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REMARKS ON THETA SERIES

By

Atsushi SHIMIZU

In this note we prove two propositions about theta series. The field of real numbers will be denoted by $\boldsymbol{R}$, the field of complex numbers by $\boldsymbol{C}$, and Siegel upper half domain of degree n by H_n. And $\exp(2\pi\sqrt{-1}\,z)$ will be denoted by $\boldsymbol{e}(z)$. A theta series $\vartheta\left[\begin{smallmatrix} k' \\ k'' \end{smallmatrix}\right](z|x)$ of theta chararcteristic $\left(\begin{smallmatrix} k' \\ k'' \end{smallmatrix}\right)$ is defined by

$$\vartheta\left[\begin{matrix} k' \\ k'' \end{matrix}\right](z|x)=\sum_{r\in \boldsymbol{Z}^n}\boldsymbol{e}\Big(\frac{1}{2}{}^t(r+k')z(r+k')+{}^t(r+k')(x+k'')\Big)$$

where $\left(\begin{smallmatrix} k' \\ k'' \end{smallmatrix}\right)\in\boldsymbol{R}^{2n}$ and $(z, x)\in H_n\times\boldsymbol{C}^n$.

One of the two propsitions is the follows;

PROPOSITION 1. *Let* $f_i(z, x)=\vartheta\left[\begin{smallmatrix} k'_i \\ k''_i \end{smallmatrix}\right](z|x)$ *be m theta series of theta characteristic* $\left(\begin{smallmatrix} k'_i \\ k''_i \end{smallmatrix}\right)$ $(i=1, 2, \cdots, m)$. *If* $\left(\begin{smallmatrix} k'_i \\ k''_i \end{smallmatrix}\right)\not\equiv\left(\begin{smallmatrix} k'_j \\ k''_j \end{smallmatrix}\right)$ (mod $\boldsymbol{Z}^{2n}$) *for any* i, j $(1\leqq i<j\leqq m)$ *then* $f_1, f_2, \cdots, f_m$ *are linearly independent over* $\boldsymbol{C}$.

In a special case, when $\left(\begin{smallmatrix} k'_i \\ k''_i \end{smallmatrix}\right)\equiv\left(\begin{smallmatrix} k'_j \\ k''_j \end{smallmatrix}\right)$ (mod $\boldsymbol{Q}^{2n}$) for any i, j $(1\leqq i<j\leqq m)$, the result was given in S. Koizumi's lecture at University of Tsukuba.

And the other is the following:

PROPOSITION 2. *If* $\vartheta\left[\begin{smallmatrix} k' \\ k'' \end{smallmatrix}\right](z|0)$ *is identically zero as a function of* $z\in H_n$, *then* $2k'\equiv 2k''\equiv 0$ (mod$\cdot\boldsymbol{Z}^{2n}$) *and* $2{}^tk'k''\not\equiv 0$ (mod $\boldsymbol{Z}$).

In case that $\left(\begin{smallmatrix} k' \\ k'' \end{smallmatrix}\right)\in\boldsymbol{Q}^{2n}$, the result has been already known (Igusa [1], p. 174 Theorem 1). And we can see easily that the converse of proposition 2 is true.

Proposition 2 was proved first by A. Seyama [3] and later by the author independently, but since the two proofs are quite different in principle, the author believes that his proof is worth publishing.

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1. PROOF OF PROPOSITION 1. The proof will be by induction on m. The result is trivial for $m=1$, so assume then that the result is true for $m-1$, and consider the proposition as stated above for m functions. Suppose the result is not true, and there exist m complex numbers $a_1, a_2, \cdots, a_m$ which are not all zero, such that $a_1f_1+a_2f_2+\cdots a_mf_m=0$. From the induction hypothesis, $a_i\neq 0$ for any i $(1\leqq i\leqq m)$. Then, for any $s', s''\in \boldsymbol{Z}^n$ we have

$$0=\sum_{i=1}^{m} a_if_i(z, x+zs'+s'')=\sum_{i=1}^{m} a_if_i(z, x)\boldsymbol{e}\Big(-{}^ts'x-\frac{1}{2}{}^ts'zs'-{}^tk_i''s'+{}^tk_i's''\Big).$$

Hence

$$\sum_{i=1}^{m} a_if_i(z, x)\boldsymbol{e}(-{}^tk_i''s'+{}^tk_i's'')=0.$$

On the other hand

$$\sum_{i=1}^{m} a_if_i(z, x)\boldsymbol{e}(-{}^tk_m''s'+{}^tk_m's'')=0.$$

Then we have

$$\sum_{i=1}^{m-1} a_i\{\boldsymbol{e}(-{}^tk_i''s'+{}^tk_i's'')-\boldsymbol{e}(-{}^tk_m''s'+{}^tk_m's'')\}f_i(z, x)=0.$$

It follows from the induction hypothesis that

$$\boldsymbol{e}(-{}^tk_i''s'+{}^tk_i's'')=\boldsymbol{e}(-{}^tk_m''s'+{}^tk_m's'') \quad \text{for any } s', s''\in \boldsymbol{Z}^n \ (1\leqq i\leqq m-1).$$

Therefore

$$\boldsymbol{e}(-{}^t(k_i''-k_m'')s'+{}^t(k_i'-k_m')s'')=1 \quad \text{for any } s', s''\in \boldsymbol{Z}^n \ (1\leqq i\leqq m-1).$$

This implies $k_i''\equiv k_m''$, $k_i'\equiv k_m'$ (mod $\boldsymbol{Z}^n$) $(1\leqq i\leqq m-1)$. But this contradicts the hypothesis, so the proof is completed.

REMARK. Let $f_1, f_2, \cdots, f_m$ be automorphic functions which respectively belong to factors of automorphy $\rho_1, \rho_2, \cdots, \rho_m$ which are different from one another. Similarly to the above proof we can prove the following proposition: if ρ_i/ρ_j is a constant factor of automorphy for any i, j then $f_1, f_2, \cdots, f_m$ are linearly independent over $\boldsymbol{C}$.

But in general $f_1, f_2, \cdots, f_m$ are not always linearly independent. For example, define a factor of automorphy ρ on $\boldsymbol{C}$ with respect to an analytic transformation group $G=\{1, -1\}$ by $\rho(1, z)=1$ and $\rho(-1, z)=e^z$, and put $\rho_1=1$, $\rho_2=\rho$, $\rho_3=\rho^2$. Then $f_1=-1$, $f_2=1+e^{-z}$, $f_3=-e^{-z}$ are automorphy functions which respectively belong to ρ_1, ρ_2, ρ_3 and are apparently linearly dependent.

2. Now in order to prove proposition 2 let us prepare two lemmas. Let $k'=\begin{pmatrix} k'_1 \\ k'_2 \\ \vdots \\ k'_n \end{pmatrix}$, $k''=\begin{pmatrix} k''_1 \\ k''_2 \\ \vdots \\ k''_n \end{pmatrix}\in \boldsymbol{R}^n$, $x=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\in \boldsymbol{C}^n$ and put $k^{*\prime}_m=\begin{pmatrix} k'_1 \\ \vdots \\ k'_m \end{pmatrix}$, $k^{*\prime\prime}_m=\begin{pmatrix} k''_1 \\ \vdots \\ k''_m \end{pmatrix}\in \boldsymbol{R}^m$, $x^*_m=\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}\in \boldsymbol{C}^m$, and $k^{m\prime}_*=\begin{pmatrix} k'_{m+1} \\ \vdots \\ k'_n \end{pmatrix}$, $k^{m\prime\prime}_*=\begin{pmatrix} k''_{m+1} \\ \vdots \\ k''_n \end{pmatrix}\in \boldsymbol{R}^{n-m}$, $x^m_*=\begin{pmatrix} x_{m+1} \\ \vdots \\ x_n \end{pmatrix}\in \boldsymbol{C}^{n-m}$, where m is some integer such that $1\leqq m<n$.

LEMMA 1. *Let m be any integer such that $1\leqq m<n$. For any $z^*_m\in H_m$, $z^m_*\in H_{n-m}$,*

$$\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}\left(\begin{pmatrix} z^*_m & 0 \\ 0 & z^m_* \end{pmatrix}\middle| x\right)=\vartheta\begin{bmatrix} k^{*\prime}_m \\ k^{*\prime\prime}_m \end{bmatrix}(z^*_m\,|\,x^*_m)\vartheta\begin{bmatrix} k^{m\prime}_* \\ k^{m\prime\prime}_* \end{bmatrix}(z^m_*\,|\,x^m_*)\,.$$

LEMMA 2. *Let u be any element of $\boldsymbol{GL}(n, Z)$. Then*

$$\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}(z\,|\,x)=\vartheta\begin{bmatrix} u^{-1}k' \\ {}^t uk'' \end{bmatrix}({}^t uzu\,|\,{}^t ux)\,.$$

Hence if $\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}(z\,|\,0)$ is identically zero as a function of $z\in H_n$, so is $\vartheta\begin{bmatrix} u^{-1}k' \\ {}^t uk'' \end{bmatrix}(z\,|\,0)$.

These two lemmas are easily proved.

PROOF OF PROPOSITION 2. When $n=1$, since

$$\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}(z\,|\,0)=\vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(z\,|\,zk'+k'')\boldsymbol{e}\Big(\frac{1}{2}k'zk'+k'k''\Big)\,,$$

both $\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}(z\,|\,0)$ and $\vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(z\,|\,zk'+k'')$ are identically zero at the same time. But $\vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(z\,|\,x)$ is zero if and only if $x\in\Big(\frac{1}{2}+\boldsymbol{Z}\Big)z+\Big(\frac{1}{2}+\boldsymbol{Z}\Big)=\Big\{az+b\,|\,a\equiv\frac{1}{2},$ $b\equiv\frac{1}{2}\ (\mathrm{mod}\ \boldsymbol{Z})\Big\}$. Hence $\vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(z\,|\,zk'+k'')$ is identically zero if and only if $zk'+k''\in\Big(\frac{1}{2}+\boldsymbol{Z}\Big)z+\Big(\frac{1}{2}+\boldsymbol{Z}\Big)$ for any $z\in H_1$. This implies that if $\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}(z\,|\,0)$ is identically zero, $k'\equiv\frac{1}{2}$ and $k''\equiv\frac{1}{2}$ (mod $\boldsymbol{Z}$) and the result is true in this case.

When $n=2$, by Lemma 1

$$\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}\left(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\middle| 0\right)=\vartheta\begin{bmatrix} k'_1 \\ k''_1 \end{bmatrix}(z_1\,|\,0)\vartheta\begin{bmatrix} k'_2 \\ k''_2 \end{bmatrix}(z_2\,|\,0)\quad\text{for any } z_1,\ z_2\in H_1\,.$$

Since the result is true when $n=1$ it follows that

$$k_1' \equiv k_1'' \equiv \frac{1}{2} \quad \text{or} \quad k_2' \equiv k_2'' \equiv \frac{1}{2} \ (\text{mod } \boldsymbol{Z}).$$

We may assume $k_1' \equiv k_1'' \equiv \frac{1}{2}$. Then putting $u = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, by Lemma 2 $\vartheta\begin{bmatrix} u^{-1}k' \\ {}^t uk'' \end{bmatrix}$ $(z|0)$ is identically zero, where $u^{-1}k' = \begin{pmatrix} k_1' - k_2' \\ -k_1' + 2k_2' \end{pmatrix}$ and ${}^t u k'' = \begin{pmatrix} 2k_1'' + k_2'' \\ k_1'' + k_2'' \end{pmatrix}$ Then similarly to above we gain

$$k_1' - k_2' \equiv 2k_1'' + k_2'' \equiv \frac{1}{2} \quad \text{or} \quad -k_1' + 2k_2' \equiv k_1'' + k_2'' \equiv \frac{1}{2} \ (\text{mod} \cdot \boldsymbol{Z}),$$

therefore

$$k_2' \equiv 0, \quad k_2'' \equiv \frac{1}{2} \quad \text{or} \quad 2k_2' \equiv 0, \quad k_2'' \equiv 0 \ (\text{mod } \boldsymbol{Z}),$$

and in either case the result is true.

Now assuming $n > 2$, we shall complete the proof by induction on n. Suppose $\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}(z|0)$ is identically zero.

Case 1) If there exists i $(1 \leqq i \leqq n)$ such that $2k_i' \not\equiv 0$ or $2k_i'' \not\equiv 0$ or $2^t k_i' k_i'' \equiv 0$ (mod $\boldsymbol{Z}$), we may assume $i = 1$. Then by Lemma 1

$$\vartheta\begin{bmatrix} k_1^{*\prime} \\ k_2^{*\prime\prime} \end{bmatrix}(z_1^* | 0) \vartheta\begin{bmatrix} k_*^{1\prime} \\ k_*^{1\prime\prime} \end{bmatrix}(z_*^1 | 0) = 0 \quad \text{for any } z_1^* \in H_1, \ z_*^1 \in H_{n-1}.$$

Since $\vartheta\begin{bmatrix} k_1^{*\prime} \\ k_1^{*\prime\prime} \end{bmatrix}(z_1^* | 0)$ is not identically zero, $\vartheta\begin{bmatrix} k_*^{1\prime} \\ k_*^{1\prime\prime} \end{bmatrix}(z_*^1 | 0)$ is identically zero, $\vartheta\begin{bmatrix} k_*^{1\prime} \\ k_*^{1\prime\prime} \end{bmatrix}(z_*^1 | 0)$ is identically zero hence we have $2k_*^{1\prime} \equiv 2k_*^{1\prime\prime} \equiv 0$ (mod $\boldsymbol{Z}^{n-1}$) and $2^t k_*^{1\prime} k_*^{1\prime\prime} \not\equiv 0$ (mod $\boldsymbol{Z}$) from the induction hypothesis. Then there exists an integer $j \geqq 2$ such that $k_j \equiv k_j'' \equiv \frac{1}{2}$ (mod $\boldsymbol{Z}$) (if not, $2^t k_*^{1\prime} k_*^{1\prime\prime} \equiv 0$ (mod $\boldsymbol{Z}$)). We may assume $j = 2$. Now by Lemma 1,

$$\vartheta\begin{bmatrix} k_2^{*\prime} \\ k_2^{*\prime\prime} \end{bmatrix}(z_2^* | 0) \vartheta\begin{bmatrix} k_*^{2\prime} \\ k_*^{2\prime\prime} \end{bmatrix}(z_*^2 | 0) = 0 \quad \text{for any } z_2^* \in H_2, \ z_*^2 \in H_{n-2},$$

Since $2^t k_*^{2\prime} k_*^{2\prime\prime} \equiv 2^t k_*^{1\prime} k_*^{1\prime\prime} - \frac{1}{2} \equiv 0$ (mod $\boldsymbol{Z}$), $\vartheta\begin{bmatrix} k_*^{2\prime} \\ k_*^{2\prime\prime} \end{bmatrix}(z_*^2 | 0)$ is not identically zero. This implies that $2k_1' \equiv 2k_1'' \equiv 0$ and $2k_1' k_1'' \equiv 0$ (mod $\boldsymbol{Z}$), hence $2k' \equiv 2k'' \equiv 0$ (mod $\boldsymbol{Z}^n$) and $2^t k' k'' = 2k_1' k_1'' + 2^t k_*^{1\prime} k_*^{1\prime\prime} \not\equiv 0$ (mod$\cdot \boldsymbol{Z}$).

Case 2) Now we assume for any i $(1 \leqq i \leqq n)$ $2k_1' \equiv 2k_i'' \equiv 0$, $2k_i' k_i'' \not\equiv 0$ (mod $\boldsymbol{Z}$), that is, $k_i' \equiv k_i'' \equiv \frac{1}{2}$ (mod $\boldsymbol{Z}$) for any i. In this case, if $\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}(z|0)$ is identically zero, n must be odd. In fact, if n is even by lemma 1

$$\vartheta\begin{bmatrix} k_2^{*\prime} \\ k_2^{*\prime\prime} \end{bmatrix}(z_2^*|0)\vartheta\begin{bmatrix} k_*^{2\prime} \\ k_*^{2\prime\prime} \end{bmatrix}(z_*^2|0)=0 \quad \text{for any } z_2^*\in H_2,\ z_*^2\in H_{n-2}.$$

But from the induction hypothesis neither $\vartheta\begin{bmatrix} k_2^{*\prime} \\ k_2^{*\prime\prime} \end{bmatrix}(z_2^*|0)$ nor $\vartheta\begin{bmatrix} k_*^{2\prime} \\ k_*^{2\prime\prime} \end{bmatrix}(z_*^2|0)$ is identically zero. This contradicts the hypothesis. Now if n is odd, clearly $2k'\equiv 2k''\equiv 0 \pmod{\boldsymbol{Z}^n}$ and $2{}^tk'k''\not\equiv 0 \pmod{\boldsymbol{Z}}$. Hence the proof is completed.

References

[1] Igusa, J., Theta functions. Die Grundlehren der Math. Wiss., bd. 194, Springer-Verlag, Berlin (1972).

[2] Koizumi, S., Theta fnnctions. (in Japanese) Lecture note, Sophia. Univ. Tokyo (1978).

[3] Seyama, A., On the vanishing of theta constants. (to appear)