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ON THE EQUATIONS DEFINING A PROJECTIVE CURVE EMBEDDED BY A NON-SPECIAL DIVISOR

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Introduction. Let C be a complete reduced irreducible curve of arithmetic genus g over an algebraically closed field K. Let L be a very ample invertible sheaf of degree d on C, and let ϕ_L: $C \subsetneq \boldsymbol{P}^{h^0(L)-1}$ be the projective embedding by means of a basis of $\Gamma(L)$. Then the following results are known:

(A) Assume that C is smooth over K.

(0) (D. Mumford [5]) L is normally generated, if $d \geq 2g+1$.

(1) (B. Saint-Donat [7]) The largest homogeneous ideal I defining $\phi_L(C)$, i.e., $I = \mathrm{Ker}[S\Gamma(L) \rightarrow \bigoplus_{m \geq 0} \Gamma(L^m)]$, is generated by its elements of deree 2, if $d \geq 2g+2$.

(2) (B. Saint-Donat [7]) I is generated by its elements of degree 2 and 3, if $d \geq 2g+1$.

(B) (T. Fujita [1]) The statements (0) and (1) in (A) are true without the assumption that C is smooth over K.

The purposes of the present paper are that we improve the second result (2) of Saint-Donat and that we construct some related examples (corollary 1.4, Example 2.4 and Proposition 3.1).

Notation and Terminology. We fix an algebraically closed field K of characteristic $p \geqq 0$ throughout the paper. We use the word "variety" to mean a reduced irreducible scheme of finite type and proper over K, and "curve" to mean a variety of dimention 1.

For a finite dimensional vector space V over K, $S^m V$ means the m-th symmetric power of V and SV means the symmetric algebra of V, i.e., $SV = \bigoplus_{m \geqq 0} S^m V$.

Let L be an invertible sheaf on a projective variety X. We denote by L^m the m-th tensor product $L^{\otimes m}$. For the vector space of global sections $\Gamma(L)$, we define I and I_m $(m \geqq 1)$, by

$$I = I(L) = \mathrm{Ker}[S\Gamma(L) \rightarrow \bigoplus_{m \geqq 0} \Gamma(L^m)],$$

and

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$$I_m = I_m(L) = \mathrm{Ker}[S^m\Gamma(L) \to \Gamma(L^m)].$$

Let $L_1, \cdots, L_m$ be invertible sheaves on X. Then $\mathcal{R}(L_1, \cdots, L_m)$ means the kernel of the natural map:

$$\Gamma(L_1)\otimes\cdots\otimes\Gamma(L_m) \to \Gamma(L_1\otimes\cdots\otimes L_m).$$

§ 1. Generality.

Let X be a projective variety, and let L be an ample invertible sheaf on X. If the canonical map $\Gamma(L)^{\otimes m} \to \Gamma(L^m)$ is surjective for all positive integers m, then L is called a normally generated ample invertible sheaf.

We will establish a criterion for surjectivity of the natural map $I_m(L)\otimes\Gamma(L) \to I_{m+1}(L)$ for a normally generated ample invertible sheaf L.

LEMMA 1.1. *Let V be a finite dimensional vectar space, and let r be a positive integer greater than* 1. *Then we have*

$$\mathrm{Ker}[V^{\otimes(r+1)} \to S^{r+1}V] = \mathrm{Ker}[V^{\otimes r} \to S^r V]\otimes V + V\otimes \mathrm{Ker}[V^{\otimes r} \to S^r V].$$

A proof of the lemma is easy, so we omit its proof.

PROPOSITION 1.2. *Let L be a normally generated ample invertible sheaf on a variety X. If m is a positive integer greater than* 1, *then the following conditions are equivalent*:

(1) $\Gamma(L)\otimes\mathcal{R}(L^{m-1}, L) \xrightarrow{\xi} \mathcal{R}(L^m, L)$ *is surjective,*

(2) $\mathcal{R}(\overbrace{L, \cdots, L}^{m+1}) = \mathcal{R}(\overbrace{L, \cdots, L}^{m})\otimes\Gamma(L) + \Gamma(L)\otimes\mathcal{R}(\overbrace{L, \cdots, L}^{m})$,

(3) $I_m(L)\otimes\Gamma(L) \xrightarrow{q} I_{m+1}(L)$ *is surjective.*

PROOF[(*)]. We consider the following exact diagram

$$\begin{array}{ccccccc} 0 \to \mathcal{R}(\overbrace{L, \cdots, L}^{m})\otimes\Gamma(L) & \xrightarrow{\alpha} & \mathcal{R}(\overbrace{L, \cdots, L}^{m+1}) & \longrightarrow & \mathcal{R}(L^m, L) & \to 0 \\ & & \uparrow{\scriptstyle \xi'} & & \uparrow{\scriptstyle \xi} & \\ & & \Gamma(L)\otimes\mathcal{R}(\overbrace{L, \cdots, L}^{m}) & \to & \Gamma(L)\otimes\mathcal{R}(L^{m-1}, L) & \to 0 \\ & & \uparrow & & & \\ & & 0 & & & \end{array}$$

It is easy to check that $\mathcal{R}(\overbrace{L, \cdots, L}^{m+1}) = Im(\alpha) + Im(\xi')$ if and only if ξ is surjective.

Next, we will prove the equivalence (2)$\Longleftrightarrow$(3). Note that the canonical map π_r: $\mathcal{R}(\overbrace{L, \cdots, L}^{r}) \to I_r(L)$ is surjective for any integers $r \geqq 2$. For a given $f \in I_{m+1}(L)$,

(*) The proof of the first part (1)$\Leftrightarrow$(2), has been fairly simplified by an idea of Dr. Sekiguchi.

We can find $s\in\mathcal{R}(\overbrace{L, \cdots, L}^{m+1})$ such that $\pi_{m+1}(s)=f$. By (2), we have $s=\sum_i \beta_i\otimes s_i+\sum_j t_j\otimes\gamma_j$ for suitable elements β_i, $\gamma_j\in\mathcal{R}(\overbrace{L, \cdots, L}^{m})$ and s_i, $t_j\in\Gamma(L)$, so we have $f=q(\sum_i \pi_m(\beta_i)\otimes s_i+\sum_j \pi_m(\gamma_j)\otimes t_j)$. Hence (2) implies (3). To prove the implication (3)$\Rightarrow$(2), it suffices to show the inclusion relation

$$\mathcal{R}(\overbrace{L, \cdots, L}^{m+1})\subset\mathcal{R}(\overbrace{L, \cdots, L}^{m})\otimes\Gamma(L)+\Gamma(L)\otimes\mathcal{R}(\overbrace{L, \cdots, L}^{m}).$$

Let s be an element of $\mathcal{R}(\overbrace{L, \cdots, L}^{m+1})$. Then by (3), there exist $t_j\in\mathcal{R}(\overbrace{L, \cdots, L}^{m})$ and $s_j\in\Gamma(L)$ such that $\pi_{m+1}(s)=q(\sum_j \pi_m(t_j)\otimes s_j)$.
Hence

$$s-\sum_j t_j\otimes s_j\in\operatorname{Ker}(\Gamma(L)^{\otimes(m+1)}\to S^{m+1}\Gamma(L)).$$

Since by Lemma 1.1,

$$\begin{aligned}&\operatorname{Ker}(\Gamma(L)^{\otimes(m+1)}\to S^{m+1}\Gamma(L))\\ &=\operatorname{Ker}(\Gamma(L)^{\otimes m}\to S^m\Gamma(L))\otimes\Gamma(L)+\Gamma(L)\otimes\operatorname{Ker}(\Gamma(L)^{\otimes m}\to S^m\Gamma(L))\\ &\subset\mathcal{R}(\overbrace{L, \cdots, L}^{m})\otimes\Gamma(L)+\Gamma(L)\otimes\mathcal{R}(\overbrace{L, \cdots, L}^{m}),\end{aligned}$$

so we have

$$s\in\mathcal{R}(\overbrace{L, \cdots, L}^{m})\otimes\Gamma(L)+\Gamma(L)\otimes\mathcal{R}(\overbrace{L, \cdots, L}^{m}).$$

Q.E.D.

COROLLARY 1.3. *Let L be a normally generated ample invertible sheaf on an* n*-dimensional variety X. Assume that $H^i(X, L^j)=(0)$ for any integers $i, j\geq 1$. Then the homogeneous ideal* I(L) *is generated by* $\mathrm{I}_2, \cdots, \mathrm{I}_{n+3}$.

PROOF. By Propostion 1.2, it suffices to prove that the natural map $\Gamma(L)\otimes\mathcal{R}(L^{m-1}, L)\to\mathcal{R}(L^m, L)$ is surjective for any integer $m\geqq n+3$. It is just the theorem of Mumford [5, Theorem 5].

COROLLARY 1.4. *Let L be a normally generated ample invertible sheaf on a curve C. Assume that $H^1(C, L)=(0)$. Then $I(L)$ is generated by I_2 and I_3.*

PROOF. By Proposition 1.2 and Corollary 1.3, it suffices to show that the natural map $\Gamma(L)\otimes\mathcal{R}(L^2, L)\to\mathcal{R}(L^3, L)$ is surjective. It is a direct consequence of the following lemma.

LEMMA 1.5. (T. Fujita [1, Lemma 1.8]) *Let L, M and N be invertible sheaves on a curve C. Assume that $H^1(C, M\otimes L^{-1})=(0)$ and that $\Gamma(L)$ is base point free and that the natural map $\Gamma(M\otimes L^{-1})\otimes\Gamma(N)\to\Gamma(M\otimes N\otimes L^{-1})$ is surjective. Then the natural map $\Gamma(L)\otimes\mathcal{R}(M, N)\to\mathcal{R}(L\otimes M, N)$ is surjective.*

REMARK 1.6. Let L be an invertible sheaf of degree d on a curve C. If $d \geqq 2g+1$, then L is a normally generated ample invertible sheaf with $H^1(C, L)=(0)$. Therefore by Corollary 1.4, $I(L)$ is generated by I_2 and I_3,

This is another proof of the second result of Saint-Donat (c.f Introduction (2)).

§ 2. Example I.

In this section we use the word "curve" to mean a smooth curve over K. We assume that the characteristic of the ground field K is not 2. The purpose of this section is to show that the first result of Saint-Donat (see Introduction (1)) is the best possible for each genus $g \geqq 1$, namely, there exists a curve C of genus g with invertible sheaf L on C of degree $2g+1$ such that the homogeneous ideal $I(L)$ is not generated by $I_2(L)$.

REMARK 2.1. Let C be a curve of genus 1 or 2, and let L be an invertible sheaf of degree $2g+1$ on C. Then the homogeneous ideal $I(L)$ is not generated by $I_2(L)$.

Indeed, C is embedded by $\Gamma(L)$ to $\boldsymbol{P}^2$ if the genus is 1 (resp. to $\boldsymbol{P}^3$ if the genus is 2), but the dimension of $I_2(L)$ is 0 if the genus is 1 (resp. is 1 if the genus is 2).

From now on, we fix a hyper-elliptic curve C of genus $g \geqq 3$. Let $K(C)$ be the function field of C. Since the characteristic of the ground field K is not 2, there exist functions $x, y \in K(C)$ such that $K(C)=K(x, y)$ with a relation

$$y^2=(x-a_1)\cdot(x-a_2)\cdots(x-a_{2g+1}).$$

Let P_∞ be the closed point on C such that $x(P_\infty)=\infty$, and let $L=\mathcal{O}_C((2g+1)P_\infty)$ For any divisor D on C, we regard $\mathcal{O}_C(D)$ as a subsheaf of $K(C)$ in the canonical way. Then we have that the $g+2$ functions

$$\{1, x, \cdots, x^g, y\}$$

forms a basis of $\Gamma(L)$ and that the $\frac{1}{2}(g+2)(g+3)$ elements

$$\left\{\begin{array}{llllll}
1\odot 1 & & & & & \\
1\odot x & x\odot x & & & & \\
1\odot x^2 & x\odot x^2 & x^2\odot x^2 & & & \\
\vdots & \vdots & \vdots & & & \\
1\odot x^g & x\odot x^g & x^2\odot x^g & \cdots\cdots & x^g\odot x^g & \\
1\odot y & x\odot y & x^2\odot y & \cdots\cdots & x^g\odot y & y\odot y
\end{array}\right\}$$

forms a basis of $S^2\Gamma(L)$, where the symbol $\odot$ means a symmetric product.

PROPOSITION 2.2. *The vector space*

$$I_2(L)=\mathrm{Ker}[S^2\Gamma(L)\to\Gamma(L^2)]$$

is generated by $\{x^i\odot x^j-x^{i-1}\odot x^{j+1}\mid 1\leqq i\leqq j\leqq g-1\}$ *over* K.

PROOF. It is easy to show that the above set is included in I_2. Let V be a subspace of I_2 generated by the above set, and let W be asubspace of $S^2\Gamma(L)$ generated by the following elements:

$$\left\{\begin{array}{llll} 1\odot 1 & 1\odot x, & \cdots\cdots, & 1\odot x^g \\ x\odot x^g & x^2\odot x^g, & \cdots\cdots, & x^g\odot x^g \\ 1\odot y & x\odot y, & \cdots\cdots, & x^g\odot y \\ y\odot y & & & \end{array}\right\}.$$

Then the natural map $W\to S^2\Gamma(L)/V$ is surjective. Indeed, if $i\leqq g-j$, then

$$x^i\odot x^j\equiv x^{i-1}\odot x^{j+1}\equiv\cdots\cdots\equiv 1\odot x^{j+i}\quad\mathrm{mod.}\ V,$$

and if $i>g-j$, then

$$x^i\odot x^j\equiv x^{i+1}\odot x^{j-1}\equiv\cdots\cdots\equiv x^{i+j-g}\odot x^g\quad\mathrm{mod.}\ V.$$

Hence we have $\dim[S^2\Gamma(L)]-\dim(V)\leqq\dim(W)$, so we have

$$\dim(V)\geqq\frac{1}{2}g(g-1)=\dim(I_2).$$

Since $I_2\supset V$, we have $I_2=V$. Q.E.D.

COROLLARY 2.3. *Let* $\{X_0, X_1, \cdots, X_g, Y\}$ *be a homogeneous coordinate of the projective space* $\boldsymbol{P}^{g+1}$ *corresponding to a basis* $\{1, x, \cdots, x^g, y\}$ *of* $\Gamma(L)$. *Then the vector space of quadrics vanishing on* $\phi_L(C)$ *is generated by the quadrics*

$$\{X_iX_j-X_{i-1}X_{j+1}\mid 1\leqq i\leqq j\leqq g-1\}$$

over K.

EXAMPLE 2.4. Let (C, L) be the above curve with invertible sheaf. Then the degree L is $2g+1$, but the homogeneous ideal $I(L)$ is not generated by $I_2(L)$.

In fact, if the homogeneous ideal $I(L)$ is generated by I_2, then

$$\phi_L(C)=\bigcap_{1\leqq i\leqq j\leqq g-1}V(X_iX_j-X_{i-1}X_{j+1})$$

by Corollary 2.3, where $V(X_iX_j-X_{i-1}X_{j+1})$ is the set of zeros of $X_iX_j-X_{i-1}X_{j+1}$ in $\boldsymbol{P}^{g+1}$. Let H be the linear subvariety of $\boldsymbol{P}^{g+1}$ defined by the equations:

$$X_0-X_1,\ X_1-X_2,\ \cdots,\ X_{g-1}-X_g.$$

Then $H \cong \boldsymbol{P}^1$, and

$$H \subset \bigcap_{1 \leqq i \leqq j \leqq g-1} V(X_i X_j - X_{i-1} X_{j+1}).$$

Hence we have $H = \phi_L(C)$, because $H \subset \phi_L(C)$ and $\phi_L(C)$ is irreducible. This contradicts $g \geqq 1$.

§ 3. Example II.

We continue assuming that the characteristic of the ground field K is not 2 and that a "curve" means a smooth curve over K.

In this section we will show that there are many examples of curves of genus g with invertible sheaf of degree $2g$ on which Corollary 1.4 works effectively. Note that since the degree of L is $2g$, the condition $H^1(C, L) = (0)$ in Corollary 1.4 is automatically satisfied. Therefore our problem is reduced to constructing many curves of genus g which have a normally generated ample invertible sheaf of degree $2g$.

PROPOSITION 3.1. *Let C be a curve of genus $g \geqq 5$. Suppose that there exists an invertible sheaf M of degree $g-1$ on C such that $\Gamma(M)$ is a base point free pencil. Then almost all invertible sheaves of degree $2g$ on C are ample with normal generation.*

The following lemma, B. Saint-Donat [8] called it "base point free pencil trick", plays an important role in the proof of our proposition.

LEMMA 3.2. (Mumford [5, p. 57], Saint-Donat [8, Lemma 2.6]) *Let M and N be invertible sheaves on a curve. Suppose that $\Gamma(M)$ is a base point free pencil. Then we have an isomorphism*

$$\mathrm{Ker}[\Gamma(M) \otimes \Gamma(N) \to \Gamma(M \otimes N)] \cong \Gamma(N \otimes M^{-1}).$$

We will use the following notation.

$\mathrm{Pic}^d(C)$: the connected component of the Picard scheme of C whose member represents an invertible sheaf of degree d,

G_d^r: the closed subvariety of $\mathrm{Pic}^d(C)$ representing the set of invertible sheaves of degree d and of projective dimension $\geqq r$,

F_d^r: the closed subvariety of $\mathrm{Pic}^d(C)$ defined by the image of the morphism

$$G_{d-1}^r \times C \ni (L, P) \longrightarrow L(P) \in \mathrm{Pic}^d(C)$$

(if $G_{d-1}^r = \phi$, then F_d^r means the void subset).

Note that $F_d^r \subset G_d^r$ and that if $r \geqq 1$, the set $G_d^r - F_d^r$ represents the set of invertible sheaves free from base points, of degree d and of proejctive dimension r.

PROOF OF PROPOSITION 3.1. There exists an invertible sheaf M_0 of degree $g-1$ such that $\Gamma(M_0)$ is a base point free pencil and $M_0^2 \neq \omega$, where ω is the canonical sheaf on C. Indeed, since $G_{g-1}^1 - F_{g-1}^1$ is non-empty open in G_{g-1}^1 by our assumption and since

$$\dim G_{g-1}^1 \geqq g-4 \geqq 1 \quad [4, \text{Theorem } 1],$$

$G_{g-1}^1 - F_{g-1}^1$ has infinitely many elements. So there exists such an invertible sheaf. We put

$$V = G_{g+1}^1 - F_{g+1}^1 = \mathrm{Pic}^{g+1}(C) - F_{g+1}^1, \text{ and}$$
$$U = \{N \otimes M_0 \mid N \in V\} \subset \mathrm{Pic}^{2g}(C).$$

Obviously, V is non-empty open in $\mathrm{Pic}^{g+1}(C)$. Hence U is non-empty open in $\mathrm{Pic}^{2g}(C)$. We will show that any invertible sheaf in U is ample with normal generation. Let L be an invertible sheaf in U. By the generalized lemma of Castelnuvo [5, Theorem 2], we have natural map $\Gamma(L^m) \otimes \Gamma(L) \to \Gamma(L^{m+1})$ is surjective for $m \geqq 2$. Therefore it suffices to show that the natural map $\Gamma(L) \otimes \Gamma(L) \to \Gamma(L^2)$ is surjective. Consider the commutative diagram

$$\begin{array}{ccc}
\Gamma(L \otimes M_0^{-1}) \otimes \Gamma(M_0) \otimes \Gamma(L) & \xrightarrow{1 \otimes \phi_1} & \Gamma(L \otimes M_0^{-1}) \otimes \Gamma(M_0 \otimes L) \\
\downarrow & & \downarrow \phi_2 \\
\Gamma(L) \otimes \Gamma(L) & \longrightarrow & \Gamma(L^2),
\end{array}$$

where ϕ_1 is the natural map $\Gamma(M_0) \otimes \Gamma(L) \to \Gamma(M_0 \otimes L)$. By Lemma 3.2, we have $\mathrm{Ker}\, \phi_1 \cong \Gamma(L \otimes M_0^{-1})$ and $\mathrm{Ker}\, \phi_2 \cong \Gamma(M_0^2)$. Therefore we have

$$\dim(\mathrm{Ker}\, \phi_1) = \dim[\Gamma(L \otimes M_0^{-1})] = 2,$$
$$\dim[\Gamma(M_0) \otimes \Gamma(L)] = 2(g+1),$$
$$\dim[\Gamma(M_0 \otimes L)] = 2g,$$
$$\dim(\mathrm{Ker}\, \phi_2) = \dim[\Gamma(M_0^2)] = g-1 \quad (\text{Note that } M_0^2 \neq \omega),$$
$$\dim[\Gamma(L \otimes M_0^{-1}) \otimes \Gamma(M_0 \otimes L)] = 4g \quad \text{and}$$
$$\dim[\Gamma(L^2)] = 3g+1.$$

Hence ϕ_1 and ϕ_2 are surjective, and hence the natural map $\Gamma(L) \otimes \Gamma(L) \to \Gamma(L^2)$ is surjective. Q.E.D.

Next, we will give a sufficient condition for a curve to have an invertible sheaf M of degree $g-1$ such that $\Gamma(M)$ is a base point free pencil. Our result on it is a direct consequence of the following theorem of Martens and Mumford.

THEOREM OF MARTENS AND MUMFORD [6, Appendix]. *Let C be a curve of genus $g \geqq 5$. Then there exists integer d, $3 \leqq d \leqq g-2$, such that* $\dim G_d^1 \geqq d-3$ *if* *and only if C is hyperelliptic, or trigonal, or double covering of an elliptic curve* ($g\geqq 6$), *or non-singular plane quintic.*

PROPOSITION 3.3. *Let C be a curve of genus $g\geqq 5$ neither hyperelliptic, nor trigonal, nor double covering of an elliptic curve* ($g\geqq 6$), *nor non-singular plane quintic. Then there exists an invertible sheaf M of degree $g-1$ on C such that $\Gamma(M)$ is a base point free pencil.*

PROOF. We must prove that $G_{g-1}^1-F_{g-1}^1\neq\phi$ in our case. For this, it suffices to show that $\dim G_{g-1}^1>\dim F_{g-1}^1$. By the results of Martens, Kleiman and Laksov [4, Theorem 1 and 3, Theorem 5], we have

$$g-3\geqq\dim G_{g-1}^1\geqq g-4, \text{ and}$$
$$g-4\geqq\dim G_{g-2}^1\geqq g-6.$$

Note that if $G_{g-2}^1\neq\phi$, then

$$\dim F_{g-1}^1=\dim G_{g-2}^1+1 \quad [4, \text{ p. } 115]$$

and that if $G_{g-2}^1=\phi$, then $F_{g-1}^1=\phi$. Suppose that $\dim G_{g-1}^1=\dim F_{g-1}^1$. Then $\dim G_{g-2}^1\geqq g-5$. This contradicts the theorem of Martens and Mumford. Q.E.D.

Finally, we state an elementary remark relative to our topic.

REMARK 3.4. If C is a curve of genus $g\geqq 4$, then there exists a non-special very ample invertible sheaf on C which is not normally generated.

Indeed, for a non-special normally generated ample invertible sheaf L, we have

$$\deg L\geqq g+\frac{1}{2}+\sqrt{2g+\frac{1}{4}}$$

because $\dim S^2\Gamma(L)\geqq\dim\Gamma(L^2)$. On the other hand, by the theorem of Halphen [2, Theorem 1.2], there exists a non-special very ample invertible sheaf of degree d, if $d\geqq g+3$.

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