) The surface $X$ when $i = 1$ (d) The surface $X$ when $i = g$

The second is the following lemma about the action of $\mathcal{I}_{g,1}$ on $\pi$. Choose a standard basis $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ for $\pi$ (as in Figure 3(b)) such that $\alpha_i$ and $\beta_i$ are freely homotopic into $T_i'$ for $1 \le i \le g$. Our proof of Lemma 4.2 would be much simpler if the image of $\text{Mod}_{g,1}$ in $\text{Aut}(\pi)$ contained the inner automorphisms – since inner automorphisms act trivially on homology, this would imply that the $\mathcal{I}_g$–orbits of $\{[x, y] \mid x, y \in \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}\}$ generate $[\pi, \pi]$. However, the image of $\text{Mod}_{g,1}$ in $\text{Aut}(\pi)$ does not contain the inner automorphisms since $\text{Mod}_{g,1}$ fixes the loop $\delta = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$ depicted in Figure 3(b). The following lemma is a weak replacement for this.

**Lemma 4.4** *Let $i$ be either 1 or $g$. Consider $h \in \text{H}_1(T_i'; \mathbb{Z})$. There then exists some $w \in \langle \alpha_i, \beta_i, \delta \rangle$ and $f \in \mathcal{I}_{g,1}$ such that $[w] = h$ and such that $f(a_j) = a_j^w$ and $f(b_j) = b_j^w$ for $1 \le j \le g$ with $j \ne i$.*

**Proof** Let $X$ be a regular neighborhood of the curves $\alpha_i \cup \beta_i \cup \partial \Sigma_{g,1}$ depicted in Figure 3(b). Thus $X \cong \Sigma_{1,2}$, the surface $T_i'$ is homotopic into $X$, and the image of $\pi_1(X, *)$ in $\pi$ is $\langle \alpha_i, \beta_i, \delta \rangle$. Let $Y = \overline{\Sigma_{g,1} \setminus X}$, so $Y \cong \Sigma_{g-1,1}$ and $X \cap Y \cong S^1$. The key property of $X$ is as follows (this is where we use the assumption that $i$ is either 1 or $g$). There exists some $*' \in X \cap Y$, a properly embedded arc $\eta$ in $X$ from $*$ to $*'$, and elements

$$\{\alpha_j', \beta_j' \mid 1 \le j \le g,\ j \ne i\} \subset \pi_1(Y, *')$$

such that $\alpha_j = \eta \cdot \alpha_j' \cdot \eta^{-1}$ and $\beta_j = \eta \cdot \beta_j' \cdot \eta^{-1}$ for $1 \le j \le g$ with $j \ne i$. See Figure 3(c) for the case $i = 1$ and Figure 3(d) for the case $i = g$.

By Lemma 3.2, there exists an oriented properly embedded arc $\eta'$ in $X$ whose endpoints are the same as those of $\eta$ such that the homology class of $w := \eta \cdot (\eta')^{-1} \in \pi$ in $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$ is $h$. Observe that $w \in \langle \alpha_i, \beta_i, \delta \rangle$. Also,

$$\eta' \cdot \alpha_j' \cdot (\eta')^{-1} = w^{-1} \cdot \eta \cdot \alpha_j' \cdot \eta^{-1} \cdot w = \alpha_j^w$$

for $j \neq i$, and similarly for $\beta_j$. It is thus enough find some $f \in \mathcal{I}(\Sigma_g, X)$ such that $f(\eta) = \eta'$.

The "change of coordinates principle" from [5, Section 1.3] implies that there exists some $f' \in \mathrm{Mod}(\Sigma_g, X)$ such that $f'(\eta) = \eta'$. Briefly, an Euler characteristic calculation shows that cutting $X$ open along either $\eta$ or $\eta'$ results in a surface homeomorphic to $\Sigma_{1,1}$. Choosing an orientation-preserving homeomorphism between these two cut-open surfaces and gluing the boundary components back together in an appropriate way, we obtain some $f' \in \mathrm{Mod}(\Sigma_g, X)$ such that $f'(\eta) = \eta'$. See [5, Section 1.3] for more details and many other examples of arguments of this form.

The mapping class $f'$ need not lie in Torelli; however, it satisfies $f'([\alpha_j]) = [\alpha_j]$ and $f'([\beta_j]) = [\beta_j]$ for $j \neq i$ and $f'(\mathrm{H}_1(T_i';\mathbb{Z})) = \mathrm{H}_1(T_i';\mathbb{Z})$. Since the image of $\mathrm{Mod}(T_i')$ in $\mathrm{Aut}(\mathrm{H}_1(T_i';\mathbb{Z})) = \mathrm{Aut}(\mathbb{Z}^2)$ is $\mathrm{SL}_2(\mathbb{Z})$, we can choose some $f'' \in \mathrm{Mod}(\Sigma_g, T_i')$ such that $f'([\alpha_i]) = f''([\alpha_i])$ and $f'([\beta_i]) = f''([\beta_i])$. It follows that $f := f' \cdot (f'')^{-1}$ lies in $\mathcal{I}(\Sigma_g, X)$ and satisfies $f(\eta) = \eta'$, as desired. □

**Proof of Lemma 4.2** The generating set for $[F_n, F_n]$ in Theorem 4.3 depends on an ordering of the generators for $F_n$. It seems hard to prove the lemma using the generating set corresponding to the standard ordering

$$(x_1, x_2, \dots, x_{2g}) = (\alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$$

of the generators for $\pi \cong F_{2g}$. However, consider the following nonstandard ordering on the generators for $\pi$:

$$(x_1, x_2, \dots, x_{2g}) = (\alpha_2, \beta_2, \alpha_1, \beta_1, \alpha_3, \beta_3, \alpha_4, \beta_4, \dots, \alpha_g, \beta_g).$$

Let $S$ be the generating set for $[\pi, \pi]$ given by Theorem 4.3 using this ordering of the generators. All the elements of $S$ lie in $K_2$ except for

$$(3) \qquad [\alpha_2, \zeta]^{\alpha_2^{n_2} \beta_2^{m_2} \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \cdots \beta_g^{m_g}} \quad \text{and} \quad [\beta_2, \zeta']^{\beta_2^{m_2} \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \cdots \beta_g^{m_g}};$$

here $\zeta \in \{\beta_2, \alpha_1, \beta_1, \alpha_3, \dots, \beta_g\}$ and $\zeta' \in \{\alpha_1, \beta_1, \alpha_3, \dots, \beta_g\}$ and $n_i, m_i \in \mathbb{Z}$. Letting $T \subset S$ be the elements in (3), we must show that every $t \in T$ can be expressed as a product of elements in the $\mathcal{I}_{g,1}$–orbit of the set $\cup_{i=1}^{g} K_i$. Consider $t \in T$, so either $t = [\alpha_2, \zeta]^{\alpha_2^{n_2} \beta_2^{m_2} \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \cdots \beta_g^{m_g}}$ or $t = [\beta_2, \zeta]^{\beta_2^{m_2} \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \cdots \beta_g^{m_g}}$. There are two cases.

**Case 1** $\zeta \notin \{\alpha_1, \beta_1\}$.

We will do the case where $t = [\alpha_2, \zeta]^{\alpha_2^{n_2} \beta_2^{m_2} \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \cdots \beta_g^{m_g}}$; the other case is treated in a similar way. Set $t' = [\alpha_2, \zeta]^{\alpha_2^{n_2} \beta_2^{m_2} \alpha_3^{n_3} \cdots \beta_g^{m_g}}$, so $t' \in K_1$. By Lemma 4.4, there exists some $w \in \{\alpha_1, \beta_1, \delta\}$ and $f \in \mathcal{I}_{g,1}$ such that $[w] = [\alpha_1^{n_1} \beta_1^{m_1}]$ and such that $f(a_j) = a_j^w$ and $f(b_j) = b_j^w$ for $j > 1$. This implies that $f(t') = [\alpha_2, \zeta]^{\alpha_2^{n_2} \beta_2^{m_2} \alpha_3^{n_3} \cdots \beta_g^{m_g} w}$. Now, $\alpha_3^{n_3} \cdots \beta_g^{m_g} w$ and $\alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \cdots \beta_g^{m_g}$ are homologous, so there exists some $\theta \in [\pi, \pi]$ such that $\alpha_3^{n_3} \cdots \beta_g^{m_g} w \theta = \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \cdots \beta_g^{m_g}$. Moreover, since $w \in \langle a_1, b_1, \delta \rangle$ we have $\theta \in K_2$. Observe now that

$$\theta^{-1} \cdot f(t') \cdot \theta = [\alpha_2, \zeta]^{\alpha_2^{n_2} \beta_2^{m_2} \alpha_3^{n_3} \cdots \beta_g^{m_g} w \theta} = [\alpha_2, \zeta]^{\alpha_2^{n_2} \beta_2^{m_2} \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \cdots \beta_g^{m_g}} = t.$$

We have thus found the desired expression for $t$.

**Case 2** $\zeta' \in \{\alpha_1, \beta_1\}$.

This case is similar to Case 1. The only difference is that the $\alpha_g^{n_g} \beta_g^{m_g}$ term of $t$ is deleted to form $t'$ instead of the $\alpha_1^{n_1} \beta_1^{m_1}$ term. □

**Proof of Lemma 4.1** Let $I$ be the subgroup of $\mathcal{I}_g$ generated by $\cup_{i=1}^{g} \mathcal{I}(\Sigma_g, S_i)$. Using the notation of Section 2, there is a surjection $\rho\colon \mathcal{I}_{g,\gamma} \to (\mathcal{I}_g)_\gamma$ induced by a continuous map $\phi\colon \Sigma_{g,\gamma} \to \Sigma_g$. Define $X = \phi^{-1}(S_1)$, so $X \cong \Sigma_{g-1,1}$. Letting $\mathcal{I}(X)$ be the Torelli group of $X$, Lemma 2.3 gives a decomposition $\mathcal{I}_{g,\gamma} = K_{g,\gamma} \ltimes \mathcal{I}(X)$. Clearly $\rho(\mathcal{I}(X)) = \mathcal{I}(\Sigma_g, S_1) \subset I$. Also, Lemma 4.2 implies that $K_{g,\gamma}$ is generated by the $\mathcal{I}(X)$–conjugates of a set $S \subset K_{g,\gamma}$ such that $\rho(S) \subset I$. We conclude that $\rho(\mathcal{I}_{g,\gamma}) \subset I$, as desired. □

## 5 Proof of main theorem

We finally prove our main theorem. The key is the following standard lemma, whose proof is similar to that given in [19, (1) of Appendix to Section 3] and is thus omitted.

**Lemma 5.1** *Consider a group $G$ acting without inversions on a connected graph $X$. Assume that $X/G$ consists of a single edge $\overline{e}$. Let $e$ be a lift of $\overline{e}$ to $X$ and let $v$ and $v'$ be the endpoints of $e$. Then $G$ is generated by $G_v \cup G_{v'}$.*

To apply this, we will need the following lemma.

**Lemma 5.2** *Let $a, b \in \mathrm{H}_1(\Sigma_g; \mathbb{Z})$ satisfy $i_a(a, b) = 1$. Then $\mathcal{H}_{a,b}/\mathcal{I}_g$ is isomorphic to a graph with a single edge.*

The proof is similar to the proofs of [16, Lemma 6.2] and [18, Lemma 6.9], and is thus omitted.

**Proof of Theorem B** Let $R'_1, \ldots, R'_g$ and $R_{ijk}$ be the subsurfaces of $\Sigma_g$ from the introduction. Let $\Gamma$ be the subgroup of $\mathcal{I}_g$ generated by $\bigcup_{1 \le i<j<k \le g} \mathcal{I}(\Sigma_g, R_{ijk})$. Our goal is to prove that $\Gamma = \mathcal{I}_g$.

The proof will be by induction on $g$. The base case $g = 3$ is trivial, so assume that $g \geq 4$ and that the theorem is true for all smaller $g$ such that $g \geq 3$. Choose simple closed curves $\alpha$ and $\beta$ in $R'_1$ such that $i_g(\alpha, \beta) = 1$. Observe that $R'_1$ is a closed regular neighborhood of $\alpha \cup \beta$. Set $a = [\alpha]$ and $b = [\beta]$. Clearly $\mathcal{I}_g$ acts on $\mathcal{H}_{a,b}$ without inversions. Lemmas 3.1 and 5.2 show that the action of $\mathcal{I}_g$ on $\mathcal{H}_{a,b}$ satisfies the other conditions of Lemma 5.1. We deduce that $\mathcal{I}_g$ is generated by the union $(\mathcal{I}_g)_\alpha \cup (\mathcal{I}_g)_\beta$ of the stabilizer subgroups of $\alpha$ and $\beta$.

Recall that $S_i = \overline{\Sigma_g \setminus R'_i}$ for $1 \leq i \leq g$. By Lemma 4.1, both $(\mathcal{I}_g)_\alpha$ and $(\mathcal{I}_g)_\beta$ are contained in the subgroup generated by $\cup_{i=1}^{g} \mathcal{I}(\Sigma_g, S_i)$. We must prove that $\mathcal{I}(\Sigma_g, S_i) \subset \Gamma$ for $1 \leq i \leq g$. We will do the case $i = g$; the other cases are similar. We have a Birman exact sequence

$$1 \longrightarrow \pi_1(U\Sigma_{g-1}) \longrightarrow \mathcal{I}(\Sigma_g, S_g) \longrightarrow \mathcal{I}_{g-1} \longrightarrow 1.$$

By induction, the subset $\bigcup_{1 \le i<j<k \le g-1} \mathcal{I}(\Sigma_g, R_{ijk})$ of $\mathcal{I}(\Sigma_g, S_g)$ projects to a generating set for $\mathcal{I}_{g-1}$. Also, it is clear that the disc-pushing subgroup $\pi_1(U\Sigma_{g-1})$ of $\mathcal{I}(\Sigma_g, S_g)$ is generated by elements that lie in $\bigcup_{1 \le i<j<g} \mathcal{I}(\Sigma_g, R_{ijg})$. We conclude that $\mathcal{I}(\Sigma_g, S_g) \subset \Gamma$, as desired. □

# References


[1] **J S Birman**, *On Siegel's modular group*, Math. Ann. 191 (1971) 59–68 MR0280606

[2] **T Brendle**, **B Farb**, *personal communication*

[3] **K S Brown**, *Presentations for groups acting on simply-connected complexes*, J. Pure Appl. Algebra 32 (1984) 1–10 MR739633

[4] **B Farb**, *Some problems on mapping class groups and moduli space*, from: "Problems on mapping class groups and related topics", Proc. Sympos. Pure Math. 74, Amer. Math. Soc., Providence, RI (2006) 11–55 MR2264130

[5] **B Farb**, **D Margalit**, *A Primer on Mapping Class Groups*, to be published by Princeton University Press

[6] **R Hain**, *Fundamental groups of branched coverings and the Torelli group in genus 3*, in preparation

[7] **A Hatcher**, **W Thurston**, *A presentation for the mapping class group of a closed orientable surface*, Topology 19 (1980) 221–237 MR579573

[8] **D Johnson**, *Conjugacy relations in subgroups of the mapping class group and a group-theoretic description of the Rochlin invariant*, Math. Ann. 249 (1980) 243–263 MR579104

[9] **D Johnson**, *The structure of the Torelli group. I. A finite set of generators for $\mathcal{I}$*, Ann. of Math. (2) 118 (1983) 423–442 MR727699

[10] **D Johnson**, *A survey of the Torelli group*, from: "Low-dimensional topology (San Francisco, Calif., 1981)", Contemp. Math. 20, Amer. Math. Soc., Providence, RI (1983) 165–179 MR718141

[11] **D Johnson**, *The structure of the Torelli group. III. The abelianization of $\mathcal{T}$*, Topology 24 (1985) 127–144 MR793179

[12] **D Margalit**, **A Hatcher**, *Generating the Torelli group*, in preparation

[13] **D McCullough**, **A Miller**, *The genus 2 Torelli group is not finitely generated*, Topology Appl. 22 (1986) 43–49 MR831180

[14] **G Mess**, *The Torelli groups for genus 2 and 3 surfaces*, Topology 31 (1992) 775–790 MR1191379

[15] **J Powell**, *Two theorems on the mapping class group of a surface*, Proc. Amer. Math. Soc. 68 (1978) 347–350 MR0494115

[16] **A Putman**, *Cutting and pasting in the Torelli group*, Geom. Topol. 11 (2007) 829–865 MR2302503

[17] **A Putman**, *A note on the connectivity of certain complexes associated to surfaces*, Enseign. Math. (2) 54 (2008) 287–301 MR2478089

[18] **A Putman**, *An infinite presentation of the Torelli group*, Geom. Funct. Anal. 19 (2009) 591–643 MR2545251

[19] **J-P Serre**, *Trees*, Springer, Berlin (1980) MR607504 Translated from the French by John Stillwell

[20] **W Tomaszewski**, *A basis of Bachmuth type in the commutator subgroup of a free group*, Canad. Math. Bull. 46 (2003) 299–303 MR1981684


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