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ALGEBRAIC INDEPENDENCE OF INFINITE PRODUCTS GENERATED BY FIBONACCI NUMBERS

By

Takeshi KUROSAWA, Yohei TACHIYA, and Taka-aki TANAKA

Abstract. The aim of this paper is to establish necessary and sufficient conditions for certain infinite products generated by Fibonacci numbers and by Lucas numbers to be algebraically independent.

1. Introduction and the Results

Let α and β be real algebraic numbers with $|\alpha| > 1$ and $\alpha\beta = -1$. We define

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n \quad (n \geq 0). \tag{1}$$

If $\alpha = (1 + \sqrt{5})/2$, we have $U_n = F_n$ and $V_n = L_n$ $(n \geq 0)$, where the sequences $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ are Fibonacci numbers and Lucas numbers defined, respectively, by $F_{n+2} = F_{n+1} + F_n$ $(n \geq 0)$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ $(n \geq 0)$, $L_0 = 2$, $L_1 = 1$. Let $d \geq 2$ be a fixed integer. For an arbitrary nonzero integer a, the second author [3] proved that the infinite products

$$\prod_{\substack{k=1 \\ U_{d^k} \neq -a}}^{\infty} \left(1 + \frac{a}{U_{d^k}}\right) \quad \text{and} \quad \prod_{\substack{k=1 \\ V_{d^k} \neq -a}}^{\infty} \left(1 + \frac{a}{V_{d^k}}\right)$$

are transcendental numbers, except for only two algebraic numbers $\prod_{k=1}^{\infty}(1 - 1/V_{2^k})$ and $\prod_{k=1}^{\infty}(1 + 2/V_{2^k})$ (cf. Remark 1 below).

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The purpose of this paper is to establish necessary and sufficient conditions for the infinite products

$$\prod_{\substack{k=1\\U_{d^k}\neq -a_i}}^{\infty}\left(1+\frac{a_i}{U_{d^k}}\right)\quad (i=1,\dots,m)\qquad\text{or}\qquad \prod_{\substack{k=1\\V_{d^k}\neq -a_i}}^{\infty}\left(1+\frac{a_i}{V_{d^k}}\right)\quad (i=1,\dots,m)$$

with nonzero integers $a_1,\dots,a_m$ to be algebraically independent.

THEOREM 1. *Let* $\{U_n\}_{n\geq 0}$ *be the sequence defined by* (1) *and* d *an integer greater than* 1. *Let* $a_1,\dots,a_m$ *be nonzero distinct integers. Then the numbers*

$$\prod_{\substack{k=1\\U_{d^k}\neq -a_i}}^{\infty}\left(1+\frac{a_i}{U_{d^k}}\right)\quad (i=1,\dots,m)$$

are algebraically independent.

EXAMPLE 1. Let $\{F_n\}_{n\geq 0}$ be the Fibonacci numbers. For any nonzero distinct integers $a_1,\dots,a_m$, the numbers

$$\prod_{\substack{k=1\\F_{d^k}\neq -a_i}}^{\infty}\left(1+\frac{a_i}{F_{d^k}}\right)\quad (i=1,\dots,m)$$

are algebraically independent. In particular, the numbers $\prod_{k=2}^{\infty}(1-1/F_{2^k})$ and $\prod_{k=1}^{\infty}(1+1/F_{2^k})$ are algebraically independent.

COROLLARY 1. *Let* $\{U_n\}_{n\geq 0}$ *and* d *be as in Theorem* 1. *Let* $a_1,\dots,a_m$ *be distinct integers. Then the numbers*

$$\prod_{\substack{k=1\\U_{d^k}\neq -a_i,\,-a_i-1}}^{\infty}\left(1+\frac{1}{U_{d^k}+a_i}\right)\quad (i=1,\dots,m) \tag{2}$$

are algebraically independent.

THEOREM 2. *Let* $\{V_n\}_{n\geq 0}$ *be the sequence defined by* (1) *and* d *an integer greater than* 1. *Let* $a_1,\dots,a_m$ *be nonzero distinct integers. Then the numbers*

$$\prod_{\substack{k=1\\V_{d^k}\neq -a_i}}^{\infty}\left(1+\frac{a_i}{V_{d^k}}\right)\quad (i=1,\dots,m)$$

are algebraically dependent if and only if $d = 2$ *and at least one of the following two properties are satisfied*:

(i) *For some* j $(1 \le j \le m)$, $a_j = -1$ *or* 2.

(ii) *The set* $\{a_1, \dots, a_m\}$ *contains integers* $b_1, \dots, b_l$ $(l \ge 3)$ *with* $b_1 \le -3$ *satisfying*

$$b_2 = -b_1, \quad b_j = b_{j-1}^2 - 2 \quad (j = 3, \dots, l-1), \quad b_l = -b_{l-1}^2 + 2. \quad (3)$$

Example 2. Let $\{L_n\}_{n \ge 0}$ be the Lucas numbers. For an arbitrary integer $m \ge 3$, the numbers

$$\prod_{k=1}^{\infty}\left(1 + \frac{1}{L_{2^k}}\right), \quad \prod_{k=1}^{\infty}\left(1 + \frac{3}{L_{2^k}}\right), \quad \prod_{k=1}^{\infty}\left(1 + \frac{4}{L_{2^k}}\right), \dots, \quad \prod_{k=1}^{\infty}\left(1 + \frac{m}{L_{2^k}}\right)$$

are algebraically independent, while $\prod_{k=1}^{\infty}(1 + 2/L_{2^k}) = \sqrt{5}$.

Example 3. The transcendental numbers

$$\rho_1 = \prod_{k=1}^{\infty}\left(1 - \frac{5}{L_{2^k}}\right), \quad \rho_2 = \prod_{k=1}^{\infty}\left(1 + \frac{5}{L_{2^k}}\right), \quad \text{and} \quad \rho_3 = \prod_{k=1}^{\infty}\left(1 - \frac{23}{L_{2^k}}\right)$$

are algebraically dependent with $\operatorname{trans.deg}_{\mathbf{Q}} \mathbf{Q}(\rho_1, \rho_2, \rho_3) = 2$ and $4\sqrt{5}\rho_1\rho_2 + \rho_3 = 0$.

Remark 1. If $d = 2$ and if the property (i) in Theorem 2 is satisfied, then the corresponding infinite products are algebraic. Indeed, the second author [3] obtained

$$\prod_{k=1}^{\infty}\left(1 - \frac{1}{V_{2^k}}\right) = \frac{\alpha^4 - 1}{\alpha^4 + \alpha^2 + 1} \quad \text{and} \quad \prod_{k=1}^{\infty}\left(1 + \frac{2}{V_{2^k}}\right) = \frac{\alpha^2 + 1}{\alpha^2 - 1}.$$

On the other hand, if $d = 2$ and if there exist integers $b_1, \dots, b_l$ $(l \ge 3)$ satisfying the recurrence relation (3), letting

$$\Psi_i(x) = \prod_{k=0}^{\infty}\left(1 + \frac{b_i x^{2^k}}{1 + x^{2^{k+1}}}\right) \quad (i = 1, \dots, l)$$

and using $\prod_{k=1}^{\infty}(1 + x^{2^k}) = 1/(1 - x^2)$, we have

$$\Psi_1(x)\Psi_2(x) = (1-x^2)^2 \prod_{k=0}^{\infty}(1 - b_3 x^{2^{k+1}} + x^{2^{k+2}}),$$

$$\Psi_1(x)\Psi_2(x)\Psi_3(x) = \frac{(1-x^2)^3}{1 - b_3 x + x^2} \prod_{k=0}^{\infty}(1 - b_4 x^{2^{k+1}} + x^{2^{k+2}}),$$

$$\dots,$$

$$\Psi_1(x)\Psi_2(x)\cdots\Psi_{l-2}(x) = \frac{(1-x^2)^{l-2}}{\prod_{j=3}^{l-2}(1 - b_j x + x^2)} \prod_{k=0}^{\infty}(1 - b_{l-1} x^{2^{k+1}} + x^{2^{k+2}}),$$

and

$$\Psi_1(x)\Psi_2(x)\cdots\Psi_{l-1}(x) = \frac{(1-x^2)^{l-2}\Psi_l(x)}{(1 + b_l x + x^2)\prod_{j=3}^{l-1}(1 - b_j x + x^2)}.$$

Noting that

$$v_i := \prod_{\substack{k=1 \\ V_{2^k} \neq -b_i}}^{\infty} \left(1 + \frac{b_i}{V_{2^k}}\right) = \Psi_i(\alpha^{-2^N}) \prod_{\substack{k=1 \\ V_{2^k} \neq -b_i}}^{N-1} \left(1 + \frac{b_i}{V_{2^k}}\right) \quad (i = 1, \dots, l)$$

for large N, we see $v_l^{-1} \prod_{j=1}^{l-1} v_j \in \mathbf{Q}(\alpha)$.

REMARK 2. In a similar but simpler way to Remark 1 and the proof of Theorem 2 we can show the following: Let γ be an algebraic number with $0 < |\gamma| < 1$ and d an integer greater than 1. Let $a_1, \dots, a_m$ be nonzero distinct integers. Then the numbers

$$\prod_{\substack{k=1 \\ a_i\gamma^{d^k} \neq -1}}^{\infty} (1 + a_i \gamma^{d^k}) \qquad (i = 1, \dots, m)$$

are algebraically dependent if and only if $d = 2$ and at least one of the following two properties are satisfied:

(i) For some j $(1 \le j \le m)$, $a_j = 1$.
(ii) The set $\{a_1, \dots, a_m\}$ contains integers $b_1, \dots, b_l$ $(l \ge 3)$ with $b_1 \le -2$ satisfying

$$b_2 = -b_1, \quad b_j = b_{j-1}^2 \quad (j = 3, \dots, l-1), \quad b_l = -b_{l-1}^2.$$

2. Preparation for the Proof

Let K be an algebraic number field, $K(x)$ the field of rational functions over K, and $K[[x]]$ the ring of formal power series with coefficients in K. We define the subgroup H_d of the group $K(x)^{\times}$ of nonzero elements of $K(x)$ by

$$H_d = \left\{ \frac{g(x^d)}{g(x)} \,\middle|\, g(x) \in K(x)^{\times} \right\}. \tag{4}$$

We use the following lemmas for proving the theorems.

Lemma 1 (Kubota [1, Corollary 8]). *Let $f_1(x), \dots, f_m(x) \in K[[x]] \setminus \{0\}$ satisfy the functional equations*

$$f_i(x^d) = c_i(x) f_i(x), \quad c_i(x) \in K(x)^{\times} \quad (i = 1, \dots, m). \tag{5}$$

Then $f_1(x), \dots, f_m(x)$ are algebraically independent over $K(x)$ if and only if the rational functions $c_1(x), \dots, c_m(x)$ are multiplicatively independent modulo H_d.

Lemma 2 (Kubota [1], see also Theorem 3.6.4 in Nishioka [2]). *Suppose that the functions $f_1(x), \dots, f_m(x) \in K[[x]]$ converge in $|x| < 1$ and satisfy the functional equations (5) with $c_i(x)$ defined and nonzero at $x = 0$. Let γ be an algebraic number with $0 < |\gamma| < 1$ such that $c_i(\gamma^{d^k})$ are defined and nonzero for all $k \geq 0$. If $f_1(x), \dots, f_m(x)$ are algebraically independent over $K(x)$, then the values $f_1(\gamma), \dots, f_m(\gamma)$ are algebraically independent.*

3. Proofs of Theorems and Corollary 1

Proof of Theorem 1. Define

$$\Phi_i(x) = \prod_{k=0}^{\infty} \left(1 + \frac{(\alpha - \beta) a_i x^{d^k}}{1 - (-1)^d x^{2d^k}} \right) \quad (i = 1, \dots, m).$$

Take an integer N such that $|U_{d^k}| > \max\{|a_1|, \dots, |a_m|\}$ for all $k \geq N$. Then

$$\Phi_i(\alpha^{-d^N}) = \prod_{k=N}^{\infty} \left(1 + \frac{a_i}{U_{d^k}} \right) \quad (i = 1, \dots, m),$$

so that

$$\prod_{\substack{k=0 \\ U_{d^k} \neq -a_i}}^{\infty} \left(1 + \frac{a_i}{U_{d^k}} \right) = \Phi_i(\alpha^{-d^N}) \prod_{\substack{k=0 \\ U_{d^k} \neq -a_i}}^{N-1} \left(1 + \frac{a_i}{U_{d^k}} \right) \quad (i = 1, \dots, m). \tag{6}$$

Suppose that the numbers (6) are algebraically dependent. Then so are the values $\Phi_1(\alpha^{-d^N}), \dots, \Phi_m(\alpha^{-d^N})$. Since $\Phi_1(x), \dots, \Phi_m(x)$ satisfy the functional equations $\Phi_i(x^d) = c_i(x)\Phi_i(x)$ with

$$c_i(x) = \frac{1 - (-1)^d x^2}{1 + (\alpha - \beta)a_i x - (-1)^d x^2} \quad (i = 1, \dots, m),$$

the functions $\Phi_1(x), \dots, \Phi_m(x)$ are algebraically dependent over $K(x)$ by Lemma 2 with $K = \mathbf{Q}(\alpha)$. Then by Lemma 1 the rational functions $c_1(x), \dots, c_m(x)$ are multiplicatively dependent modulo H_d, namely there exist integers $e_1, \dots, e_m$, not all zero, and $g(x) \in K(x)^\times$ such that $\prod_{i=1}^m c_i(x)^{e_i} = g(x^d)/g(x)$. Then, renumbering the a_i, we may assume that there exist coprime polynomials $A(x), B(x) \in K[x]$ such that

$$A(x)B(x^d)\prod_{i=1}^{h} P_i(x)^{e_i} = (1 - (-1)^d x^2)^e A(x^d)B(x)\prod_{i=h+1}^{s} P_i(x)^{e_i}, \tag{7}$$

where $h, e_i, e \in \mathbf{Z}$ with $h, e_i \geq 1$ and $P_i(x) = 1 + (\alpha - \beta)a_i x - (-1)^d x^2$. Since α and β are real with $\alpha\beta = -1$, we have $|\alpha - \beta| = |\alpha| + |\beta| > 2$, so that the roots α_i and β_i ($|\alpha_i| \geq |\beta_i|$) of $P_i(x) = 0$ are real with $|\alpha_i| > 1 > |\beta_i|$. Since we admit the case of $e < 0$, renumbering the a_i again if necessary, we may assume $|\alpha_1| \geq |\alpha_i|$ and $|\beta_1| \leq |\beta_i|$ $(i = 2, 3, \dots, s)$. Noting that $P_i(x)$ and $P_j(x)$ $(i \neq j)$ are coprime and substituting $x = \alpha_1$ into (7), we have $A(\alpha_1^d)B(\alpha_1) = 0$.

If $A(\alpha_1^d) = 0$, substituting $x = \alpha_1^d$ into (7) and noting that $A(x)$ and $B(x)$ are coprime, we get $A(\alpha_1^{d^2}) = 0$. Repeating this process, we obtain $A(\alpha_1^{d^l}) = 0$ for all $l \geq 1$, which is impossible. Thus we have $B(\alpha_1) = 0$. If $d \geq 3$ or $\alpha_1 < 0$, substituting $x = \alpha_1^{1/d}$ into (7) and noting that the roots of $P_i(x) = 0$ are real, we get $B(\alpha_1^{1/d}) = 0$ and so $B(\alpha_1^{1/d^l}) = 0$ for all $l \geq 0$. This is impossible and hence we obtain $d = 2$ and $\alpha_1 > 0$. Substituting $x = \beta_1$ into (7), we have $\beta_1 > 0$ by the same way as above, which contradicts $\alpha_1\beta_1 = -1$. □

PROOF OF THEOREM 2. We define

$$\Psi_i(x) = \prod_{k=0}^{\infty}\left(1 + \frac{a_i x^{d^k}}{1 + (-1)^d x^{2d^k}}\right) \quad (i = 1, \dots, m),$$

which satisfy the functional equations $\Psi_i(x^d) = c_i(x)\Psi_i(x)$ with

$$c_i(x) = \frac{1 + (-1)^d x^2}{1 + a_i x + (-1)^d x^2} \quad (i = 1, \dots, m).$$

Then we have

$$\prod_{\substack{k=1\\ V_{d^k}\neq -a_i}}^{\infty}\left(1+\frac{a_i}{V_{d^k}}\right)=\Psi_i(\alpha^{-d^N})\prod_{\substack{k=1\\ V_{d^k}\neq -a_i}}^{N-1}\left(1+\frac{a_i}{V_{d^k}}\right)\quad (i=1,\dots,m) \tag{8}$$

for large N. Suppose that the numbers (8) are algebraically dependent. By Lemmas 1 and 2 with $K=\mathbf{Q}$, the rational functions $c_1(x),\dots,c_m(x)$ defined above are multiplicatively dependent modulo H_d. Hence by the same way as in the proof of Theorem 1 there exist coprime polynomials $A(x)$, $B(x)$ in $\mathbf{Z}[x]$ such that

$$A(x)B(x^d)\prod_{i=1}^{h}P_i(x)^{e_i}=(1+(-1)^d x^2)^e A(x^d)B(x)\prod_{i=h+1}^{s}P_i(x)^{e_i}, \tag{9}$$

where $h,e_i,e\in\mathbf{Z}$ with $h,e_i\geq 1$ and $P_i(x)=1+a_ix+(-1)^d x^2$. Let α_i be one of the roots of $P_i(x)=0$ with $|\alpha_i|\geq 1$. If $d\geq 3$ is odd, then the roots of $P_i(x)=0$ are real and $|\alpha_i|>1$. Hence we can deduce a contradiction by a similar way to the proof of Theorem 1.

Now we suppose that $d\geq 2$ is even. Then the equation (9) is expressed as

$$A(x)B(x^d)\prod_{i=1}^{h}P_i(x)^{e_i}=(1+x^2)^e A(x^d)B(x)\prod_{i=h+1}^{s}P_i(x)^{e_i} \tag{10}$$

with $P_i(x)=1+a_ix+x^2$. Comparing the orders at $x=1$ of both sides of (10), we see that $a_i\neq -2$ for all i. We distinguish two cases.

Case I). $d\geq 4$ is even.

If $|a_i|\geq 3$ for some i $(1\leq i\leq s)$, noting that $|\alpha_i|>1$, we deduce a contradiction by a similar way to the proof of Theorem 1. Hence $a_1\in\{\pm 1,2\}$, so that $\alpha_1\in\{\pm\omega,-1\}$, where ω is a primitive cubic root of unity. First we consider the cases of $d\geq 8$ and $d=4$. Let ζ_d be a primitive d-th root of unity. Substituting $x=\alpha_1$ into (10), we have $A(\alpha_1^d)B(\alpha_1)=0$. If $A(\alpha_1^d)=0$, substituting $x=\zeta_d\alpha_1$ into (10) again and noting that $P_i(\zeta_d\alpha_1)\neq 0$ $(1\leq i\leq 3)$, we have $A(\zeta_d\alpha_1)=0$. Repeating this process, we obtain $A(\zeta_{d^l}\alpha_1^{1/d^{l-1}})=0$ for all $l\geq 1$, a contradiction. Similarly in the case of $B(\alpha_1)=0$ we obtain $B(\zeta_{d^l}\alpha_1^{1/d^l})=0$ for all $l\geq 1$, a contradiction.

Thus we have $d=6$. In the case of $a_1=1$, noting that $A(1)=0$ and substituting $x=-\omega,-1$ into (10), we may put $P_2(-\omega)=0$ and $P_3(-1)=0$, respectively, since $A(-\omega)A(-1) \neq 0$ by the same arguments as above. Hence the equation (10) is written as

$$A(x)B(x^6)(1+x+x^2)^{e_1}(1-x+x^2)^{e_2}(1+x)^{2e_3} = (1+x^2)^e A(x^6)B(x), \quad (11)$$

where $e = e_1 + e_2 + e_3 \geq 1$. Substituting $x = \sqrt{-1}$ into (11), we have $A(\sqrt{-1})B(-1) = 0$, which again leads to a contradiction. The proof is similar also in the cases of $a_1 = -1$ and $a_1 = 2$.

Case II). $d = 2$.

Comparing the orders at $x = \omega$ of both sides of (10), we see that $a_1 \neq 1$. For the case of $a_1 = -1$, using

$$1 + \frac{-x}{1+x^2} = \frac{g(x)}{g(x^2)}, \quad g(x) = \frac{1-x^2}{1+x+x^2},$$

we have

$$\Psi_1(x) = \prod_{k=0}^{\infty}\left(1 + \frac{-x^{2^k}}{1+x^{2^{k+1}}}\right) = \frac{1-x^2}{1+x+x^2}, \quad |x| < 1. \quad (12)$$

Similarly for the case of $a_1 = 2$, we have

$$\Psi_1(x) = \prod_{k=0}^{\infty}\left(1 + \frac{2x^{2^k}}{1+x^{2^{k+1}}}\right) = \frac{1+x}{1-x}, \quad |x| < 1. \quad (13)$$

Thus the property (i) in Theorem 2 is yielded.

Now we consider the remaining case of $|a_1| \geq 3$. Then α_1 is a real quadratic number with $|\alpha_1| > 1$. Since $c_i(x)P_i(x) = (1-x^4)/(1-x^2) \in H_2$, where H_2 is defined by (4), and since $c_1(x), \ldots, c_m(x)$ are multiplicatively dependent modulo H_2, the polynomials $P_1(x), \ldots, P_m(x)$ are multiplicatively dependent modulo H_2. Hence, changing the indices i if necessary, we have

$$A(x)B(x^2)\prod_{i=1}^{h} P_i(x)^{e_i} = A(x^2)B(x)\prod_{i=h+1}^{s} P_i(x)^{e_i} \quad (14)$$

with $|\alpha_1| \geq |\alpha_i|$ $(i = 2, \ldots, s)$. Substituting $x = \alpha_1$ into (14), we get $A(\alpha_1^2)B(\alpha_1) = 0$. Suppose that $A(\alpha_1^2) = 0$. Then we see inductively that $A(\alpha_1^{2^l}) = 0$ for all $l \geq 1$, which is impossible, so that $B(\alpha_1) = 0$. If $\alpha_1 < 0$, then $P_i(\alpha_1^{1/2}) \neq 0$ $(1 \leq i \leq s)$, so that $B(\alpha_1^{1/2^l}) = 0$ for all $l \geq 1$, which is also impossible. Thus we obtain $\alpha_1 > 0$. In what follows, we denote by $\alpha_1^{1/2^N}$ the positive root of $x^{2^N} - \alpha_1 = 0$ $(N = 1, 2, \ldots)$. Substituting $x = -\alpha_1^{1/2}$ into (14) and noting that $B(-\alpha_1^{1/2}) \neq 0$, we see that there exists an i_1 $(h+1 \le i_1 \le s)$ with $P_{i_1}(-\alpha_1^{1/2}) = 0$. On the other hand, substituting $x = \alpha_1^{1/2}$ into (14), we have $B(\alpha_1^{1/2}) \prod_{i=h+1}^{s} P_i(\alpha_1^{1/2}) = 0$. If there exists a j with $P_j(\alpha_1^{1/2}) = 0$, we put $i_2 = j$. Otherwise, we see $B(\alpha_1^{1/2}) = 0$ and hence, by the same argument as above, there exists an i_2 $(h+1 \le i_2 \le s)$ with $P_{i_2}(-\alpha_1^{1/4}) = 0$. Repeating this process, we get for some $t \ge 1$

$$P_{i_k}(-\alpha_1^{1/2^k}) = 0 \quad (k = 1, 2, \dots, t), \quad P_{i_{t+1}}(\alpha_1^{1/2^t}) = 0.$$

Noting that $\alpha_1 + \alpha_1^{-1} = -a_1$ and $P_{i_k}(x) = 1 + a_{i_k}x + x^2$, we see that

$$-\alpha_1^{1/2^k} - \alpha_1^{-1/2^k} = -a_{i_k} \quad (k = 1, 2, \dots, t), \quad \alpha_1^{1/2^t} + \alpha_1^{-1/2^t} = -a_{i_{t+1}}.$$

Therefore we can choose from the set $\{a_1, \dots, a_m\}$ the integers $b_1, \dots, b_l$ $(l \ge 3)$ with $b_1 \le -3$ satisfying

$$b_2 = -b_1, \quad b_j = b_{j-1}^2 - 2 \quad (j = 3, \dots, l-1), \quad b_l = -b_{l-1}^2 + 2,$$

which implies the property (ii) in Theorem 2. The converse follows from (12), (13), and Remark 1. $\square$

Proof of Corollary 1. Let $M = \max\{|a_1|, \dots, |a_m|\}$ and let $N \ge 1$ be an integer such that $|U_{d^k}| > M + 1$ for all $k \ge N$. Assume on the contrary that the numbers (2) are algebraically dependent. Then so are the numbers

$$\eta_j := \prod_{k=N}^{\infty} \left(1 + \frac{1}{U_{d^k} - M - 1 + j}\right) \quad (j = 1, \dots, 2M+1),$$

and so there exists a nonzero polynomial $f(x_1, \dots, x_{2M+1}) \in \mathbf{Z}[x_1, \dots, x_{2M+1}]$ such that $f(\eta_1, \dots, \eta_{2M+1}) = 0$. Let $F > 0$ be the total degree of $f(x_1, \dots, x_{2M+1})$ and define $g(y_1, \dots, y_{2M+1}) \in \mathbf{Z}[y_1, \dots, y_{2M+1}]$ by

$$\frac{g(y_1, \dots, y_{2M+1})}{(y_1 y_2 \cdots y_{2M})^F} = f\left(\frac{y_2}{y_1}, \dots, \frac{y_M}{y_{M-1}}, \frac{1}{y_M}, y_{M+1}, \frac{y_{M+2}}{y_{M+1}}, \dots, \frac{y_{2M+1}}{y_{2M}}\right). \quad (15)$$

We note that $g(y_1, \dots, y_{2M+1}) \ne 0$. Indeed, substituting $y_i = \prod_{k=i}^{M} x_k^{-1}$ $(i = 1, \dots, M)$ and $y_i = \prod_{k=M+1}^{i} x_k$ $(i = M+1, \dots, 2M+1)$ into (15), we see that the right-hand side coincides with $f(x_1, \dots, x_{2M+1}) \ne 0$. Let

$$\xi_j = \begin{cases} \displaystyle\prod_{k=N}^{\infty} \left(1 + \frac{-M-1+j}{U_{d^k}}\right) & (j = 1, \dots, M), \\ \displaystyle\prod_{k=N}^{\infty} \left(1 + \frac{-M+j}{U_{d^k}}\right) & (j = M+1, \dots, 2M+1). \end{cases}$$

Noting that

$$1 + \frac{1}{U_{d^k} - M - 1 + j} = \left(1 + \frac{-M + j}{U_{d^k}}\right)\left(1 + \frac{-M - 1 + j}{U_{d^k}}\right)^{-1},$$

we have $\eta_j = \xi_{j+1}/\xi_j$ $(j = 1, \dots, M-1)$, $\eta_M = 1/\xi_M$, $\eta_{M+1} = \xi_{M+1}$, and $\eta_j = \xi_j/\xi_{j-1}$ $(j = M+2, \dots, 2M+1)$. Therefore we obtain $g(\xi_1, \dots, \xi_{2M+1}) = (\xi_1 \cdots \xi_{2M+1})^F f(\eta_1, \dots, \eta_{2M+1}) = 0$, so that $\xi_1, \dots, \xi_{2M+1}$ are algebraically dependent, which contradicts Theorem 1. $\square$

References

[1] K. K. Kubota, On the algebraic independence of holomorphic solutions of certain functional equations and their values, Math. Ann. **227** (1977), 9–50.

[2] K. Nishioka, Mahler Functions and Transcendence, Lecture Notes in Math. 1631, Springer, 1996.

[3] Y. Tachiya, Transcendence of certain infinite products, J. Number Theory **125** (2007), 182–200.

Takeshi Kurosawa
Department of Mathematical Information Science
Tokyo University of Science, 1-3 Kagurazaka
Shinjyuku-ku, Tokyo 162-8601, Japan
E-mail address: tkuro@rs.kagu.tus.ac.jp

Yohei Tachiya
Department of Mathematics, Keio University
Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan
E-mail address: bof@math.keio.ac.jp

Taka-aki Tanaka
Department of Mathematics, Keio University
Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan
E-mail address: takaaki@math.keio.ac.jp