TSUKUBA J. MATH.
Vol. 22 No. 3 (1998), 559–574

ON QUASILINEAR HYPERBOLIC EQUATIONS WITH DEGENERATE PRINCIPAL PART

By

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§ 1. Introduction

As it is well known, the Cauchy problem for a nonlinear strictly hyperbolic system of first order

$$\begin{cases} u_t = f(t,x,u,u_x), \\ u(0,x) = u_0(x) \end{cases}$$

is locally solvable in C^∞, or even in Sobolev spaces $H^s(\boldsymbol{R}^n)$ of order large enough ($s > 1 + n/2$). More generally, this is true for any symmetrizable pseudodifferential nonlinear system of first order, or higher order strictly hyperbolic equation. These results include as a special case the higher order hyperbolic equations.

Indeed, the corresponding linearized system

$$u_t = g_1(t,x,v,v_x)u_x + g_2(t,x,v,v_x)u + h(t,x,v,v_x)$$

satisfies an energy estimate of the form

$$\|u(t,\cdot)\|_s \leq C(\|v(t,\cdot)\|_s)[\|u_0\|_s + 1], \quad 0 \leq t \leq T$$

($\|\cdot\|_s$ denotes the $H^s(\boldsymbol{R}^n)$ norm). Using such estimates, the local existence for the nonlinear system is essentially a consequence of the contraction mapping principle (see [Di]). We should remark, however, that the local existence in Sobolev spaces of lower order is a delicate problem. It is possible to improve the preceding result slightly by using paradifferential tecniques, but if the Sobolev order s is too small the local existence does not hold in general, even for the simplest cases such as the semilinear wave equation (see e.g. [L]).

On the other hand, the question of local existence for degenerate hyperbolic equations or systems is much more difficult, and still largely open. In the following, we shall restrict ourselves to the second order equations of the form

Received May 21, 1997.

$$u_{tt} = \sum_{ij} (a_{ij}(t,x)u_{xi})_{xj} + f(t,x,u,u_x). \tag{1.1}$$

This equation is weakly hyperbolic if the form $\sum a_{ij}\xi_i\xi_j$ is nonnegative (for simplicity we consider real valued functions). It is always possible to solve locally this equation, or even the above general first order system, in Gevrey classes of suitable order; for the most general results in this directions see [K]. But for the local existence in C^∞, already in the linear case there are two main obstructions:

1) Lower order terms must be dominated in a suitable sense by the principal part of the equation (so-called Levi conditions). For instance, the equation

$$u_{tt} = u_x$$

is not solvable in C^∞, but only in Gevrey classes of order less than 2.

2) The oscillations in time of the coefficients of the elliptic part can destroy the solvability in C^∞. For instance, in [CS] an example is constructed of the form

$$u_{tt} = a(t)u_{xx}$$

which is not locally solvable; the function $a(t)$ is C^∞, satisfies $a(0) = 0$, $a(t) > 0$ for $t > 0$, and has an infinite number of oscillations as $t \to 0+$.

Thus some additional assumptions are needed, taking into account the above obstructions. One of the first results in this direction was proved by O. Oleinik [O], regarding the linear equation

$$u_{tt} = \sum_{ij} (a_{ij}(t,x)u_{x_i})_{x_j} + \sum_j b_j(t,x)u_{x_j} + f(t,x). \tag{1.2}$$

Under the assumption

$$\alpha\Big(\sum b_j\xi_j\Big)^2 \cdot t \le A\sum_{ij} a_{ij}\xi_i\xi_j + \partial_t \sum_{ij} a_{ij}\xi_i\xi_j \tag{1.3}$$

the Cauchy problem for (1.2) is solvable in C^∞. More precisely, an estimate of the following form holds:

$$\|u(t,\cdot)\|_s \le M\left[\|u_0\|_{s+p+4} + \|u_1\|_{s+p+3} + \sup_{0\le t\le T} \sum_{j+k\le s+p+2} \|\partial_t^j f(t,\cdot)\|_k\right] \tag{1.4}$$

where u_0, u_1 are the initial data. Note that, in contrast with the strictly hyperbolic case, there is a *loss of derivatives*, i.e., the solution is less regular than the data and the right hand member f; oddly enough, the loss is connected to the size of the coefficient α in (1.3), that is to say,

$$p > \frac{1}{2\alpha} - 3. \tag{1.5}$$

This is not an artificial consequence of the tecnique, but an actual phenomenon, as shown by the explicit example of Qi Min-You [Q]

$$u_{tt} - t^2 u_{xx} = a u_x, \quad u(0,x) = \mu(x), \quad u_t(0,x) = 0$$

with $a = 4n+1$, $n \in \boldsymbol{N}$, whose solution is

$$u(t,x) = \sum_{k=0}^{n} \frac{\sqrt{\pi} t^{2k}}{k!(n-k)!\Gamma(k+1/2)} \frac{\partial^k}{\partial x^k} \mu(x + t^2/2).$$

This example shows that assumption (1.3) (with (1.5)) is in some sense close to optimal.

The loss of derivatives makes it impossible to prove the local existence in the nonlinear case (1.1) as simply as for the strictly hyperbolic equations. Indeed, one cannot find a Banach space where to apply the contraction mapping principle. However, it is possible to overcome this difficulty by resorting to the Nash-Moser theorem. Thus we shall consider the Cauchy problem

$$u_{tt} = \sum_{ij} (a_{ij}(t,x) u_{x_i})_{x_j} + f(t,x,u,u_x) \tag{1.6}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{1.7}$$

under the assumptions

$$0 \le \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \le \Lambda \tag{1.8}$$

and

$$\alpha \left(\sum_{i=1}^{n} f_{x_i}(t,x,y,p) \xi_i \right)^2 \cdot t \le A \sum_{i,j=1}^{n} a_{ij}(t,x) \xi_i \xi_j + \partial_t \sum_{i,j=1}^{n} a_{ij}(t,x) \xi_i \xi_j \tag{1.9}$$

for some $A, \alpha > 0$, for all t, x, y, p, and we shall prove:

Theorem 1. *Assume that* (1.8), (1.9) *hold and the coefficients in* (1.6) *are* C^∞ *functions. Then, for any initial data* $u_0, u_1 \in C_0^\infty(\boldsymbol{R}^n)$, *Problem* (1.6), (1.7) *has a unique* (*local*) *solution in* $C^\infty([0, T_0], \boldsymbol{R}^n)$.

Remark 2. By similar techniques, it is possible to prove an analogous result by replacing assumption (1.9) with

$$\alpha(T-t)\left(\sum_{i=1}^{n} f_{x_i}\xi_i\right)^2 \leq A\sum_{i,j=0}^{n} a_{ij}\xi_i\xi_j - \partial_t \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j + (\alpha(T-t))^{-1}\sum_{i,j=1} a_{ij}\xi_i\xi_j \tag{1.10}$$

(see Remark 2.3). Under this assumption we can prove the local unique solvability near $t = T$ (and of course near any $t = t_0$).

REMARK 3. To our knowledge, the Nash-Moser theorem was applied in the study of nonlinear non-strictly hyperbolic equations for the first time by Iwasaki [I1], [I2], and later by Gourdin [G]. These results concern equations with constant multiplicity. In [D], [DT] the Nash-Moser theorem was used to solve nonlinear hyperbolic equations of second order with variable multiplicity. We finally recall that a partial result in the direction of the present paper was proved in [DM].

REMARK 4. It should be noticed that if the elliptic part vanishes of infinite order, then assumption (1.9) (resp. (1.10)) is no longer optimal. For more details, see [RY], [T].

REMARK 5. Theorem 1 can be extended without difficulty to the more general equations

$$u_{tt} = \sum_{ij}(a_{ij}(t,x)u_{x_i})_{x_j} + f(t,x,u,u_x,u_t)$$

under the same assumptions.

§2. The linear theorem

The basic tools in the proof of Theorem 1 are Oleinik's estimate for the linear equation, and Nash-Moser theorem. For the convenience of the reader, in this section we briefly recall the main indeas of Oleinik's result ([O]), with minor modifications in view of the application to the nonlinear case.

Thus we consider the Cauchy problem

$$L(u) \equiv u_{tt} - \sum_{i,j=1}^{n}(a_{ij}(t,x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(t,x)u_{x_i} + b_0(t,x)u_t + c(t,x)u = f(t,x) \tag{2.1}$$

$$u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x) \tag{2.2}$$

and we assume that it is weakly hyperbolic, i.e.,

$$0 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq \Lambda. \tag{2.3}$$

Moreover, we shall assume that Oleinik's condition is satisfied, namely that

$$\alpha\left(\sum_{i=1}^{n} b_i\xi_i\right)^2 \cdot t \leq A \sum_{i,j=1}^{n} a_{ij}\xi_i x_j + \partial_t \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j. \tag{2.4}$$

Then it is possible to prove the following *a priori* estimate (by $\boldsymbol{T}^n$ we denote the n-dimensional torus):

Lemma 2.1. *Assume the coefficients in* (2.1) *are* C^∞, *and satisfy* (2.3), (2.4), *where* $A > 0$, $\alpha > (2p+6)^{-1}$ *for* $0 \leq t \leq t_0$, $t_0 = const. > 0$, *while the constant* α *can be arbitrary for* $t_0 \leq t \leq T$.

Then any C^∞ *solution* $u(t,x)$ *of* (2.1), (2.2) *on* $[0,T] \times \boldsymbol{R}^n$ (*resp. on* $[0,T] \times \boldsymbol{T}^n$) *satisfies, for* $0 \leq t \leq T$ *and any* $k \geq 0$, *the estimate*

$$\sum_{j=0}^{k} \|\partial_t^j u(t,\cdot)\|_{k-j} \leq C_{p+3}\Bigg[\|\phi\|_{k+p+4} + \|\psi\|_{k+p+3} + \max_{0\leq t\leq T} \sum_{j=0}^{k+p-1} \|\partial_t^j f\|_{k+2p+4-j} + C_{p+k+3}\Bigg] \tag{2.5}$$

where $\|\cdot\|_s$ *is the norm of* $H^s(\boldsymbol{R}^n)$ (*resp.* $H^s(\boldsymbol{T}^n)$), *while*

$$C_r = \mu(r)\left(1 + \max_{t,x}\max_{|\alpha|\leq r}\max_{\sigma} |\partial_{t,x}^\alpha \sigma(t,x)|\right),$$

with (t,x) *varying on* $[0,T] \times \boldsymbol{R}^n$ (*or* $[0,T] \times \boldsymbol{T}^n$), *and* σ *varying among the coefficients* a_{ij}, b_i, b_0, c ($\mu(r)$ *depends on* r *only*).

Remark 2.2. Estimate (2.5) is slightly modified with respect to the original Oleinik's Lemma, in order to make more explicit the dependence on the coefficients as k increases.

Before sketching the proof of the Lemma, we notice two consequences of it. First, (2.5) implies the global unique solvability in C^∞ or in Sobolev spaces $H^s(\boldsymbol{R}^n)$ or $H^s(\boldsymbol{T}^n)$ for (2.1), (2.2). This can be proved in several (standard) ways: for instance, one can approximate (2.1) by a sequence of strictly hyperbolic equations, e.g., replacing a_{ij} by $a_{ij} + \varepsilon\delta_{ij}$; notice that the modified coefficients satisfy assumption (2.4) with the same constants. These approximating equations have global solution by the classical theory, and (2.5) holds uniformly for the solutions; by a compactness argument, we obtain in the limit the unique solution to (2.1), (2.2), provided the initial data belong to the suitable Sobolev spaces. Second, the same kind of approximations implies that equation (2.1) has the property of the finite speed of propagation, with speed $\sqrt{\Lambda}$ as in the strictly hyperbolic case.

SKETCH OF THE PROOF OF LEMMA 1.1. To fix the ideas, we shall work on $\boldsymbol{T}^n$; the case of $\boldsymbol{R}^n$ is analogous.

Assume $\tilde{u}$ is a solution of (2.1), (2.2), and define

$$v_p(t,x) = \phi(x) + t\psi(x) + \frac{t^2}{2!}\partial_t^2\tilde{u}(0,x) + \cdots + \frac{t^{p+2}}{(p+2)!}\partial_t^{p+2}\tilde{u}(0,x);$$

then the function $u = \tilde{u} - v_p$ satifies the equation

$$L(u) = f - L(v_p) \equiv \mathscr{F}(t,x). \tag{2.6}$$

Let

$$w = \int_t^\tau u(\sigma,x)\,d\sigma.$$

Multiply (2.6) by $we^{\theta t}$, where $\theta > 0$ is a constant, and integrate over the domain

$$G_\tau = [0,\tau] \times \boldsymbol{T}^n$$

Suppose $0 \le t \le t_0$. We transform the individual terms of the equality so obtained: we have

$$\int_{G_\tau} u_{tt} we^{\theta t} = \frac{1}{2}u(\tau,x)^2 e^{\theta\tau} + \int_{G_\tau} u\left(\theta^2 w - \frac{3}{2}u\right)e^{\theta t},$$

$$\int_{G_\tau} (a_{ij}u_{x_i})_{x_j} we^{\theta t} = -\frac{1}{2}\int_{G\tau} (\theta a_{ij} + \partial_t a_{ij}) w_{x_i} w_{x_j} e^{\theta t} - \frac{1}{2}\int a_{ij} w_{x_i}(0,x) w_{x_j}(0,x)\,dx,$$

$$\int_{G_\tau} b_0 u_t we^{\theta t} = \int_{G_\tau} (u^2 b_0 - uwb_{0_t} - uw\theta b_0)e^{\theta t}$$

$$\int_{G\tau} b_i u_{x_i} we^{\theta t} = -\int_{G_\tau} b_{i_{x_i}} uwe^{\theta t} - \int_{G_\tau} b_i w_{x_i} ue^{\theta t}.$$

In particular, the last inequality implies

$$\left|\int_{G_\tau} b_i u_{x_i} we^{\theta t}\right| \le M_2\tau^2 \int_{G_\tau} u^2 t^{-1} e^{\theta t} + \frac{1}{2}\alpha \int_{G_\tau} t b_i^2 w_{x_i}^2 e^{\theta t} + \frac{1}{2\alpha}\int_{G_\tau} u^2 t^{-1} e^{\theta t}.$$

As to the last term, it is easy to see that

$$\iint \mathscr{F} w e^{\theta t}\, dt\, dx = (-1)^{p+1} \iint \partial_t^{p+1} \mathscr{F}\, W_{p+1}\, dx\, dt, \tag{2.7}$$

with

$$W_0 = w e^{\theta t}, \quad W_{\nu+1} = \int_t^\tau W_\nu(\sigma, x)\, d\sigma, \quad \nu = 0, 1, \ldots, p,$$

and that

$$|W_{p+1}|^2 \le \tau^{2p+3} e^{2\theta T} \int_0^\tau u^2(\sigma, x)\, d\sigma,$$

$$\left| \iint \mathscr{F} w e^{\theta t}\, dt\, dx \right| \le \delta \iint u^2 t^{-1} e^{\theta t}\, dx\, dt + \frac{e^{2\theta T} \tau^{2p+6}}{4\delta} \max_{0 \le \sigma \le t_0} \int |\partial_t^{p+1} \mathscr{F}(\sigma, x)|^2\, dx, \tag{2.8}$$

where δ is any constant such that $\alpha^{-1} + 2\delta < 2p + 6$. Using the estimates (2.7) and (2.8) and the above identities, we deduce from (2.6), provided $\theta \ge A$, that, for $\tau \le t_0$,

$$\tau y'(\tau) \le (\alpha^{-1} + 2\delta) y(\tau) + M_1 \tau y(\tau) + M_2 \tau^{2p+6} \max_{0 \le \sigma \le t_0} \int |\partial_t^{p+1} \mathscr{F}(\sigma, x)|^2\, dx, \tag{2.9}$$

where

$$y(\tau) = \int_{G_\tau} u^2 t^{-1} e^{\theta t}\, dx\, dt$$

and M_1, M_2 are constants depending on one derivative of the coefficients. Now we need the following generalized form of Gronwall's lemma:

Lemma 2. *Assume that* $y \in C^1([0, T])$ *satisfies*

$$y'(\tau) \le \frac{a}{\tau} y(\tau) + b y(\tau) + f(\tau) \tag{2.10}$$

for some $a, b \in \boldsymbol{R}$, $f \in L^\infty([0, T])$. *Suppose that*

(1) $\dfrac{y(\varepsilon)}{\varepsilon^a} \to 0$ *for* $\varepsilon \to 0$

(2) $|f(s)| \le c s^{a-1+\delta}$ *for some* $c, \delta > 0$.

Then

$$y(\tau) \leq \tau^a \int_0^\tau e^{b(\tau-s)} f(s) s^{-a}\, ds. \tag{2.11}$$

PROOF. If we multiply (2.10) by the function

$$g(\tau) = e^{-\int_\varepsilon^\tau (a/s+b)ds} \equiv \left(\frac{\varepsilon}{\tau}\right)^a e^{-b(\tau-\varepsilon)}$$

we easily obtain

$$[g(\tau)y]' \leq g(\tau)f(\tau).$$

We integrate from ε to τ, to obtain

$$g(\tau)y(\tau) \leq g(\varepsilon)y(\varepsilon) + \int_\varepsilon^\tau g(s)f(s)\, ds$$

and hence

$$y(\tau) \leq \frac{g(\varepsilon)}{g(\varepsilon)} y(\varepsilon) + \int_\varepsilon^\tau \frac{g(s)}{g(\tau)} f(s)\, ds$$

which can be written $(g(\varepsilon) = 1)$

$$y(\tau) \leq \tau^a e^{b(\tau-\varepsilon)} \frac{y(\varepsilon)}{\varepsilon^a} + \tau^a \int_\varepsilon^\tau \frac{f(s)}{s^a} e^{b(\tau-s)}\, ds, \quad 0 \leq \tau \leq T.$$

Letting $\varepsilon \to 0$, we obtain the thesis.

Applying Lemma 2 to (2.9) we get immediately

$$\begin{aligned} y(\tau) \leq \tau^{\alpha^{-1}+2\delta} C \int_0^\tau s^{2p+6-1-\alpha^{-1}-2\delta}\, ds \cdot \max_{0\leq\sigma\leq t_0} \int |\partial_t^{p+1}\mathscr{F}(\sigma,x)|^2\, dx\, d\tau \\ + C\tau^{2p+6} \max_{0\leq\sigma\leq t_0} \int |\partial_t^{p+1}\mathscr{F}(\sigma,x)|^2\, dx \end{aligned}$$

and therefore

$$\|u(t,\cdot)\|_{L^2}^2 \leq C\tau^{2p+6} \max_{0\leq\sigma\leq t_0} \int |\partial_t^{p+1}\mathscr{F}(\sigma,x)|^2\, dx$$

which implies the thesis for $k = 0$, and $0 \leq t \leq t_0$. The proof for $t \geq t_0$ is similar, only easier (the assumption (2.4) is stronger).

The higher order derivatives $D_x^\alpha u$ can be estimated in a similar way, by differentiating the equation and proceeding inductively. We skip the computations, which are similar to the above ones; the only subtle point is that the terms

$$\sum \int_{G_\tau} \partial_{x_r} a_{ij} D^{\gamma - e_r} u_{x_i} D^\gamma w e^{\theta t}$$

(here e_r is the multiindex with 1 in place r and 0 elsewhere) must be estimated by the following Oleinik's lemma (see [O]): for any $v \in C^2(\boldsymbol{R}^n)$, $r = 1, \dots, n$,

$$\left(\sum_{ij} (\partial_{x_r} a_{ij}) v_{x_i x_j} \right)^2 \leq M \sum_{ijk} a_{ij} v_{x_k x_i} v_{x_k x_j}$$

with M depending on the supremum of the second derivatives of the a_{ij}.

Finally, by differentiating the equation also with respect to time, it is easy to conclude the proof.

REMARK 2.4. An analogous result holds when assumption (1.3) is replaced with

$$\alpha(T - t) \left(\sum_{i=1}^n b_i \xi_i \right)^2 \leq A \sum_{i,j=1}^n a_{ij} \xi_i \xi_j - \partial_t \sum_{i,j=1}^n a_{ij} \xi_i \xi_j + (\alpha(T - t))^{-1} \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \tag{2.11}$$

where $A > 0$, while $\alpha > (2p + 1)^{-1}$ if $t_1 \leq t \leq T$, $t_1 = \text{const} < T$, and $\alpha > 0$ arbitrary for $0 \leq t \leq t_1$. In this case the following a priori estimate holds for a solution to (2.1), (2.2):

$$\int_0^t \sum_{j=0}^k \| \partial_t^j u(\sigma, \cdot) \|_{H^{k-j}}^2 \, d\sigma \leq C \left[\| \phi \|_{H^{k+p+1}}^2 + \| \psi \|_{H^{k+p}}^2 + \sum_{j=0}^{k+p} \int_0^t \| \partial_t^j f(\sigma, \cdot) \|_{H^{k+p-j}}^2 \, d\sigma \right]. \tag{2.12}$$

§ 3. Proof of the main theorem

We can now conclude the proof of Theorem 1. The essential definitions and results of the Nash-Moser theory are given in the Appendix at the end of the paper; in particular we refer to it for the notions of tame space, tame map, smooth tame map, and the statement of the Nash-Moser theorem in the form we shall use here.

We shall perform the proof on $\boldsymbol{T}^n$, since we need to work on a compact manifold; the proof on $\boldsymbol{R}^n$ follows without difficulty from this one by a standard localization argument (indeed, as we observed in Section 2, equation (2.1) enjoys the finite speed of propagation property). Let F be the Fréchet space $C^\infty([0,1]\times \boldsymbol{T}^n)$ endowed with the grading

$$|u(t,x)|_m = \sup_{0\le t\le 1}\sum_{j=0}^m \|\partial_t^j u(t,\cdot)\|_{H^{m-j}(\boldsymbol{T}^n)}.$$

With this grading F is a tame space. We consider the nonlinear application $P: F\to F$ defined as

$$(Pu)(t,x) = u(t,x) - u_0(x) - tu_1(x) - \int_0^t (t-s)\sum_{i,j=1}^n (a_{ij}(s)u_{x_i}(s,x))_{x_j}\,ds$$
$$- \int_0^t (t-s)f(s,x,u(s,x),u_x(s,x))\,ds.$$

Proving the local existence for the problem (1.6), (1.7) is equivalent to showing that the image of P contains a function which vanishes for $0\le t\le \varepsilon$, for some $\varepsilon>0$. To this end we shall apply the Nash-Moser theorem.

P is a smooth tame map because it is a composition of linear and nonlinear differential operators, and of integrations (see the remarks in Section 3). Its Fréchet derivative DP can be computed explicitly as follows: for all $u,h\in F$

$$DP(u)h = h(t,x) - \int_0^t (t-s)A(s)h(s,x)\,ds$$
$$- \int_0^t (t-s)\left[f_u(s,x,u,u_x)h(s,x) + \sum_{i=1}^n f_{u_{x_i}}(s,x,u,u_x)h_{x_i}(s,x)\right]ds;$$

where we write for brevity

$$A(s)v = \sum_{i,j=1}^n (a_{ij}(s,x)\partial_{x_i}v)_{x_j}.$$

We now verify the basic assumption of Nash-Moser theorem, i.e., that the equation

$$DP(u)h = k$$

can be solved in h for any $u,k\in F$, and the solution is a smooth tame map of u,k. This equation is equivalent to the Cauchy problem

$$h_{tt} = A(t)h + \sum_{i=1}^n f_{u_{x_i}}(u,u_x)h_{x_i} + f_u(u,u_x)h + k_{tt}$$

$$h(0,x) = k(0,x),\quad h_t(0,x) = k_t(0,x);$$

notice that this problem has a global smooth solution, thanks to Oleinik's result (see Lemma 2.1 and Remark 2.2).

The nth-order Fréchet derivative of VP can be expressed as follows: the function

$$h^{(n)} = D^n VP(u)\{k_1, \ldots, k_{n+1}\}$$

(a nonlinear function of u and a $(n+1)$-multilinear function of $k_1, \ldots, k_{n+1}$), is the solution of the Cauchy problem

$$\begin{aligned} h_{tt}^{(n)} &= A(t)h^{(n)} + \sum_{i=1}^{n} f_{u_{x_i}}(t,x,u,u_x)h_{x_i}^{(n)} + f_u(t,x,u,u_x)h^{(n)} \\ &\quad + F_n(t,x,u,u_x)\{k_1,\ldots,k_{n+1}\} \\ & h^{(n)}(0,x) = h_t^{(n)}(0,x) = 0 \end{aligned}$$

where the map F_n, nonlinear in u, u_x and linear in $k_1, \ldots, k_{n+1}$, is defined recursively as follows:

$$\begin{aligned} &F_n(t,x,u,u_x)\{k_1,\ldots,k_{n+1}\} \\ &\quad = \sum_i \left[k_{n+1} f_{uu_{x_i}} + \sum_j \partial_{x_j} k_{n+1} f_{u_{x_j}u_{x_i}} \right] \partial_{x_i} D^{n-1} VP(u)\{k_1,\ldots,k_n\} \\ &\qquad + \left[k_{n+1} f_{uu} + \sum_j \partial_{x_j} k_{n+1} f_{uu_{x_j}} \right] D^{n-1} VP(u)\{k_1,\ldots,k_n\} \\ &\qquad + k_{n+1}\partial_u F_{n-1}\{k_1,\ldots,k_n\} + \sum_i \partial_{x_i} k_{n+1} \partial_{u_{x_i}} F_{n-1}\{k_1,\ldots,k_n\} \end{aligned}$$

($D^0 VP \equiv VP$, $F_0\{k\} \equiv k_{tt}$) and hence can be easily expressed as a linear combination of derivatives of $D^j VP$ for $j = 0, \ldots, n-1$.

We must show that $D^n VP$ is tame for all n; this can be easily obtained by a repeated application of the following Lemma.

LEMMA 3.1. *For any $u, \phi \in F$, denote by $h = T_0(u)\phi$ the solution of the Cauchy problem*

$$\begin{gathered} h_{tt} = A(t)h + \sum_{i=1}^{n} f_{u_{x_i}}(t,x,u,u_x)h_{x_i} + f_u(t,x,u,u_x)h + \phi \\ h(0,x) = h_t(0,x) = 0 \end{gathered}$$

and by $\tilde{h} = T(u)\phi$ *the solution of*

$$\tilde{h}_{tt} = A(t)\tilde{h} + \sum_{i=1}^{n} f_{u_{x_i}}(t, x, u, u_x)\tilde{h}_{x_i} + f_u(t, x, u, u_x)\tilde{h} + \phi_{tt}$$

$$\tilde{h}(0, x) = \phi(0, x), \quad \tilde{h}_t(0, x) = \phi_t(0, x).$$

Then the mappings $T, T_0 : F \times F \to F$ *are well defined and tame.*

PROOF. T_0 and T are well defined by Oleinik's result. To prove that T is tame we have to find some $r, b \in N$ such that the following estimate holds for any $u, \phi \in F$ and for any $m \geq b$

$$|T(u)\phi|_m \leq c_m(1 + |(u, \phi)|_{m+r})$$

($|(u, \phi)|_m = |u|_m + |\phi|_m$ is the grading on $F \times F$). By estimate (2.5), using the Gagliardo-Nirenberg inequalities and the Sobolev embedding, we easily obtain

$$\begin{aligned} |T(u)\phi|_m = |\tilde{h}|_m &= \sup_{0 \leq t \leq 1} \sum_{j=0}^{m} \|\partial_t^j \tilde{h}(t, \cdot)\|_{m-j} \\ &\leq C(|u|_N)\Bigg[\|\phi(0, \cdot)\|_{m+p+4} + \|\phi_t(0, \cdot)\|_{m+p+3} \\ &\qquad + \max_{0 \leq t \leq 1} \sum_{j=0}^{k+p-1} \|\partial_t^j \phi_{tt}\|_{m+2p+4-j} + |u|_{m+N}\Bigg] \end{aligned}$$

for some N independent of m. If u varies in an arbitrary set, bounded for the $|\cdot|_N$ norm, this implies

$$|T(u)\phi|_m \leq c_m(1 + |(u, \phi)|_{m+r}), \quad m \geq N$$

with $r = \max\{N, 2p + 6\}$. The proof that T_0 is smooth tame is analogous.

We remark now that the map VP coincides with T, while the derivatives $D^j VP$ can be obtained by repeated compositions of T_0, T, according to the recursive expression given above, and hence are tame, being composition of tame maps. This proves that VP is smooth tame.

We can thus apply the Nash-Moser Theorem, and we obtain that $P : F \to F$ is locally invertible.

We shall now construct a function $w \in F$ with the property

$$\partial_t^j Pw(t, x)|_{t=0} = 0 \quad \forall j \in N. \tag{3.1}$$

Indeed, it is clear that (4.1) is equivalent to an infinite set of conditions on $g_j(x) = \partial_t^j w(0,x)$: for $j = 0, 1$ we obtain $g_0 = u_0$, $g_1 = u_1$, and for $j \geq 2$

$$g_j(x) = \partial_t^{j-2}[A(t)w + f(t,x,w,w_x)]_{t=0}.$$

Hence it is sufficient to construct a function $w(t,x)$ with the assigned traces g_j at $t = 0$, which is standard (see e.g. [Hö, Th.1.2.6]).

We know that P is a bijection of a neighbourhood U of w, satisfying (3.1), onto a neighbourhood V of Pw. By the definition of the topology of F, possibly restricting V we can assume that it is a $|\cdot|_k$-neighbourhood of Pw, for some k. Now, let $\rho(s)$ be a C^∞ function on $\boldsymbol{R}$ such that $0 \leq \rho \leq 1$, $\rho \equiv 0$ for $s \leq 1$, $\rho \equiv 1$ for $s \geq 2$, and define

$$\phi_\varepsilon(t,x) = \int_0^t (t-s)^k \rho\left(\frac{s}{\varepsilon}\right) \partial_t^{k+1}(Pw(s,x))\, ds.$$

The function ϕ_ε vanishes for $0 \leq t \leq \varepsilon$. Moreover,

$$Pw - \phi_\varepsilon = \int_0^t (t-s)^k \left(1 - \rho\left(\frac{s}{\varepsilon}\right)\right) \partial_t^{k+1}(Pw(s,x))\, ds$$

(we use here the elementary identity $v(t,x) \equiv \int_0^t (t-s)^k \partial_t^{k+1} v(s,x)\, ds$, valid for any function v such that $\partial_t^j v(0,x) = 0$ for $0 \leq j \leq k$). This implies easily, for any $h \leq k$ and any α, the inequalities

$$|\partial_x^\alpha \partial_t^h (Pw - \phi_\varepsilon)| \leq c_{\alpha,h}(a, u_0, u_1, f)\varepsilon. \tag{3.2}$$

If ε is small enough, (3.2) implies that $\phi_\varepsilon \in V$, hence $u = P^{-1}\phi_\varepsilon$ is the required solution.

Uniqueness follows from estimate (2.5) by standard linearization arguments.

REMARK 3.2. As stated in the Introduction, an analogous result can be proved by similar techniques using estimate (2.12) instead of (2.5).

Appendix: The Nash-Moser theorem

We give here a short account of the Nash-Moser theory. We refer to Hamilton's paper [H] for a detailed discussion of the definitions and for the proof of the results cited in this section.

A *graded* (*Fréchet*) space is a Fréchet space whose topology is generated by a *grading*, i.e. an increasing sequence of seminorms $|\cdot|_n$, $|f|_n \leq |f|_{n+1}$ for all $f \in F$ and $n = 0, 1, 2, \ldots$. A linear map $L: F \to G$ of one graded space into another is a *tame linear map* if for some $r, b \in N$ the following estimate holds:

$$|Lf|_n \le C_n |f|_{n+r}, \quad f \in F, n \ge b \tag{A.1}$$

where the constant C_n depends only on n. The number b is called the *base* and r the *degree* of the *tame estimate* (A.1).

Thus a tame map is a map of "finite order" in the sense of the gradings (as in the preceding definition, we shall not use different notations for the gradings of different graded spaces, as far as there is no risk of misunderstanding). Note that, in particular, tameness implies continuity.

To introduce the notion of tame space, we must define first the space of *exponentially decreasing sequences* $\Sigma(B)$ on a Banach space B. This is the graded space of all sequences of vectors in B, such that, for $n \ge 0$,

$$|\{v_k\}|_n \equiv \sum_{k=0}^{\infty} e^{nk} \|v_k\|_B < \infty \tag{A.2}$$

endowed with the grading $|\cdot|_n$ defined in (A.2).

DEFINITION A.1. A graded space F is *tame* if, for some Banach space B, there exist two tame linear maps $L_1 : F \to \Sigma(B)$ and $L_2 : \Sigma(B) \to F$ such that $L_2 L_1$ is the identity on F.

The tameness property is stable under usual operations (direct sum, product etc.).

The most important examples of tame spaces are the spaces of C^∞ functions on manifolds:

PROPOSITION A.2. *Let X be a smooth compact manifold, with or without boundary. Then $C^\infty(X)$, equipped with one of the gradings*

$$|f|_n = \sup_{|\alpha| \le n} \sup_{x \in X} |D^\alpha f(x)|$$

or

$$|f|_n^2 = \sum_{|\alpha| \le n} \|D^\alpha f(x)\|_{L^2(X)}^2 \tag{A.3}$$

is a tame space.

PROOF. See [H, pp. 137–8]. The definition of tameness for nonlinear maps is slightly more involved than for the linear ones.

DEFINITION A.3. Let $P : U \subseteq F \to G$ be a nonlinear map from a subset U of the graded space F to the graded space G. P satisfies a *tame estimate* of degree r and base b if, for any $f \in U$, $n \geq b$,

$$|P(f)|_n \leq C_n(1 + |f|_{n+r}) \tag{A.4}$$

for some constant C_n depending only on n. A map P defined on a open set is said to be *tame* if it satisfies a tame estimate in the neighbourhood of each point (with constants r, b and C_n which may depend on the neighbourhood).

We remark that a linear map is tame if and only if it is a tame linear map; moreover, the composition of tame maps is tame ([H, pp. 141–2]).

DEFINITION A.4. Let F, G be graded spaces, U an open subset of F. A map $P : U \to G$ is *smooth tame* if it is C^∞ and and D^nP is tame for all $n \geq 0$.

We remark that sums and compositions of smooth tame maps are smooth tame. Moreover, nonlinear partial differential operators on $C^\infty(X)$ are smooth tame maps, X smooth manifold with or without boundary.

Finally, the following theorem is the fundamental result of the Nash-Moser theory:

THEOREM A.5 [Nash-Moser]. *Let F, G be tame spaces, U an open subset of F, $P : U \to G$ a smooth tame map. Assume that the equation $DP(u)h = k$ has a unique solution $h \equiv VP(u)k$ for all $u \in U$, $u \in G$, and that $VP : U \times G \to F$ thus defined is smooth tame. Then P is locally invertible, and each local inverse is smooth tame.*

References

[CS] F. Colombini, S. Spagnolo, An example of a weakly hyperbolic Cauchy problem not well posed in C^∞, Acta Math. **148** (1982), 243–253

[D] P. D'Ancona, Local existence for semilinear weakly hyperbolic equations with time dependent coefficients, Nonlin. Analysis T.M.A. **21** (1993), 685–696

[DM] P. D'Ancona, R. Manfrin, Sufficient conditions for the local solvability of some quasilinear equations of weakly hyperbolic type, Boll. UMI **8-B** (1994), 731–754

[DT] P. D'Ancona, P. Trebeschi, On the local solvability of a semilinear weakly hyperbolic equation, (in preparation)

[Di] P. A. Dionne, Sur les problèmes hyperboliques bien posées, J. Anal. Math. **10** (1962–3), 1–90

[G] D. Gourdin, Opérateurs faiblement hyperboliques non linéaires, C. R. Acad. Sci. Paris Ser. I 306, **15** (1988), 659–662

[H] Richard S. Hamilton, The inverse function theorem of Nash and Moser, Bull. A.M.S. **7** (1982), 65–222

[Hö] L. Hörmander, The analysis of linear partial differential operators I. Grundlehren der mathematischen Wissenschaften, Springer, Berlin.

[I1] N. Iwasaki, Effectively hyperbolic equations, J. Math. Kyoto Univ. **25** (1985), 727–743

[I2] N. Iwasaki, The strongly hyperbolic equation and its applications, RIMS Kyoto Univ. (520) (1985), 1–38

[K] K. Kajitani, Local solution of Cauchy problem for nonlinear hyperbolic systems in Gevrey classes, Hokkaido Math. J. **12** (1983), 434–460

[L] H. Lindblad, A sharp counterexample to local existence of low regularity solutions to nonlinear wave equations, Duke Math. J. **72** (1993), 503–539

[O] O. Oleinik, On the Cauchy problem for weakly hyperbolic equations, Comm. P.A.M. **23** (1970), 569–586

[Q] Qi Min-You, On the Cauchy problem for a class of hyperbolic equations with initial data on the parabolic degenerating line, Acta Math. Sinica **8** (1958), 521–529

[RY] M. Reissig, K. Yagdjian, Levi conditions and global Gevrey regularity for the solutions of quasilinear weakly hyperbolic equations, Math. Nachrichten **178** (1996), 285–307

[T] S. Tarama, On the second order hyperbolic equations degenerating in the infinite order—Example—, Math. Japonica **42** (1995), 523–534

[Y] K. Yagdjian, The Cauchy problem for weakly hyperbolic equations in Gevrey classes (in Russian), Izv. Akad. Nauk Armenia **13** (1978), 3–22