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ON THE SUM OF DIGITS OF PRIMES IN IMAGINARY QUADRATIC FIELDS

By

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1. Introduction. Let $r\geqq 2$ be a fixed integer. Any positive integer n can be uniquely written in the form

$$n=\sum_{j=1}^{k} a_{j}r^{k-j}=a_{1}a_{2}\cdots a_{k}\,, \tag{1}$$

where each a_j is one of $0, 1, \cdots, r-1$ and

$$k=k(n)=\left[\frac{\log n}{\log r}\right]+1\,, \tag{2}$$

where $[u]$ is the integral part of the real number u. We put

$$s(n)=\sum_{j=1}^{k} a_{j}\,.$$

I. Kátai [1] proved, assuming the validity of density hypothesis for the Riemann zeta function, that

$$\sum_{p\leqq x} s(p)=\frac{r-1}{2\log r}\,x+O\Big(\frac{x}{(\log\log x)^{1/3}}\Big)\,,$$

where in the sum p runs through the prime numbers. The second-named author [6] proved, without any hypothesis, the result of Kátai with an improved remainder term

$$O\Big(x\Big(\frac{\log\log x}{\log x}\Big)^{1/2}\Big)\,. \tag{3}$$

His method is to appeal to a simple combinatorial inequality (see Lemma in § 4), and the deepest result on which he depends is the prime number theorem in a weak form

$$\sum_{p\leqq x} 1=\frac{x}{\log x}+O\Big(\frac{x}{(\log x)^{2}}\Big)\,. \tag{4}$$

E. Heppner [2] independently proved a more general result by making use of a Chebyshev's inequality to the sum of independent random variables (cf. [5] p. 387, Theorem 2): Let B be a set of positive integers such that

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$$\log \frac{x}{B(x)} = o(\log x)\,,$$

where

$$B(x) = \sum_{\substack{n \leq x \\ n \in B}} 1\,.$$

Then

$$\sum_{\substack{n \leq x \\ n \in B}} s(n) = \frac{r-1}{2} \frac{\log x}{\log r} B(x)\left(1 + O\left(\left(\frac{\log\log x + \log \frac{x}{B(x)}}{\log x}\right)^{1/2}\right)\right).$$

This together with (4) implies (3).

In the present paper we shall show that the estimate (3) is also valid, in some sense, for primes in each imaginary quadratic field $\boldsymbol{Q}(\sqrt{-m})$, where m is any positive square free integer.

2. Representation of integers in $\boldsymbol{Q}(\sqrt{-m})$ in the scale of r. Let $\mathfrak{o}$ be the ring of all integers in $\boldsymbol{Q}(\sqrt{-m})$. Any $\alpha \in \mathfrak{o}$ can be expressed in a unique way as

$$\alpha = a + b\omega \quad (a,\ b \in \boldsymbol{Z})\,,$$

where

$$\omega = \begin{cases} \sqrt{-m} & \text{if} \quad -m \equiv 2,\ 3 \pmod 4\,, \\ \dfrac{1+\sqrt{-m}}{2} & \text{if} \quad -m \equiv 1 \quad \pmod 4\,, \end{cases}$$

and $\boldsymbol{Z}$ denotes as usual the set of all rational integers. So by means of the expessions

$$|a| = a_1 a_2 \cdots a_{k(|a|)}\,, \quad |b| = b_1 b_2 \cdots b_{k(|b|)}$$

given by (1), we can define coordinatewisely the representation of $\alpha \in \mathfrak{o}$ in the scale of r; i. e.

(5) $$\alpha = \sum_{j=1}^{k} \alpha_j r^{k-j} = \alpha_1 \alpha_2 \cdots \alpha_k\,,$$

where

(6) $$k = k(\alpha) = \max\,\{k(|a|),\ k(|b|)\}\,, \quad k(0) = 1\,,$$

$$\alpha_j = \operatorname{sgn}(a) a_j + \operatorname{sgn}(b) b_j \omega\,,$$

and $\operatorname{sgn}(c) = c/|c|$ if $c \neq 0$, $=0$ otherwise. We define

$$s(\alpha) = \sum_{j=1}^{k} \alpha_j\,.$$

We write

$$\mathcal{A}_1 = \{a + b\omega \,|\, a,\ b \in \boldsymbol{Z};\ a \geqq 0,\ b \geqq 0\}\,,$$

$$\mathcal{A}_2=\{-a+b\omega \mid a+b\omega\in\mathcal{A}_1\}\ ,$$

$$\mathcal{A}_3=\{-a-b\omega \mid a+b\omega\in\mathcal{A}_1\}\ ,$$

$$\mathcal{A}_4=\{a-b\omega \mid a+b\omega\in\mathcal{A}_1\}\ ,$$

so that $\mathfrak{o}=\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3\cup\mathcal{A}_4$. We denote by $\mathcal{B}_i$ the set of all 'digits' α_j needed for the expressions (5) of all $\alpha\in\mathcal{A}_j$. Then

$$\mathcal{B}_1=\{c+d\omega \mid c,\ d=0,\ 1,\ \cdots,\ r-1\}\ ,$$

$$\mathcal{B}_2=\{-c+d\omega \mid c+d\omega\in\mathcal{B}_1\}\ ,$$

$$\mathcal{B}_3=\{-c-d\omega \mid c+d\omega\in\mathcal{B}_1\}\ ,$$

$$\mathcal{B}_4=\{c-d\omega \mid c+d\omega\in\mathcal{B}_1\}\ ,$$

and card $\mathcal{B}_i=r^2$ $(1\leqq i\leqq 4)$. So we may say that the r-adic expression (5) of $\alpha\in\mathfrak{o}$ is a kind of representation in the scale of r^2. For any fixed $\beta\in\mathcal{B}_i$ we denote by $F(\alpha,\ \beta)$ the number of β appearing in the expression (5) of an integer $\alpha\in\mathcal{A}_1$. By definition

$$s(\alpha)=\sum_{\beta\in\mathcal{B}_i}\beta F(\alpha,\ \beta)\quad(\alpha\in\mathcal{A}_i) \tag{7}$$

and

$$\begin{aligned}F(a+b\omega,\ c+d\omega)&=F(-a+b\omega,\ -c+d\omega)\\&=F(-a-b\omega,\ -c-d\omega)=F(a-b\omega,\ c-d\omega)\quad(a,\ b\in\boldsymbol{Z})\,.\end{aligned} \tag{8}$$

The norm of $\alpha=a+b\omega\in\mathfrak{o}$ is a rational integer

$$N(\alpha)=\begin{cases}a^2+mb^2 & \text{if}\quad -m\equiv 2,\ 3\pmod 4\\[2mm] a^2+ab+\dfrac{m+1}{4}b^2 & \text{if}\quad -m\equiv 1\pmod 4\end{cases}$$

so that for $\alpha\neq 0$

$$\left|k(\alpha)-\frac{\log N(\alpha)}{2\log r}\right|\leqq c_1\,, \tag{9}$$

where c_1 is a constant depending only on m, since by definition

$$\left|k(a)-\frac{\max(\log|a|,\ \log|b|)}{\log r}\right|\leqq 1$$

(we mean that $\max(\log 0,\ x)=x$) and

$$\begin{aligned}&|2\max(\log|a|,\ \log|b|)-\log N(\alpha)|\\&\leqq\begin{cases}\log(1+m) & \text{if}\quad -m\equiv 2,\ 3\pmod 4\,,\\[2mm] \log\left(2+\dfrac{m+1}{4}\right) & \text{if}\quad -m\equiv 1\pmod 4\,.\end{cases}\end{aligned}$$

3. A prime number theorem (A. Mitsui [3], [4]). An integer $\alpha \in \mathfrak{o}$ is said to be prime if (α) is an prime ideal in $\boldsymbol{Q}(\sqrt{-m})$. Let θ_1, θ_2 be two real numbers such that $0 \leqq \theta_1 < \theta_2 \leqq 2\pi$. Then

$$\sum_{\substack{\alpha:\text{ prime} \\ N(\alpha)\leqq x \\ \theta_1 \leqq \arg \alpha \leqq \theta_2}} 1 = \frac{(\theta_2-\theta_1)w}{2\pi h}\int_2^x \frac{dt}{\log t} + O(x\exp(-c_2(\log x)^{3/5}(\log\log x)^{-1/5})), \tag{10}$$

where h is the class number of $\boldsymbol{Q}(\sqrt{-m})$ and

$$w = \begin{cases} 4 & \text{if } m=1, \\ 6 & \text{if } m=3, \\ 2 & \text{otherwise} \end{cases}$$

We note that a weaker estimate $O(x/(\log x)^2)$ is sufficient for the proof of our theorem.

4. A combinatorial lemma (I. Shiokawa [6]). Let $\beta_1, \cdots, \beta_g$ be given g symbols and let A^j be the set of all sequences of these symbols of length $j \geqq 1$. Denote by $F_j(\alpha, \beta)$ the number of any fixed symbol β appearing in a sequence $\alpha \in A^j$. Then for any ε with $0<\varepsilon<1/2$ there exist a positive integer j_0 independent of ε such that the number of sequences $\alpha \in A^j$ satisfying

$$\left| F_j(\alpha, \beta) - \frac{j}{g} \right| > j^{1/2+\varepsilon}$$

is less that $jg^j \exp(-c_3 j^{2\varepsilon})$ for all $j \geqq j_0$, where c_3 is an absolute constant.

5. Theorem. Let $\varphi_1=0$, $\varphi_5=2\pi$, $\varphi_2=\arg\omega$, $\varphi_3=\pi$, and $\varphi_4=\varphi_2+\pi$. Then for any θ_1, θ_2 satisfying $\varphi_j \leqq \theta_1 < \theta_2 \leqq \varphi_{j+1}$ for some j we have

$$\sum_{\substack{\alpha:\text{ prime} \\ N(\alpha)\leqq x \\ \theta_1 \leqq \arg \alpha \leqq \theta_2}} s(\alpha) = \frac{(\theta_2-\theta_1)w}{2\pi h}\frac{(r-1)}{4\log r}\lambda_j x + O\left(x\left(\frac{\log\log x}{\log x}\right)^{1/2}\right), \tag{11}$$

where

$$\lambda_j = \begin{cases} 1+\omega & \text{if } j=1, \\ -1+\omega & \text{if } j=2, \\ -1-\omega & \text{if } j=3, \\ 1-\omega & \text{if } j=4, \end{cases}$$

and the O-constant depends at most on r and m.

6. Proof of Theorem. By (7) and (8) we may assume $j=1$. We define for $\alpha\in\mathcal{A}_1$ and $\beta\in\mathcal{B}_1$

$$D(\alpha,\beta)=\left|F(\alpha,\beta)-\frac{k(\alpha)}{r^2}\right|. \tag{12}$$

Put for brevity

$$C(x)=\{\alpha\in\mathfrak{o}\mid\alpha:\text{ prime, } N(\alpha)\leqq x,\ \theta_1\leqq\arg\alpha\leqq\theta_2\}.$$

Then by (7) and (12)

$$\begin{aligned}\sum_{\alpha\in C(x)}s(\alpha)&=\sum_{\beta\in\mathcal{B}_1}\beta\sum_{\alpha\in C(x)}F(\alpha,\beta)\\&=\frac{r-1}{2}\lambda_1\sum_{\alpha\in C(x)}k(\alpha)+O\Big(\sum_{\beta\in\mathcal{B}_1}\sum_{\alpha\in C(x)}D(\alpha,\beta)\Big).\end{aligned}\tag{13}$$

By (9) and (10) we have

$$\sum_{\alpha\in C(x)}k(\alpha)=\frac{(\theta_2-\theta_1)w}{2\pi h}\frac{x}{2\log r}+O\Big(\frac{x}{\log x}\Big). \tag{14}$$

Put $D(\alpha)=D(\alpha,\beta_0)$, where β_0 is any fixed integer in $\mathcal{B}_1$. We have from (9), (10), and (12)

$$\begin{aligned}\sum_{x\in C(\alpha)}D(\alpha)&\leqq\sum_{\alpha\in C(x)}k(\alpha)^{1/2+\varepsilon}+\sum_{\substack{\alpha\in C(x)\\ D(\alpha)>k(\alpha)^{1/2+\varepsilon}}}D(\alpha)\\&=O(\sum_{\alpha\in C(x)}(\log N(\alpha))^{1/2+\varepsilon})=O(\sum_{\substack{\alpha\in\mathcal{A}_1\\ N(\alpha)\leqq x\\ D(\alpha)>k(\alpha)^{1/2+\varepsilon}}}D(\alpha)\\&=O(x\log x)^{\varepsilon-1/2})+O(\log x\sum_{\substack{\alpha\in\mathcal{A}_1\\ N(\alpha)\leqq x\\ D(\alpha)>k(\alpha)^{1/2+\varepsilon}}}1).\end{aligned}\tag{15}$$

Besides, using (9),

$$\sum_{\substack{\alpha\in\mathcal{A}_1\\ N(\alpha)\leqq x\\ D(\alpha)>k(\alpha)^{1/2+\varepsilon}}}1\leqq\sum_{j\leqq l(x)}\sum_{\substack{\alpha\in\mathcal{A}_1\\ k(\alpha)=j\\ D(\alpha)>j^{1/2+\varepsilon}}}1,$$

where

$$l(x)=\frac{\log x}{2\log r}+c_1.$$

Applying now the lemma in §4 with $g=r^2$ and $A^1=\mathcal{B}_1$, we get

$$\sum_{\substack{\alpha\in\mathcal{A}_1\\ k(\alpha)=j\\ D(\alpha)>j^{1/2+\varepsilon}}}1<jr^{2j}\exp(-c_3j^{2\varepsilon})$$

for all $j\geqq j_0$, which leads to

$$\sum_{\substack{\alpha\in\mathcal{A}_1\\ N(\alpha)\leqq x\\ D(\alpha)>k(\alpha)^{1/2+\varepsilon}}}1=O(1)+\sum_{j_0<j\leqq l(x)}jr^{2j}\exp(-c_3j^{2\varepsilon}) \tag{16}$$

$$=O(1)+\sum_{j_0<j\leq l(x)/2}+\sum_{l(x)/2<j\leq l(x)}$$

$$=O\left(x(\log x)^2\exp\left(-\frac{c_3}{4}\left(\frac{\log x}{\log r}\right)^{2\varepsilon}\right)\right).$$

where the O-constant is uniform in ε.

If we take a constant $c_4=c_4(r)$ large enough and choose $\varepsilon=\varepsilon(x, r)$ with $0<\varepsilon<1/2$ in such a way that

$$(\log x)^{2\varepsilon}=c_4\log\log x$$

we obtain from (15) and (16)

$$\sum_{\alpha\in C(x)}D(\alpha, \beta_0)=O\left(x\left(\frac{\log\log x}{\log x}\right)^{1/2}\right).$$

This together with (13) and (14) yealds the theorem.

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