p-SUBGROUPS OF COMPACT LIE GROUPS AND TORSION OF INFINITE HEIGHT IN $H^*(BG)$, II

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1. The main purpose of this paper is to describe the size of the p-torsion subring of $H^*(BG,\mathbf{Z})$ where G is a compact Lie group. We construct a subring of $H^*(BG,\mathbf{Z})$ which is a direct sum of reduced polynomial algebras over $\mathbf{Z}/p^i\mathbf{Z}$ for various i (Corollary 1.7). This is accomplished by using transfer results largely developed in Part 1 of this paper [2]. A detection result of Quillen implies this is the best possible result in a suitable sense (Corollary 1.8).

The integral cohomology of the classifying space of a compact Lie group is extremely complicated in general. Even the torsion free quotient ring may behave oddly. For example if $G=\mathrm{Spin}(12)$, the quotient ring is not isomorphic to the invariants of the cohomology of the classifying space of a maximal torus under the action of the Weyl group [4]. In theory if one knows the rational and mod p cohomologies of BG, in addition to all the Bocksteins, one can derive a great deal of information about the integral cohomology of BG. This is often very difficult in practice however. One might also think that knowing $H^*(G)$ one could easily determine $H^*(BG)$ by a spectral sequence argument. The point is that the calculations become extremely difficult. The transfer techniques used here, however, are straightforward and produce a subring of interest.

A result we use several times is the following theorem of Quillen [5], which we shall summarize briefly. First there is a detection part of the theorem. This says that any non-nilpotent element in $H^*(BG,\mathbf{Z}/p\mathbf{Z})$ is detected on some maximal elementary abelian p-subgroup of G. The second result concerns the existence of non-nilpotent elements. Let $x\in\bigoplus H^*(BL,\mathbf{Z})$, where the direct sum is over all conjugacy classes of maximal elementary abelian p-subgroups of G, with L being a representative subgroup. If the coordinates of x are compatible with the obvious necessary conditions imposed by the relationship between the elementary abelian p subgroups of G and G, then some pth power of x is the image of an element of $H^*(BG,\mathbf{Z}/p\mathbf{Z})$.

We shall only use the detection part of Quillen's theorem in this paper. We note that Quillen and Venkov have given a short proof of this part of the theorem for finite G [6]. One may wonder to what extent the full theorem of Quillen can be used to prove results like Corollary 1.7 concerning the existence of p-torsion elements in $H^*(BG,\mathbf{Z})$ even though Quillen's theorem provides no information about the Bocksteins. The answer is somewhat technical. If a maximal elementary abelian p-subgroup has p-rank larger than the rank of G, Quillen's theorem can in fact be used to show the existence of p-torsion elements which are detected on this subgroup. This takes a little work however. If the p-rank of the maximal elementary abelian p-subgroup L is not greater than the rank of G, but L is not contained in a maximal

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torus, Quillen's theorem does not seem to provide enough information to show the existence of p-torsion. Part a) of Theorem 1.3 does follow indirectly from the full Quillen theorem but this makes no comment about p-torsion. Hence we shall only use the detection part of Quillen's theorem. The existence of elements will be deduced by transfer means (although Quillen's detection result will be used to determine facts about them). In essence Quillen's detection result provides an upper bound for the size of the p-torsion element subring of $H^*(BG, \mathbf{Z})$ and the transfer techniques we use realize that upper bound.

The second section of this paper is more technical and discusses among other things exactly which elements of $H^*(BG)$ can be produced by the transfer techniques of this paper.

We shall say an element x is *p-torsion of infinite height* if $p^i x = 0$ for some $i > 0$ and $x^n \neq 0$ for all n. The theorem given below is technical but is directly related to our discussion of p-torsion elements.

DEFINITION 1.1. Let R be a commutative ring and p be any prime. By the reduction of $R \bmod p$ we mean $R \otimes \mathbf{Z}/p\mathbf{Z}$. We say a subset S of R is *semi-algebraically independent* $\bmod p$ if the reduction of elements of S are not nilpotent and any relation among the reductions of elements in S is a sum of terms which are nilpotent. The *semitranscendental degree* $\bmod p$ of R (std) is the maximum cardinality of a subset of semialgebraically independent elements $\bmod p$.

A *reduced polynomial ring* is the subring of a polynomial ring consisting of the elements which have zero constant term.

EXAMPLE 1.2. Let R be a direct sum of two reduced polynomial rings over $\mathbf{Z}$ with r and t independent variables respectively. The std of R is $r+t$.

THEOREM 1.3. *Let G be a compact Lie group. Let $\mathfrak{L}$ be the set of conjugacy classes of maximum elementary abelian p subgroups of G.*

a) *Then the* std mod p *of $H^*(BG)$ is k, the sum of the p ranks of the conjugacy class representatives of $\mathfrak{L}$.*

Furthermore,

b) *If every element of $\mathfrak{L}$ consists of a conjugacy class of subgroups which are not contained in maximal tori, then there exist k semialgebraically independent elements* mod p *in $H^*(BG)$ which are p-torsion of infinite height.*

c) *If a conjugacy class of* p-rank t *consists of maximal elementary abelian p subgroups of maximal tori in G, then a set of k semialgebraically independent elements of $H^*(BG)$ may be chosen so that exactly $k-t$ of them are p-torsion of infinite height. The largest number of semialgebraically independent elements* mod p *in $H^*(BG)$ which are p-torsion is $k-t$.*

Proof. First we note that the std mod p of $H^*(BG)$ is $\leqslant k$. Quillen's detection result implies that $H^*(BG, \mathbf{Z}/p\mathbf{Z})$ mod nilpotents is isomorphic to a subring of $\bigoplus H^*(BL, \mathbf{Z}/p\mathbf{Z})$, where the direct sum is over $\mathfrak{L}$, L being a representative subgroup. The std mod p of $\bigoplus H^*(BL, \mathbf{Z}/p\mathbf{Z})$ is k. Hence k is an upper bound for the std mod p of $H^*(BG)$.

We use transfer techniques to produce the desired elements. Recall the situation in [2]. G is an arbitrary compact Lie group. L is a maximal elementary abelian p subgroup of G of p rank l. C is the centralizer of L in G. N is the normalizer of L in G.

DEFINITION 1.4. Let $W_L = N/C$ be a mod p *Weyl group of G.*

Let $Tr = \sum Cg : H^*(BL) \to H^*(BL)$ be the *trace homomorphism* where Cg is the conjugation homomorphism associated to an element $g \in W_L$ and where the sum is over W_L. Let $s > 0$. Let Tr^s be Tr restricted to p^s powers of elements in $H^*(BL)$.

Using a lemma concerning the image of $H^*(BC)$ in $H^*(BL)$ and the double coset theorem for the transfer [3] we showed in [2] that

$$\text{(1.5)} \qquad \operatorname{im}[\rho^*(L,G) \circ T(C,G)] \supset \operatorname{im} Tr^s$$

where $T(C,G)$ is the transfer associated to the fibre bundle $\rho(C,G) : BC \to BG$ and s is a nonnegative integer dependent on a unitary embedding of C.

In [2] we merely used the fact that $\operatorname{im} Tr^s \neq 0$ to produce a single p-torsion element of infinite height in $H^*(BG)$. In this section we exploit the fact that im Tr is an ideal in the invariants of $H^*(BL)$ under the action of W_L. In Section 2 we discuss which elements are in im Tr.

There are three cases that must be considered.

Case 1) L is contained in a maximal torus in G.

Case 2) L is not contained in a maximal torus in G but the intersection of L with the connected component of G is contained in a maximal torus.

Case 3) The intersection of L with the connected component of G is not contained in a maximal torus.

In Case 1 any element in $H^*(BG)$ which is detected on $H^*(BL)$ is torsion free since $H^*(BT)$ is torsion free (T being the maximal torus containing L). In Case 2 we showed in [2] by simple means that there is a p-torsion element of infinite height in $H^*(BG)$ which pulls back to an element of infinite height in $H^*(BL)$. Case 3 is the main case. We showed that given any element in $\operatorname{im} Tr^s$ which is of infinite height there is a p-torsion element of the form $T(C,G)(y)$ which hits it.

We now observe that im Tr is an ideal in $H^*(BL)^{W_L}$. Let A be the polynomial subalbegra of $H^*(BL)$ on l generators of degree two corresponding to the factors of L. W_L acts on A as a subgroup of $\mathrm{GL}(l, \mathbf{Z}/p\mathbf{Z})$, the full general linear group. Dickson [1] has shown that A^{GL} is a polynomial algebra on l homogeneous generators $c_1, \dots, c_l$. Pick integers a_i so that all $c_i^{a_i}$ have the same degree and $p^s \mid a_i$. Let t be a nonzero element of $A \cap \operatorname{im} Tr^s$ [2, Lemma 1.6]. Then

$$tc_1^{a_1}, \dots, tc_l^{a_l} \in \operatorname{im}[\rho^*(L,G) \circ T(C,G)]$$

by 1.5. These elements are algebraically independent. For suppose $F(tc_1^{a_1}, \dots, tc_l^{a_l}) = 0$. We can assume F is homogeneous of degree r. Hence

$$F(tc_1^{a_1}, \dots, tc_l^{a_l}) = t^r F(c_1^{a_1}, \dots, c_l^{a_l}) = 0$$

contradicting the fact that the c_i are algebraically independent.

We thus have l albegraically independent elements of the form $T(C,G)(y)$ which map to the the $tc_i^{a_i}$. We note the following useful fact.

PROPOSITION 1.6. *If K is a maximal elementary abelian p subgroup of G which is not conjugate to L, then $\rho^*(K,G)\circ T(C,G)(y)=0$ for positive dimensional y.*

We apply the double coset formula for the transfer [3] and note that all terms in the sum involve $T(K\cap C^g,K)$ for some $g\in G$. Moreover $K\cap C^g=K\cap L^g$ since L is maximal and hence contains all the elements of order p in its centralizer. Let $K\cap L^g=B$. Then $T(B,K)$ equals 0 on all positive dimensional elements ($\rho^*(B,K)$ is surjective and $T(B,K)\circ\rho^*(B,K)$ is multiplication by the index of B in K and hence equals 0 on all positive dimensional elements since B is a proper subgroup). Hence all the terms in the double coset formula are 0.

We now choose a subset S of $H^*(BG)$ which has k elements. For Cases 1 and 3 pick l elements of the form $T(C,G)(y)$ which pull back to the $tc_i^{a_i}$. The elements for Case 1 are torsion free since L is contained in a maximal torus. For Case 3 the elements can be chosen to be p-torsion of infinite height as in [2, p. 232].

For Case 2 we must alter the procedure slightly. Let $x\in H^*(BG)$ be a p-torsion element of infinite height which is detected on L [2, p. 232]. We can assume $\rho^*(L,G)(x)=w\in A^{p^s}$ where A is the polynomial subalgebra of $H^*(BL)$. Then $\{wtc_i^{a_i}\}$ is algebraically independent. There exist l elements of the form $T(C,G)(y)$ which pull back to the $tc_i^{a_i}$. Pick l elements of the form $xT(C,G)(y)$ which pull back to the $wtc_i^{a_i}$. We note that if K is a maximal elementary abelian p subgroup of G which is not conjugate to L, then $\rho^*(K,G)(xT(C,G)(y))=0$ as in 1.6.

We claim that S is semialgebraically independent mod p. Suppose F is a polynomial in reductions of elements of S. The image of F in each summand $H^*(BL)$ consists of those terms which are products of the elements associated to L (since the other elements map to 0). Since there are no relations among the elements associated to L each term of F must map to 0. By Quillen's theorem this implies each term is nilpotent.

Finally suppose L has p rank t and is contained in a maximal torus. Then no p-torsion element is detected on L. Hence by Quillen's theorem the largest number of semialgebraically independent elements mod p which are p-torsion is $k-t$. This completes the proof. □

Example 1.2 indicates that what the std mod p is really measuring is the maximum number of independent variables of a direct sum of reduced polynomial rings that can be embedded in $R\otimes\mathbf{Z}/p\mathbf{Z}$. The following shows what the shape of the polynomial subrings is in $H^*(BG)$. Corollary 1.8 shows that this result is the best possible in a suitable sense.

COROLLARY 1.7. *$H^*(BG)$ contains a subring isomorphic to $\bigoplus A_L$, where the sum is over $\mathfrak{L}$. L is a representative subgroup of a class in $\mathfrak{L}$ and has p-rank l. A_L is a reduced polynomial ring on l variables. If L is not contained in a maximal torus the coefficient ring of A_L is $\mathbf{Z}/p^i\mathbf{Z}$ for some $i>0$. If L is contained in a maximal torus the coefficient ring is $\mathbf{Z}$. (Note that the degrees of the generators of the A_L might be quite high.)*

Proof. We take a subring of the subring generated by the elements constructed in the proof of 1.3. First take high enough pth powers of the elements constructed so that the product of any two elements corresponding to different classes in $\mathfrak{L}$ is 0.

(The product of any two such elements is killed by some power of p and is not of infinite height since it is not detected on any class in $\mathcal{L}$). Second it is possible that if L is not contained in a maximal torus then some elements associated to L may be p-torsion of different pth powers. However, one may choose new elements from the subring generated by the elements previously associated to L so that the subring they generate is a reduced polynomial ring on l generators. This completes the proof. □

Any subring of $H^*(BG)$ which is isomorphic to a direct sum of reduced polynomial rings over coefficient rings of the form $\mathbf{Z}$ and $\mathbf{Z}/p^i\mathbf{Z}$ $i > 0$ must be of the following shape. First there can be at most one torsion free summand and its dimension must be less than or equal to the dimension of the maximal torus of G. This follows since the torsion free quotient ring of $H^*(BG)$ embeds in $H^*(BT)$. Second no two summands are detected on the same class in $\mathcal{L}$ since otherwise their product is not 0. Third if a summand consists of p-torsion elements the dimension of the summand is less than or equal to the maximal p-rank of those classes in $\mathcal{L}$ which detect an element of the summand. Consider the image of the generators of a summand in each conjugacy class. If these images are not algebraically independent then a nontrivial relation exists between them. This relation corresponds to a p-torsion element of infinite height in $H^*(BG)$ which goes to zero on the given conjugacy class. If such a relation held for all detecting conjugacy classes, the product of the corresponding elements would be a p-torsion element of infinite height which is not detected on any conjugacy class. This is impossible. Hence the maximum number of independent variables in a summand is at most equal to the maximal p-rank of a conjugacy class which detects it.

We thus have the following corollary.

COROLLARY 1.8. *Suppose $H^*(BG)$ contains a subring isomorphic to a direct sum of reduced polynomial rings with coefficient rings equal to either $\mathbf{Z}$ or $\mathbf{Z}/p^i\mathbf{Z}$ for $i > 0$. Then the number of p-torsion summands is at most equal to the number of conjugacy classes of maximal elementary abelian p-subgroups of G not contained in a maximal torus and the number of variables in each summand is bounded above by the maximal p-rank of the conjugacy classes that detect it, such conjugacy classes being distinct for each summand.*

If a torsion free summand exists the number of variables in it is bounded above by the dimension of a maximal torus of G. If this summand is detected on any class in $\mathcal{L}$, then the torsion free summand corresponds to the conjugacy classes that detect it as for the p-torsion summands. That is, the conjugacy classes that detect it do not detect any other summand. If however no conjugacy class detects it (which will happen if all generators are divisible by p in $H^*(BG)$) and no conjugacy class is contained in maximal tori, then it is possible to have a torsion free summand which does not correspond to any class in $\mathcal{L}$. (For example require that all generators be divisible in $H^*(BG)$ by high powers of p.) This is in a sense an extra summand.

Excepting this possibility, in which the elements in the torsion free summand are not detected on any elementary abelian p-subgroup of G, we see that the direct sum of rings generated in Corollary 1.7 is the largest possible in the sense that it achieves the maximum number of summands with the maximum number of independent variables in each summand.

2. In this section we discuss exactly which elements of $H^*(BG, \mathbf{Z})$ can be produced by the transfer techniques of this paper. We also show that if L is a maximal elementary abelian p-subgroup of a connected compact Lie group which is not contained in a maximal torus, then the normalizer of L mod the centralizer has order divisible by p. We begin with an example.

Formula 1.5 implies the existence of many elements in $H^*(BG, \mathbf{Z})$. Since $\operatorname{im} Tr$ depends only on L and W_L these elements exist in general situations. One must consider the three cases mentioned above to decide whether these elements are p-torsion or not. If L is contained in a maximal torus (Case 1) then the elements are torsion free. If the intersection of L with the connected component of G is not contained in a maximal torus (Case 3) then the elements are p-torsion. In the remaining case, where L is not contained in a maximal torus but the intersection of L with the connected component of G is contained in a maximal torus, some elements may be p-torsion whereas others are not. However, if one multiplies these elements by a p-torsion element of infinite height which is detected on L, then the products are all p-torsion of infinite height.

EXAMPLE 2.1. $U(n)$ and $O(n)$ have isomorphic maximal elementary abelian 2 subgroups L_U and L_O (namely $(\mathbf{Z}/2\mathbf{Z})^n$ embedded in the diagonal of the standard matrix representations of $U(n)$ and $O(n)$). W_L in both cases is Σ_n, the symmetric group on n letters. L_U is contained in a maximal torus in $U(n)$. Hence the elements created in this case are torsion free. L_O is not contained in a maximal torus in $O(n)$. If $n > 2$, L_O intersected with the connected component of $O(n)$ is not contained in a maximal torus. Hence all the elements created are 2-torsion of infinite height.

In [2] we used the fact that $\operatorname{im} Tr \neq 0$ to construct a single p-torsion element of infinite height in $H^*(BG)$. In Section 1 of this paper we have used the fact that im Tr is a nonzero ideal in $H^*(BL)^{W_L}$. One may ask how much additional information can be obtained by analyzing im Tr in more detail. That is, exactly which elements of $H^*(BG)$ can be produced by the transfer techniques of this paper.

We note the following property of im Tr.

PROPOSITION 2.2. *Let G be any compact Lie group which has L as a maximal elementary abelian p subgroup with* mod p *Weyl group W_L. Let K be any maximal elementary abelian p subgroup of G which is not conjugate to L. Then $\rho^*(K \cap L, L)$ annihilates positive dimensional elements in* $\operatorname{im} Tr \cap A$, *where A is the polynomial subalgebra of $H^*(BL)$ generated by the two dimensional elements.*

Proof. From 1.5 and 1.6 it follows that for $x \in \operatorname{im} Tr \cap A$, $\rho^*(K \cap L, L)(x^{p^s}) = 0$ for some s. Since $\operatorname{im} \rho^*(K \cap L, L)$ is contained in a polynomial subalgebra of $H^*(B(K \cap L))$, it follows that $\rho^*(K \cap L, L)(x) = 0$. □

The conclusion of this proposition along with the fact that im Tr is contained in the invariants essentially characterizes im Tr.

THEOREM 2.3. *If $x \in H^*(BL)^{W_L}$ and $\rho^*(K \cap L, L)(x) = 0$ for all G and K satisfying the hypotheses of* 2.2, *then some power of x is in* im Tr.

Note that Theorem 2.3 along with 1.5 tells one which sorts of elements can be created by the transfer techniques of this paper.

This theorem follows from the following result.

THEOREM 2.4. *Let G be a finite group acting on $\mathbf{F}_p[x_1, \ldots, x_n] = A$ via an embedding in* $\mathrm{GL}(n, \mathbf{F}_p)$, *which acts on the one dimensional elements of A. Let $Tr = \sum_{g \in G} g : A \to A$.*

a) *If $x \in \operatorname{im} Tr$ and g is any element of order p in G then $x \in K_g$, the ideal in A generated by $\{(g-1)(x_i) \mid i = 1, \ldots, n\}$.*

b) *If $x \in A^G$ and $x \in \bigcap K_g$, then $x^s \in \operatorname{im} Tr$ for some $s > 0$.*

REMARK. No good reference for this result exists. However, it was pointed out to us that this theorem follows more or less from the theory of covering spaces in algebraic geometry. An elementary proof also exists which in addition puts a bound on the exponent s. However, this is less conceptual than the proof using algebraic geometry and less enlightening. We shall take advantage of this opportunity to outline the conceptual proof, following closely a sketch pointed out to us by a referee.

Let $K_G = \bigcap K_g$. Restated, Theorem 2.4 states that $\operatorname{im} Tr \subset K_G \cap A^G \subset \operatorname{rad}(\operatorname{im} Tr)$, where $\operatorname{im} Tr$ is viewed as an ideal in A^G and rad is the radical. The inclusion of $\operatorname{im} Tr$ in $K_G \cap A^G$ is easy to establish. If $x \in \operatorname{im} Tr_G$, then $x \in \operatorname{im} Tr_H$ for any subgroup H. In particular if g is an element of order p in G, then $x \in \operatorname{im} Tr_G$ implies that there exists a y such that $x = (1 + g + \cdots + g^{p-1})y = (g-1)^{p-1}y$. The last equality holds since we are in characteristic p. Since $(g-1)(uv) = [(g-1)u]v + [(g-1)v]g(u)$ it follows that $x \in K_g$. This establishes part a) of the theorem.

The proof of part b) is more complicated. It is straightforward to reduce to the case where G is a p-group. We shall work in Spec. Let $V(I)$ denote the variety of the ideal I. Since $V(\operatorname{rad}(I)) = V(I)$, we seek to establish the equality of $V(\operatorname{im} Tr)$ and $V(K_G \cap A^G)$. We now appeal to the theory of covering spaces. The inclusion $A^G \to A$ corresponds to a covering map $\operatorname{Spec} A \to \operatorname{Spec} A^G$ which is generically $|G|$-sheeted but which includes, in general, a ramification locus. We claim that $V(K_G)$ is precisely the ramification locus. If $g \in G$ has order p, then $V(K_g)$ is the locus where the subgroup generated by g acts trivially. Moreover, $V(K_G) = V(\bigcap K_g) = \bigcup V(K_g)$. Hence $V(K_G)$ is the union of all points fixed by any p-cyclic subgroup of G. If G is a p-group, then any point fixed by an element of G is also fixed by a p-cyclic subgroup. Hence $V(K_G)$ is the total ramification locus. The image of the ramification locus in $\operatorname{Spec} A^G$ thus equals $V(K_G \cap A^G)$. The theorem follows by noting that $V(\operatorname{im} Tr)$ is contained in the image of the ramification locus. If a covering is unramified the image of the trace is nondegenerate. Hence the locus where the trace vanishes is supported on the image of the ramification locus.

Proof of 2.3. Let A be identified with the polynomial subalgebra of $H^*(BL)$, on which W_L acts by conjugation. Consider the semidirect product of L by W_L where the action of W_L on L is induced from that of N. Call this group D. Let Lg be a maximal elementary abelian p-subgroup of D which contains g and all elements of L which commute with g. Then Lg is not conjugate to L in D since L is normal. Hence by 2.2, $\rho^*(Lg \cap L, L)$ is 0 on $\operatorname{im} Tr \cap A$. Note that $Lg \cap L$ is exactly those elements of L which commute with g.

It is not hard to see that the kernel of $\rho^*(Lg \cap L, L)$ restricted to A is exactly K_g. Since the minimal polynomial of g divides $X^p - 1 = (X-1)^p$ there is a basis for L so that g is composed of Jordan blocks with 1's down the diagonal. $Lg \cap L$ is generated by the last elements in the Jordan blocks. The dual of each element of the basis represents a generator of A. The action of g on A is given by the transpose of the matrix. The image of $g-1$ on the generators of A consists of the duals of the generators of L not fixed by g. This is the same as the generators which are in $\ker \rho^*(Lg \cap L, L)$. It follows that if $x \in A^{W_L} \cap \ker \rho^*(Lg \cap L, L)$ for all g of order p in W_L, then $x^s \in \operatorname{im} Tr$. □

Finally an interesting and important question is what mod p Weyl groups, W_L, exist for the simply connected simple compact Lie groups. The answer is of course known for certain groups such as $O(n)$ and $U(n)$. We give one preliminary general result.

COROLLARY 2.5. *Suppose L is a maximal elementary abelian p subgroup of a connected compact Lie group G, which is not contained in a maximal torus. Then $p \mid |W_L|$.*

Proof. Suppose $p \nmid |W_L|$. Then im Tr on positive dimensional elements equals the invariants $\tilde{H}^*(BL)^{W_L}$. (In fact if x is invariant and has positive dimension then $Tr(x) = qx$ where q is prime to p). Hence given $x \in \tilde{H}^*(BL)^{W_L}$ there exists a p-torsion element of the form $T(C, G)(y)$ which hits x^r for some $r \geqslant 1$.

Let $S = \mathbf{Z}/p\mathbf{Z}$ be a subgroup of L. Then S is contained in a maximal torus by the maximal torus theorem. Since $\rho^*(S, G)$ is nontrivial by a result of Swan [7] and $\operatorname{im} \rho^*(L, G)$ is contained in the invariants, there is an element x in $\tilde{H}^*(BL)^{W_L}$ which is detected on S. No power of such an element can be the restriction of a p-torsion element since S is contained in a maximal torus. This contradiction implies that $p \mid |W_L|$. □

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