HOMOTOPY EQUIVALENCES OF PUNCTURED MANIFOLDS

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Let M be a closed (smooth or PL) manifold of dimension $n \geqslant 2$, and let W be the punctured manifold obtained by removing from M the interior of a (smoothly or PL)-imbedded n-disc D, centered at the basepoint m of M. Not surprisingly, the group $G_1(M)$ of homotopy classes of basepoint-preserving degree 1 self-homotopy-equivalences of M is closely related to the group $G(W, \partial W)$ of homotopy classes (rel ∂W) of self-homotopy-equivalences of W that fix ∂W. The main theorem of this paper makes this precise. We begin by stating the theorem and outlining various applications.

The map $r: W \to W$ is a rotation about the boundary sphere of W, described in Section 1(b). Our main theorem is

THEOREM 3.2. *There is a central extension*

$$1 \longrightarrow K \longrightarrow G(W, \partial W) \longrightarrow G_1(M) \longrightarrow 1$$

where K is the subgroup of $G(W, \partial W)$ generated by $\langle r \rangle$. For $n \geqslant 3$, $K = 0$ or $\mathbf{Z}/2$ according as $\langle r \rangle = \langle 1_W \rangle$ or $\langle r \rangle \neq \langle 1_W \rangle$. For $n = 2$, $K = 0$ if M is the 2-sphere or real projective plane, otherwise $K \cong \mathbf{Z}$.

After proving this theorem in Section 3, we apply it to the problem of deforming homotopy equivalences to homeomorphisms, showing that every element of $G_1(M)$ contains PL homeomorphisms if and only if every element of $G(W, \partial W)$ does. In Section 4, we consider the case of M aspherical. In this case, $\langle r \rangle \neq \langle 1_W \rangle$, so $G(W, \partial W)$ is determined, at least up to extension, by $\pi_1(M, m)$. In Section 5, we apply Theorem 3.2 to describe the stabilizers of certain elements in finitely-generated free groups. In the final section, we show that when $M = T^n$, the n-dimensional torus, the exact sequence of Theorem 3.2 is isomorphic to a well-known sequence involving the Steinberg group $\mathrm{St}(n, \mathbf{Z})$. Thus, the extension need not be trivial.

In the first section, we will discuss a few preliminaries, including the fact that $G(W, \partial W)$ is a group. The main lemma, from which Theorem 3.2 follows easily, is proved in Section 2. The proof uses geometric constructions to simplify a homotopy between a self-homotopy-equivalence of M and the identity map of M.

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1. Preliminaries.

1.a. *Mapping spaces of manifolds.*

We will always work only with basepoint-preserving maps of M, and use the C-O topology on mapping spaces. If $A(N)$ is a space of mappings from a manifold N to itself, and $X \subset N$, let $A(N, X) = \{f \in A(N) : f|_X \text{ is the identity map } 1_X\}$. When the

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elements of $A(M)$ are homotopy equivalences, denote by $A_1(M)$ the subspace of maps having degree 1. (The degree of a basepoint-preserving self-homotopy-equivalence of a nonorientable closed manifold can be defined using the orientable double cover. Details appear in [11].) We are primarily concerned with $A(M) = E(M)$, the space of self-homotopy-equivalences of M, which we regard as acting on the right on M. Note that any element of $E(W, \partial W)$ is homotopic (rel ∂W) to a map f such that for some collar neighborhood C of ∂W, $f|_C = 1_C$ and $f^{-1}(C) = C$. We will frequently assume this condition without mention.

Suppose $f: W \to W$ is a map with $f|_{\partial W} = 1_{\partial W}$. We let $\hat{f}: M \to M$ be the map with $\hat{f}|_W = f$ and $\hat{f}|_D = 1_D$. Obviously, $\hat{f}$ has degree 1.

LEMMA 1.1. *$f \in E(W, \partial W)$ if and only if $\hat{f} \in E_1(M)$.*

Proof. For $n \geqslant 3$, f induces an isomorphism on $\pi_1(W)$ if and only if $\hat{f}$ induces an isomorphism on $\pi_1(M)$. Also, the universal cover $\tilde{W}$ of W is obtained from the universal cover $\tilde{M}$ of M by removing the interior of the inverse image $\tilde{D}$ of D. Hence, by excision, $H_*(\tilde{W}, \partial\tilde{W}) \cong H_*(\tilde{M}, \tilde{D})$. The lift of f (respectively, $\hat{f}$) permutes the components of $\partial\tilde{W}$ (respectively, $\tilde{D}$), so looking at the long exact homology sequences of $(\tilde{W}, \partial\tilde{W})$ and $(\tilde{M}, \tilde{D})$, we see that f induces an isomorphism on all homotopy groups if and only if $\hat{f}$ does.

For $n = 2$, it is still true that f induces an isomorphism on $\pi_1(W)$ if and only if $\hat{f}$ induces an isomorphism on $\pi_1(M)$, the "if" direction using the fact that free groups are Hopfian. This proves the lemma when M is aspherical. When M is the 2-sphere, any degree 1 map of M is homotopic to 1_M, while $E(W, \partial W)$ is contractible by the Alexander trick. Finally, when M is the real projective plane, W is a Möbius band. By [4, Theorem 13.1] and [2, Theorem 3.4], any map of W which fixes ∂W is homotopic (rel ∂W) to 1_W. Therefore, in this case both f and $\hat{f}$ will be homotopic to the identity. □

Thus, we obtain an injection $E(W, \partial W) \to E_1(M)$ which is easily checked to be an imbedding. There are analogous imbeddings for the groups of homeomorphisms Homeo(M), PL homeomorphisms PL(M), and diffeomorphisms Diff(M), if by Diff($W, \partial W$) we mean the space of diffeomorphisms of W that are the identity on some neighborhood of ∂W.

It is well-known that composition of maps induces a group structure on $G(M) = \pi_0(E(M))$, with $G_1(M) = \pi_0(E_1(M))$ a subgroup of index one or two. That $G(W, \partial W) = \pi_0(E(W, \partial W))$ is also a group under composition follows from the next proposition, which was pointed out to me by F. Ancel.

PROPOSITION 1.2. *Let A be a closed collared subset of a space X, i.e., suppose there is a closed imbedding $e: A \times [0,1] \to X$ such that $e(a, 0) = a$ for each $a \in A$ and $e(A \times [0,1))$ is an open subset of X. If $f: X \to X$ is a homotopy equivalence with $f|_A = 1_A$, then there is a map $g: X \to X$ with $g|_A = 1_A$ and $fg \simeq gf \simeq 1_X$* (rel A).

Proof. Identify $A \times [0,1]$ with $e(A \times [0,1])$ for notational convenience, and let $X_t = X - (A \times [0, t))$. Define $\varphi, \psi: X \times [0,1] \to X$ by $\varphi_t|_{X_1} = 1_{X_1}$, $\varphi_t(a, s) = a$ for $0 \leqslant s \leqslant t/2$, $\varphi_t(a, s) = (a, (2s - t)/(2 - t))$ for $t/2 \leqslant s \leqslant 1$, $\psi_t|_{X_1} = 1_{X_1}$, and $\psi_t(a, s) = (a, s(1 - t/2) + t/2)$ for $0 \leqslant s \leqslant 1$. Note $\varphi_0 = \psi_0 = 1_X$, $\psi_t\varphi_t|_{X_{t/2}} = 1_{X_{t/2}}$, and $\varphi_t\psi_t = 1_X$.

Let $g_1 : X \to X$ be a homotopy inverse for f and $h : X \times [0,1] \to X$ a homotopy from 1_X to $g_1 f$. Define $g : X \to X$ by $g(x) = \psi_1 g_1 \varphi_1(x)$ for $x \in X_{1/2}$, $g(a,s) = (a, 2s)$ for $0 \leqslant s \leqslant \frac{1}{4}$, and $g(a,s) = \psi_1 h(a, 4s-1)$ for $\frac{1}{4} \leqslant s \leqslant \frac{1}{2}$. Note that $g|_A = 1_A$, and $\varphi_t g \psi_t$ defines a homotopy from g to g_1.

We now define homotopies $F : f \simeq F_1$, and $H : 1_X \simeq gF_1$, both (rel A). Let $F_t(x) = \psi_t f \varphi_t(x)$ for $x \in X_{t/2}$ and $F_t(a,s) = (a,s)$ for $0 \leqslant s \leqslant t/2$; let $H_t(x) = \psi_1 h_t \varphi_1(x)$ for $x \in X_{1/2}$, $H_t(a,s) = (a, s(t+1))$ for $0 \leqslant s \leqslant 1/(2(t+1))$, and

$$H_t(a,s) = \psi_1 h(a, 2s(t+1)-1) \quad \text{for} \quad 1/(2(t+1)) \leqslant s \leqslant \tfrac{1}{2}.$$

These homotopies show that g is a left homotopy inverse (rel A) for f.

Repeating, we obtain a left homotopy inverse (rel A) for g, say f'. Then $1_X \simeq f'g \simeq f'gfg \simeq fg$ (rel A). □

1.b. *Rotation about a sphere.*

Suppose $S^{n-1} \times I \subset M$ and $\gamma : (I, 0, 1) \to (SO(n), 1, 1)$ is a loop representing a generator of $\pi_1(SO(n), 1)$. We define a homeomorphism $h : M \to M$ by $h(x,t) = (\gamma(t)(x), t)$ for $(x,t) \in S^{n-1} \times I$ and $h(y) = y$ for $y \notin S^{n-1} \times I$. The isotopy class of h does not depend on the choice of γ, and any homeomorphism isotopic to h is called a *rotation about the sphere* $S^{n-1} \times \{0\}$. For $n \geqslant 3$, $\pi_1(SO(n), 1) \cong \mathbf{Z}/2$ so h^2 is isotopic (rel $M - (S^{n-1} \times (0,1))$) to 1_M. In fact, $h|_{S^{n-1} \times I}$ generates

$$\pi_0(\mathrm{Maps}(S^{n-1} \times I, S^{n-1} \times \partial I)) \cong \mathbf{Z}/2.$$

(For $n = 3$, see [5, p. 172]; for $n > 3$ use suspension.) For $n = 2$, $h|_{S^1 \times I}$ represents a generator of $\pi_0(\mathrm{Maps}(S^1 \times I, S^1 \times \partial I)) \cong \mathbf{Z}$, and h is usually called a *Dehn twist* about $S^1 \times \{0\}$.

When $S^{n-1} \times I$ is a collar neighborhood of ∂W in W, we denote by $r : W \to W$ a rotation about the sphere ∂W. Since any two collarings of ∂W are ambient isotopic keeping ∂W fixed, the homotopy class $\langle r \rangle \in G(W, \partial W)$ does not depend on the choice of collar.

2. Main lemma.

LEMMA 2.1. *Let $f \in E_1(M)$ with $f|_D = 1_D$ and $f^{-1}(D) = D$, and suppose $f \simeq 1_M$* (rel m). *Then there is a homotopy $F : f \simeq \hat{r}^k$* (rel D) *with $F^{-1}(D) = D \times I$, consequently $f|_W \simeq r^k$* (rel ∂W). *If $n = 2$, then $k \in \mathbf{Z}$, while if $n \geqslant 3$ then k may be chosen to be in* $\{0, 1\}$.

Proof. We first treat the case $n \geqslant 3$. Let $G : M \times I \to M$ be a homotopy (rel m) from f to 1_M. By [7, Theorem 2.3], we may assume G is a homotopy (rel D). Without changing it in a neighborhood of $M \times \partial I \cup m \times I$, change G to be a transverse to m, so that $G^{-1}(m)$ consists of $m \times I$ and a collection of contractible simple closed curves in $M \times (0,1)$. Using a standard construction, as in the proof of Theorem 4.1 of [6], we may change G by a homotopy (rel $M \times \partial I$) whose effect on $G^{-1}(m)$ is to replace its components by their connected sum. Because the simple closed curves were contractible, the arc $G^{-1}(m)$ will be homotopic (rel $m \times \partial I$) to $m \times I$. Since $n \geqslant 3$, this implies $G^{-1}(m)$ is ambient isotopic (rel $M \times \partial I$) to $m \times I$. Changing G by this ambient isotopy, we may assume $G^{-1}(m) = m \times I$. The altered homotopy G is, however, no longer a homotopy (rel D).

Since $G^{-1}(D)$ is a neighborhood of $m \times I$, we may choose a smaller concentric n-ball $D' \subset D$ so that $G(D' \times I) \subset D$. Choose coordinates on D so that D' is the ball of radius $\frac{1}{2}$, and let D'' be the ball of radius $\frac{1}{4}$. We will define an isotopy of embeddings $J_t : M \to M \times I$. If $x \notin D'$ let $J_t(x) = (x, t)$, while if $x \in D''$ let $J_t(x) = (x, 0)$. For $x \in D' - D''$, let $J_t(x) = (x, (4\|x\| - 1)t) \in D \times I$. The effect of J is to slide $(M - D') \times \{0\}$ upward, holding D'' fixed and extending linearly to $D' - D''$. We define the homotopy $K : f \simeq f_1$ (rel D'') by $K(x, t) = G(J_t(x))$. Observe that $f_1|_{(M-D') \cup D''}$ is the identity, $f_1(D' - \mathrm{int}(D'')) \subset D - m$, and $K^{-1}(m) = m \times I$. Pushing image points radially away from m, we may assume K satisfies the above conditions and also $K^{-1}(D'') = D'' \times I$ and $f_1(D' - \mathrm{int}(D'')) \subset D' - \mathrm{int}(D'')$. Now $f_1|_{D' - \mathrm{int}(D'')}$ is a self-map of $S^{n-1} \times I$ which is the identity on $S^{n-1} \times \partial I$, so it is homotopic (rel $\partial(D' - \mathrm{int}(D''))$) to the identity or a rotation about $\partial D''$. Following K by the trivial extension of such a homotopy, we obtain a homotopy F satisfying the conclusion of Lemma 2.1 with $k \in \{0, 1\}$ and D'' in place of D. A radial adjustment in a neighborhood of D completes the proof.

When $n = 2$, the previous argument breaks down because $G^{-1}(m)$ might not be isotopic to $m \times I$, so we must proceed differently. Let $g_0 = f|_W : W \to W$. By Lemma 1.1, g_0 is a homotopy equivalence. By a theorem due to Baer and Nielsen [4, Theorem 13.1], g is properly homotopic to a homeomorphism g. Therefore, there is a homotopy $K : f \simeq \hat{g}$ with $K^{-1}(m) = m \times I$. But $\hat{g} \simeq f \simeq 1_M$ (rel m) so $\hat{g}$ is isotopic to 1_M (rel m) [2, Theorem 6.3]. Following K by this isotopy, we obtain $G : f \simeq 1_M$ with $G^{-1}(m) = m \times I$. We can now continue the argument as in the case $n \geqslant 3$, with the difference that the homotopy classes of maps of $S^1 \times I$ fixed on $S^1 \times \partial I$ form an infinite cyclic group with a Dehn twist about $S^1 \times \{0\}$ as generator. □

2. The main theorem.

LEMMA 3.1. *For* $f \in E(W, \partial W)$, $\langle r \rangle \langle f \rangle = \langle f \rangle \langle r \rangle$.

Proof. We may assume $f|_{\partial W \times I} = 1_{\partial W \times I}$ and $f^{-1}(\partial W \times I) = \partial W \times I$ for some collar neighborhood of ∂W. If r' is a rotation defined using this collar, then $\langle r \rangle \langle f \rangle = \langle r' \rangle \langle f \rangle = \langle r' f \rangle = \langle f r' \rangle = \langle f \rangle \langle r \rangle$. □

THEOREM 3.2. *There is a central extension*

$$1 \longrightarrow K \longrightarrow G(W, \partial W) \longrightarrow G_1(M) \longrightarrow 1$$

where K is the subgroup of $G(W, \partial W)$ generated by $\langle r \rangle$. For $n \geqslant 3$, $K = 0$ or $\mathbf{Z}/2$ according as $\langle r \rangle = \langle 1_W \rangle$ or $\langle r \rangle \neq \langle 1_W \rangle$. For $n = 2$, $K = 0$ if M is the 2-sphere or real projective plane, otherwise $K \cong \mathbf{Z}$.

Proof. Let $\langle f \rangle \in G_1(M)$. By Lemma 1.2 of [11], we may choose f within the homotopy class so that $f|_D = 1_D$ and $f^{-1}(D) = D$. (The lemma is proved for the smooth category in [11] but the argument works in the PL category using [3, Theorem 3] in place of [13, Theorem B].) By Lemma 1.1, $f_0 = f|_W \in E(W, \partial W)$, and $\langle \hat{f}_0 \rangle = \langle f \rangle$. Therefore, $G(W, \partial W) \to G_1(M)$ is surjective. If $\langle f_0 \rangle$ is in K, then Lemma 2.1 applied to $\hat{f}_0$ shows that $\langle f_0 \rangle = \langle r \rangle^k$ for some k. Since $\langle r \rangle$ is in K, this shows that K equals the subgroup generated by $\langle r \rangle$, which is central by Lemma 3.1.

The description of K for $n \geqslant 3$ is obvious. When M is the 2-sphere or real projective plane, it is easy to see geometrically that r is isotopic to 1_W (rel ∂W). Finally, if M is a closed 2-manifold other than the 2-sphere or real projective plane, then $\pi_1(W, w)$ is free on at least two generators, and the boundary curve of W represents a nonzero element $c \in \pi_1(W, w)$. (We choose $w \in \partial W$.) The induced map

$$r_\#^k : \pi_1(W, w) \to \pi_1(W, w)$$

equals conjugation by c^k, so $\{\langle r \rangle^k \mid k \in \mathbf{Z}\}$ are distinct elements of $G(W, \partial W)$. □

Our first application of Theorem 3.2 concerns the problem of deforming homotopy equivalences to homeomorphisms.

COROLLARY 3.3. *For* $A = \mathrm{PL}$ *or* Diff, $\pi_0(A_1(M)) \to G_1(M)$ *is surjective if and only if* $\pi_0(A(W, \partial W)) \to G(W, \partial W)$ *is surjective.*

Proof. By Theorem 3.2, $G(W, \partial W) \to G_1(M)$ is surjective, so the "if" direction follows from the commutative diagram

$$\begin{array}{ccc} \pi_0(A(W, \partial W)) & \longrightarrow & \pi_0(A_1(M)) \\ \downarrow & & \downarrow \\ G(W, \partial W) & \longrightarrow & G_1(M) \end{array} .$$

Also by Theorem 3.2, the kernel of $G(W, \partial W) \to G_1(M)$ is generated by $\langle r \rangle$, which can be represented by an element of $A(W, \partial W)$. Therefore, the "only if" direction is implied by the fact that $\pi_0(A(W, \partial W)) \to \pi_0(A_1(M))$ is surjective, which follows from [13, Theorem B] for $A = \mathrm{Diff}$ and [3, Theorem 3] for $A = \mathrm{PL}$. □

4. The case of M aspherical. For $n \geqslant 3$ it can easily happen that $\langle r \rangle = \langle 1_W \rangle$. For example, this occurs when $M^n = S^p \times S^q$. There is, however, an important case for which $\langle r \rangle \neq \langle 1_W \rangle$.

LEMMA 4.1. *If M is aspherical, or a connected sum of aspherical manifolds, then* $\langle r \rangle \neq \langle 1_W \rangle$ *in* $G(W, \partial W)$.

Proof. For $n = 2$, this is part of Theorem 3.2. For $n \geqslant 3$, regard the connected sum $M \# M$ as two copies W_1 and W_2 of W with their boundaries identified. A rotation about the boundary sphere of W_1 extends using the identity on W_2 to a homeomorphism of $M \# M$. From the proofs of Theorems 4.5 and 5.3 of [10], the homotopy class of this homeomorphism is nontrivial in $G(M \# M)$. Therefore, the rotation cannot be trivial in $G(W, \partial W)$. □

When M is aspherical, the degree of an automorphism of $\pi_1(M, m)$ can be defined to be the degree of a basepoint-preserving self-homotopy-equivalence of M that induces the automorphism. Let $\mathrm{Aut}_1(\pi_1(M, m))$ be the automorphisms of degree 1. Combining Theorem 3.2 and Lemma 4.1, and the isomorphism of $G(M)$ with $\mathrm{Aut}(\pi_1(M, m))$ [14, Theorem 8.1.9], we have

COROLLARY 4.2. *Suppose M is aspherical.*

(a) *If $n = 2$, there is a central extension*

$$1 \longrightarrow \mathbf{Z} \longrightarrow G(W, \partial W) \longrightarrow \mathrm{Aut}_1(\pi_1(M, m)) \longrightarrow 1$$

in which the kernel is generated by a Dehn twist about ∂W.

(b) *If $n \geqslant 3$, there is a central extension*

$$1 \longrightarrow \mathbf{Z}/2 \longrightarrow G(W, \partial W) \longrightarrow \mathrm{Aut}_1(\pi_1(M, m)) \longrightarrow 1$$

in which the kernel is generated by a rotation about ∂W.

In the next two sections, we will see examples for which the extensions of Corollary 4.2 are nontrivial; in fact, I do not know an aspherical example for which the extension is trivial.

5. Boundary-preserving automorphisms of free groups. For $g \geqslant 1$ let F_{2g} be the free group on $2g$ generators $\{a_1, b_1, a_2, \ldots, b_g\}$ and let $c_{2g} = \prod_{i=1}^{g} [a_i, b_i]$. For $x \in F_{2g}$, let $\mathrm{Aut}(F_{2g}, x)$ be the stabilizer of x in the automorphism group of F_{2g}; that is, let $\mathrm{Aut}(F_{2g}, x) = \{\varphi \in \mathrm{Aut}(F_{2g}) \mid \varphi(x) = x\}$. In [9], an effective procedure is given for obtaining a presentation for the group of automorphisms stabilizing a finite tuple of cyclic, or ordinary, words in a finitely-generated free group. Using combinatorial methods, the authors of [1] find generators for the stabilizers of certain commutators, and obtain a simple presentation for $\mathrm{Aut}(F_2, [a_1, b_1])$. As an application of Theorem 3.2 we prove the following theorem. Let T_g be the closed orientable surface of genus g.

THEOREM 5.1. *For $g \geqslant 1$ there are nontrivial central extensions*

$$1 \longrightarrow \mathbf{Z} \longrightarrow \mathrm{Aut}(F_{2g}, c_{2g}) \longrightarrow \mathrm{Aut}_1(\pi_1(T_g)) \longrightarrow 1$$

where the kernel is generated by conjugation by c_{2g}.

Proof. Let W_g be T_g with an open disc removed, and choose a basepoint $w \in \partial W_g$. Generators for $\pi_1(W_g, w) \cong F_{2g}$ can be chosen so that the boundary curve of W_g represents c_{2g}, and in this case a Dehn twist about the boundary of W_g induces the automorphism conjugation by c_{2g}. The central extension exists by Corollary 4.2(a) and the following lemma, which is easily proved using asphericity of W_g:

LEMMA 5.2. *The function sending $\langle f \rangle$ to $f_\# : \pi_1(W_g, w) \to \pi_1(W_g, w)$ is an isomorphism from $G(W_g, \partial W_g)$ onto $\mathrm{Aut}(F_{2g}, c_{2g})$.*

The extension is nontrivial since $\mathrm{Aut}_1(\pi_1(T_g))$ contains torsion, while $\mathrm{Aut}(F_{2g}, c_{2g})$ is torsion-free by [8, Proposition I.5.5]. □

The case $g = 1$ will be discussed further in the next section.

Using nonorientable surfaces, one obtains a similar theorem for the stabilizer of $\prod_{i=1}^{k} a_i^2$ in the free group on k generators $\{a_1, a_2, \ldots, a_k\}$, for $k \geqslant 1$.

6. Homotopy equivalences of the punctured n-torus. We now specialize to the case where M is the n-torus $T^n = \prod_{i=1}^{n} S^1$. Let X^n denote the punctured n-torus $T^n - \mathrm{int}(D^n)$, with basepoint $x \in \partial X^n$. Since $\pi_1(T^n, m) \cong \mathbf{Z}^n$, Corollary 4.2 becomes

LEMMA 6.1. *There are exact sequences*

$$1 \longrightarrow \mathbf{Z} \longrightarrow G(X^2, \partial X^2) \longrightarrow \mathrm{SL}(2, \mathbf{Z}) \longrightarrow 1$$

$$1 \longrightarrow \mathbf{Z}/2 \longrightarrow G(X^n, \partial X^n) \longrightarrow \mathrm{SL}(n, \mathbf{Z}) \longrightarrow 1 \qquad (n \geqslant 3)$$

in which the kernels are generated by a rotation about the boundary sphere.

As far as I know, the next theorem arises not from any deep connection between Steinberg groups and groups of homotopy equivalences of punctured manifolds, but merely indicates the paucity of extensions of $\mathbf{Z}/2$ by $\mathrm{SL}(n, \mathbf{Z})$.

THEOREM 6.2. (a) *There is an isomorphism of exact sequences*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathrm{St}(2,\mathbf{Z}) & \longrightarrow & \mathrm{SL}(2,\mathbf{Z}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \alpha_2 & & \downarrow = & & \\ 1 & \longrightarrow & \mathbf{Z} & \longrightarrow & G(X^2, \partial X^2) & \longrightarrow & \mathrm{SL}(2,\mathbf{Z}) & \longrightarrow & 1. \end{array}$$

(b) *For $n \geqslant 3$ there are isomorphisms of exact sequences*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & \mathrm{St}(n,\mathbf{Z}) & \longrightarrow & \mathrm{SL}(n,\mathbf{Z}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \alpha_n & & \downarrow = & & \\ 1 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & G(X^n, \partial X^n) & \longrightarrow & \mathrm{SL}(n,\mathbf{Z}) & \longrightarrow & 1. \end{array}$$

Proof. For $n \geqslant 2$, regard T^n as $(\prod_{i=1}^{n}[-3,3])/\sim$, where $(x_1, x_2, \ldots, (3)_i, \ldots, x_n) \sim (x_1, x_2, \ldots, (-3)_i, \ldots, x_n)$ for each $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \prod_{i=1}^{n-1}[-3,3]$. Let $D = \{x \in T^n \mid \|x\| \leqslant 1\}$, $C = \{x \in T^n \mid 1 \leqslant \|x\| \leqslant 2\}$, and regard X^n as $T^n - \mathrm{int}(D)$. For $1 \leqslant i,\ j \leqslant n,\ i \neq j$, and $\epsilon = \pm 1$, we define $\hat{f}_{ij}^{\epsilon} : T^n \to T^n$ by

$$\hat{f}_{ij}^{\epsilon}(x) = \hat{f}_{ij}^{\epsilon}(x_1, x_2, \ldots, x_n) = \begin{cases} (x_1, \ldots, x_n) & \text{if } x \in D \\ (x_1, \ldots, x_j + \epsilon(\|x\| - 1)x_i, \ldots, x_n) & \text{if } x \in C \\ (x_1, \ldots, x_j + x_i, \ldots, x_n) & \text{if } x \notin D \cup C. \end{cases}$$

We interpret the coordinates (mod 6) so that $\hat{f}_{ij}^{\epsilon}(x) \in T^n$. It is not hard to check that $\hat{f}_{ij}^{\epsilon}$ is well-defined and continuous, and that $\hat{f}_{ij}^{1}$ and $\hat{f}_{ij}^{-1}$ are inverse homeomorphisms. Let $f_{ij} = \hat{f}_{ij}^{1}|_{X^n} : X^n \to X^n$. For $n \geqslant 2$, $\mathrm{St}(n, \mathbf{Z})$ has generators $\{x_{ij} \mid 1 \leqslant i,\ j \leqslant n$ and $i \neq j\}$. We define $\alpha_n : \mathrm{St}(n, \mathbf{Z}) \to G(X^n, \partial X^n)$ by $\alpha_n(X_{ij}) = \langle f_{ij} \rangle$.

Proof of (a): From [12, pp. 82-83], we know that $\mathrm{St}(2, \mathbf{Z})$ has presentation $\langle x_{12}, x_{21} : x_{12}x_{21}^{-1}x_{12} = x_{21}^{-1}x_{12}x_{21}^{-1} \rangle$ and that the top sequence of part (a) is exact with kernel generated by $(x_{12}x_{21}^{-1}x_{12})^4$. Let $w = (0, -1)$ be the basepoint of X^2. Define $\alpha : I \to X^2$ by $\alpha(t) = (6t, -1)$ and $\beta : I \to X^2$ by $\beta(t) = (-\sqrt{1-(6t-1)^2}\ , 6t-1)$ for $0 \leqslant t \leqslant \frac{1}{3}$ and $\beta(t) = (0, 6t-1)$ for $\frac{1}{3} \leqslant t \leqslant 1$, interpreting the coordinates (mod 6). Observe that $\pi_1(X^2, w)$ is generated by $a = \langle \alpha \rangle$ and $b = \langle \beta \rangle$, and that $[a, b]$ is represented by the boundary curve of X^2. It is easy to see that $(f_{12})_{\#}(a) = ab$, $(f_{12})_{\#}(b) = b$, $(f_{21})_{\#}(a) = a$, and $(f_{21})_{\#}(b) = ba$. Since $(f_{12}f_{21}^{-1}f_{12})_{\#} = (f_{21}^{-1}f_{12}f_{21}^{-1})_{\#}$, Lemma 5.2 implies $f_{12}f_{21}^{-1}f_{12} \simeq f_{21}^{-1}f_{12}f_{21}^{-1}$ (rel ∂X^2), so α_2 is well-defined. The right-hand square commutes, where we regard $\mathrm{SL}(2, \mathbf{Z})$ as acting on the right on $\pi_1(T^2, m)$. Moreover, $(f_{12}f_{21}^{-1}f_{12})_{\#}^4$ equals conjugation by $[a, b]$, so $(f_{12}f_{21}^{-1}f_{12})^4$ is homotopic (rel ∂X^2) to a Dehn twist about ∂X^2. Therefore, α_2 induces an isomorphism on kernels, so α_2 is an isomorphism.

Proof of (b): For $n \geqslant 3$, $\mathrm{St}(n, \mathbf{Z})$ has relations
(1) $[x_{ij}, x_{kl}] = 1$ for $j \neq k$ and $i \neq l$
(2) $[x_{ij}, x_{jk}] = x_{ik}$ for $i \neq k$.
According to Theorem 10.1 of [12], the top sequence of part (b) is exact with kernel generated by $(x_{12}x_{21}^{-1}x_{12})^4$. We first check that α_n is well-defined. For type (1) relations, we have $[f_{ij}, f_{kl}] = 1_{X^n}$ so $\alpha_n([x_{ij}, x_{kl}]) = \langle 1_W \rangle$. For type (2) relations, recalling that $E(X^n, \partial X^n)$ acts on the right, we have $([f_{ij}, f_{jk}])(x_1, \ldots, x_n) =$

$$f_{jk}^{-1}(f_{ij}^{-1}(f_{jk}(f_{ij}(x_1, \ldots, x_n)))) = \begin{cases} (x_1, \ldots, x_k + (\|x\| - 1)^2 x_i, \ldots, x_n) & \text{if} \quad x \in C \\ (x_1, \ldots, x_k + x_i, \ldots, x_n) & \text{if} \quad x \notin C. \end{cases}$$

Since

$$F_t(x_1, \ldots, x_n) = \begin{cases} (x_1, \ldots, x_k + (\|x\| - 1)^{2-t} x_i, \ldots, x_n) & \text{if} \quad x \in C \\ (x_1, \ldots, x_k + x_i, \ldots, x_n) & \text{if} \quad x \notin C \end{cases}$$

defines a homotopy (rel ∂X^n) from $[f_{ij}, f_{jk}]$ to f_{ik}, we have $\alpha_n([x_{ij}, x_{jk}]) = \alpha_n(x_{ik})$, so α_n is well-defined. It is straightforward to check that the right-hand square commutes.

We now prove inductively that $\alpha_n((x_{12}x_{21}^{-1}x_{12})^4)$ is represented by a rotation about the boundary sphere of X^n. For $n = 2$, this was established in part (a). For $n \geqslant 3$, it is convenient to regard X^n in the following way. Let $X^{n-1} \times S^1 = (X^{n-1} \times I)/\sim$, where $(x, 0) \sim (x, 1)$. Now $S^{n-2} = \partial D^{n-1} = \partial X^{n-1}$; using this identification we glue $D^{n-1} \times [\frac{1}{4}, \frac{3}{4}]$ to $X^{n-1} \times S^1$ in the obvious way to form X^n. By induction, $\alpha_{n-1}((x_{12}x_{21}^{-1}x_{12})^4)$ is represented by $r: X^{n-1} \to X^{n-1}$, a rotation about the boundary sphere of X^{n-1}. Consideration of the definition of f_{ij} shows that $\alpha_n((x_{12}x_{21}^{-1}x_{12})^4)$ has a representative which equals $r \times 1_{S^1}$ on $X^{n-1} \times S^1$ and equals the identity on $D^n \times [\frac{1}{4}, \frac{3}{4}]$. It is not difficult to see that this map is homotopic, in fact, isotopic, to a rotation about the boundary sphere of X^n. □

Recalling that each f_{ij} is a homeomorphism of X^n and noting that F_t is an isotopy from $[f_{ij}, f_{jk}]$ to f_{ik}, we can factor α_n as a composite

$$\mathrm{St}(n, \mathbf{Z}) \to \pi_0(\mathrm{Homeo}(X^n, \partial X^n)) \to G(X^n, \partial X^n).$$

We conclude

COROLLARY 6.3. *The natural homomorphism*

$$\pi_0(\mathrm{Homeo}(X^n, \partial X^n)) \to G(X^n, \partial X^n)$$

has a section.

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