AN EXAMPLE OF A PLURISUBHARMONIC MEASURE ON THE UNIT BALL IN $\mathbf{C}^2$

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1. INTRODUCTION

Let $D \subset \mathbf{C}^n$ be an open set, and let $P(D)$ denote the plurisubharmonic functions on D. Given a closed set $E \subset \partial D$, we define the function

$$(1) \qquad u_E(z) = \sup\{v(z) : v \in P(D), v \leq 1, \overline{\lim_{\zeta \to z_0}}\, v(\zeta) \leq 0 \text{ for } z_0 \in \partial D \setminus E\}$$

(see [5]). If $n = 1$, then u_E is the harmonic measure of E. For $n > 1$, this is an instance of the generalized Dirichlet problem of Bremermann with upper semicontinuous boundary values. If D is strongly pseudoconvex, then it follows from [4] and [6] that $u_E \in P(D)$, that $\lim_{\zeta \to z_0} u_E(\zeta) = 0$ if $z_0 \in \partial D \setminus E$, and $\lim_{\zeta \to z_1} u_E(\zeta) = 1$ if $z_1 \in \text{int}\, E$. It was also shown in [4] that if u_E is C^2, then

$$(2) \qquad \det\left(\frac{\partial^2 u_E}{\partial z_i \partial \bar{z}_j}\right) = 0.$$

The structure of solutions of (2) is discussed in [2]. There it is shown that if $u \in C^2(D)$, $D \subset \mathbf{C}^2$, $\partial\bar{\partial} u \neq 0$ satisfies (2), then there exists a foliation $\mathscr{M}$ of D by Riemann surfaces (1-dimensional complex submanifolds $M \subset D$) such that for each $M \in \mathscr{M}$, $u|_M$ is harmonic on M. Conversely, it is possible to try to construct u by first finding a foliation $\mathscr{M}$ of D and then prescribing that u be harmonic on each $M \in \mathscr{M}$. Any u which is harmonic along the leaves of a complex foliation will satisfy

$$(3) \qquad \det\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) \leq 0.$$

To construct the solution of (1) we will find a foliation $\mathscr{M}$ and make u be harmonic on the leaves such that $u = 1$ on $E \cap \partial M$ and $u = 0$ on $\text{int}\,(\partial D \setminus E) \cap \partial M$. The only trick is to choose $\mathscr{M}$ so that the resulting function u will be plurisubharmonic, and will thus satisfy (2).

In a related problem (see [1]), the leaves of the corresponding foliation were found by solving a free boundary problem. Here we construct u_E for special D

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and E, where there are enough symmetries that the construction may be reduced to a problem in two real variables. Our methods are elementary, but the examples given here seem to be the only ones that are known explicitly.

2. CONSTRUCTION FOR A SPHERICAL CAP

We let $D = \{(z,w) \in \mathbf{C}^2 : |z|^2 + |w|^2 < 1\}$ and $E = \{(z,w) \in \partial D : \mathrm{Re}\, z \geq 0\}$. It is easily seen that D and E are invariant under the holomorphic mappings

$$A_{s,t}(z,w) = \left(\frac{z - is}{1 + isz}, \frac{\sqrt{1 - s^2}\, e^{it} w}{1 + isz} \right)$$

for $-1 < s < 1$, $0 \leq t < 2\pi$. It follows that u_E is invariant under the $A_{s,t}$, i.e., $u_E(A_{s,t}(z,w))$ is constant in s and t. We will use coordinates $x = \mathrm{Re}\, z$, $\rho = \log w\bar{w}$, and we set

$$\Gamma = \{(x,\rho) \in \mathbf{R}^2 : (x, \exp(\rho/2)) \in D\}.$$

We will give the function $h = u_E|_\Gamma$, and then u_E will be defined on all of D by the invariance under $A_{s,t}$. For $(x,\rho) \in \Gamma$,

$$A_{s,0}(x, \exp(\rho/2)) = (x(s) + iy(s), \exp(\rho(s)/2))$$

where

$$x(s) = \frac{x(1 - s^2)}{1 + s^2x^2}, \; y(s) = \frac{-s(1 + x^2)}{1 + s^2x^2}, \; \rho(s) = \rho + \log\left(\frac{1 - s^2}{1 + s^2x^2}\right).$$

If u is invariant under $A_{s,t}$, then $\dfrac{du}{ds} = u_y = 0$ at $(x,\rho) \in \Gamma$. Taking second derivatives with respect to s, we obtain

$$\frac{d^2u}{ds^2} = 0 = u_x x_{ss} + u_{yy}(y_s)^2 + u_\rho \rho_{ss}$$

which allows us to solve for u_{yy}. Thus for $(x,\rho) \in \Gamma$,

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4}\left(h_{xx} + \frac{2x}{1 + x^2} h_x + \frac{2}{1 + x^2} h_\rho \right)$$

$$\frac{\partial^2 u}{\partial z \partial \bar{w}} = \frac{h_{x\rho}}{2\bar{w}}, \qquad \frac{\partial^2 u}{\partial w \partial \bar{w}} = \frac{h_{\rho\rho}}{w\bar{w}}. \tag{4}$$

If we set

$$M(h) = \begin{pmatrix} h_{xx} + \dfrac{2}{1+x^2}(xh_x + h_\rho) & 2\,h_{x\rho} \\ 2\,h_{x\rho} & 4\,h_{\rho\rho} \end{pmatrix}, \tag{5}$$

it follows that u is plurisubharmonic if and only if $M(h) \geq 0$, and (2) holds if and only if $\det M(h) = 0$.

Now to construct the foliation we consider the family of curves

$$\gamma(\sigma) = \{(x,\rho) \in \Gamma : \rho + f(x,\sigma) = 0\}$$

where $f(x,\sigma) = \sigma \arctan x - \log(1+x^2)$. Let $(x(\sigma), \rho(\sigma))$ denote the nonzero endpoint of $\gamma(\sigma)$, and let $L(\sigma)$ denote the hypersurface in D swept out by $\gamma(\sigma)$ under the action of $A_{s,t}$. Let $F_\sigma(z,w)$ be the function in D which is invariant under $A_{s,t}$ and such that $F_\sigma(z,w)|_\Gamma = \rho + f(x,\sigma)$. Since $f(x,\sigma)$ satisfies

$$f_{xx} + \frac{2x}{1+x^2} f_x + \frac{2}{1+x^2} = 0$$

it follows from (4) that the complex hessian of F_σ vanishes at Γ. Thus F_σ is pluriharmonic on D, and since $L(\sigma) = \{(z,w) \in D : F_\sigma(z,w) = 0\}$ we see that $L(\sigma)$ is Levi flat, so it is foliated by complex manifolds. If $G(z,w)$ is the invariant function on D such that $G|_\Gamma = \arctan x$, then by (4) G is also pluriharmonic on D, and so $G|_M$ is harmonic on M for any complex manifold $M \subset L(\sigma)$. Now we define

$$h(x,\rho) = \begin{cases} \dfrac{\arctan x}{\arctan x(\sigma)} & \text{for } (x,\rho) \in \gamma(\sigma) \\ 0 & \text{for } (x,\rho) \in \Gamma,\ x \leq 0. \end{cases} \tag{6}$$

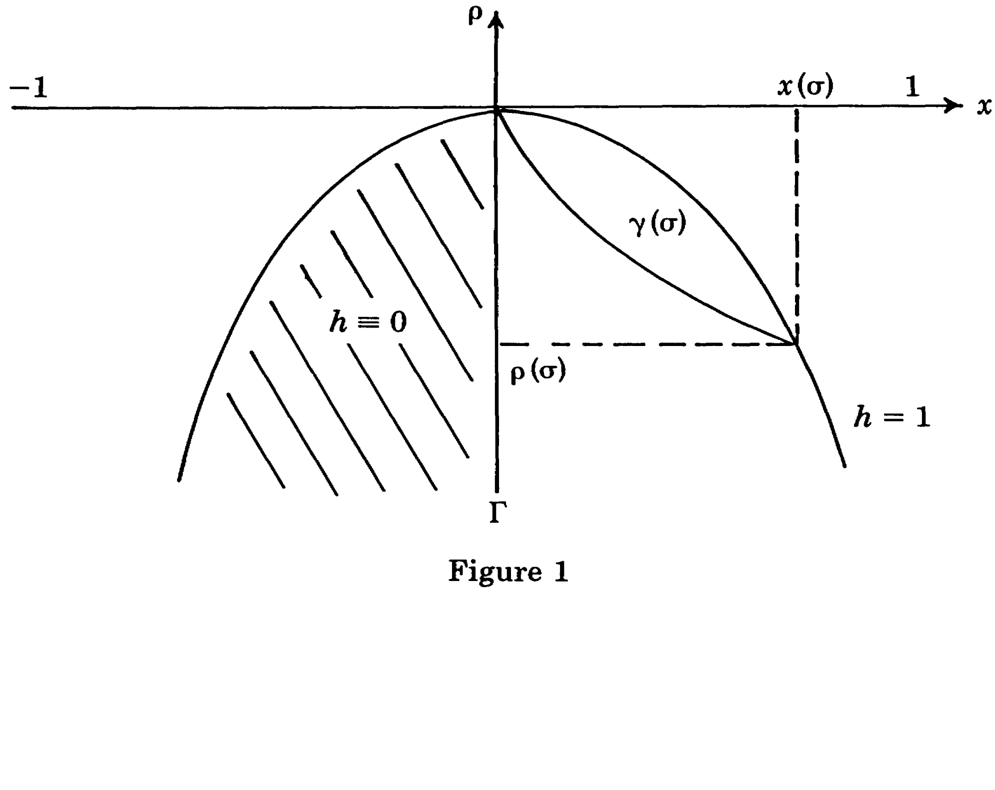

Figure 1

It follows from (6) *that* u_E *is real analytic on* $\bar{D}\setminus\{\mathrm{Re}\, z = 0\} \cup \{w = 0, \mathrm{Re}\, z \geq 0\}$ *and continuous on D.*

For the remainder of this section we will denote by u_E the function obtained from $h(x,\rho)$ in (6) by making it constant on the orbits of $A_{s,t}$, and we will show that u_E is in fact the extremal function defined in (1). First we show that $u_E \geq v$ for all v in the competing family in (1), and then we show that u_E is actually plurisubharmonic. (Clearly u_E has the correct boundary values.)

It is easily shown that the curve $s \to \dfrac{x - is}{1 + isx}$, $-1 \leq s \leq 1$ is a circular arc starting at i, passing through x, and ending at $-i$. This arc is analogous to the curves found in [1]. We will denote by $R(\sigma)$ the region in the z-plane bounded by the interval $(-i,i)$ and the arc above with $x = x(\sigma)$. There is a function $\psi(z,\sigma)$ such that $L(\sigma) = \{(z,w) : z \in R(\sigma), \log w\bar{w} + \psi(z,\sigma) = 0\}$. Since $L(\sigma)$ is Levi-flat, $\psi_{z\bar{z}} = 0$ on $R(\sigma)$. In fact $\log w\bar{w} + \psi(z,\sigma) = F_\sigma(z,w)$ although neither $\log w\bar{w}$ nor $\psi(z,\sigma)$ is invariant under $A_{s,t}$.

Let us choose a harmonic conjugate $\tilde{\psi}(z,\sigma)$ for $\psi(z,\sigma)$ on $R(\sigma)$. It follows that $L(\sigma) = \bigcup_{0\leq\theta<2\pi} M(\sigma,\theta)$, where $M(\sigma,\theta) = \{(z,w) : z \in R(\sigma), w = \exp(\psi + i\tilde{\psi} + i\theta)\}$. As was observed above, $G|_{M(\sigma,\theta)}$ is harmonic on $M(\sigma,\theta)$ so

$$\tilde{u}(z,\sigma,\theta) = u_E(z,\exp(\psi + i\tilde{\psi} + i\theta))$$

is harmonic on $R(\sigma)$. If v is any function in the family of (1), it follows that $v(z,\exp(\psi + i\tilde{\psi} + i\theta))$ is subharmonic on $R(\sigma)$ and less than or equal to $\tilde{u}(z,\sigma,\theta)$ at all boundary points of $R(\sigma)$ except perhaps $\{\pm i\}$. It follows that $v \leq u_E$ on $M(\sigma,\theta)$ for all σ,θ, and thus $v \leq u_E$ on D.

Finally we show that u_E is plurisubharmonic. To do this we fix σ and show that $M(h) \geq 0$ holds on $\gamma(\sigma)$. For each real λ, we set

$$P(x,\rho) = \frac{\arctan x}{\arctan x(\sigma)} + \lambda\,(f(x,\sigma) + \rho).$$

We may choose $\lambda > 0$ such that at $(x(\sigma),\rho(\sigma))$, ∇P is a multiple of ∇h, i.e., ∇P and ∇h are parallel at that point. (In fact we have $\nabla P = \nabla h$ there since $P = h$ on $\gamma(\sigma)$.) Now if we show that $h|_{\gamma(\sigma')} \geq P|_{\gamma(\sigma')}$ for σ' near σ, then it follows that $M(h)|_{\gamma(\sigma)} \geq M(P)|_{\gamma(\sigma)} = 0$. Comparing the second derivatives of P and $\rho - \log(1 - x^2)$ (the defining function for $\partial\Gamma$), we see that $\{P(x,\rho) = 1\}$ is less curved than Γ at $(x(\sigma),\rho(\sigma))$. It follows that $P(x,\rho) \leq 1$ on $\partial\Gamma$ near $(x(\sigma),\rho(\sigma))$ and thus $P \leq h$ on $\partial\gamma(\sigma')$ for σ' near σ. For $(x,\rho(x)) \in \gamma(\sigma')$, P and h both satisfy

$$g_{xx} + \frac{2x}{1 + x^2}\, g_x = 0,$$

a first order differential equation for g_x. If $P \leq h$ on $\partial\gamma(\sigma')$, then $P \leq h$ on $\gamma(\sigma')$ for otherwise there is an interior point where $g_x = (P - h)_x = 0$, and by unique continuation of solutions, $g_x \equiv 0$. This completes the proof.

Let us remark that by the proof above, there is an alternate formulation of u_E. First we note that $\psi(z,\sigma)$ is the harmonic function on $R(\sigma)$ such that

$$\psi(z) = \log(1 - z\bar{z})$$

for $z \in \partial R(\sigma)$. If we let $\omega(z,\sigma)$ be the bounded harmonic function (harmonic measure) on $R(\sigma)$ such that $\omega(z,\sigma) = 1$ for $z \in \partial R(\sigma) \cap \{\operatorname{Re} z > 1\}$ and $\omega(z,\sigma) = 0$ for $-i \le z \le i$, then

$$u_E(z,w) = \begin{cases} \omega(z,\sigma) & \text{if } (z, \log w\bar{w}) \in L(\sigma) \\ 0 & \text{if } \operatorname{Re} z \le 0. \end{cases}$$

3. REINHARDT DOMAINS

The only other instance in which the solution of (1) is known explicitly is where D and E are Reinhardt, i.e., invariant under $z \to (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$. As was shown in [3], we may introduce the variables

$$\xi = (\xi_1, \ldots, \xi_n) = (\log |z_1|, \ldots, \log |z_n|)$$

and reduce (1) to the corresponding problem for convex functions in ξ-space. We give a sketch of the construction in the case $n = 2$ (one can fill in the details using arguments from [3]).

If $D \subset \mathbf{C}^2$ is a strongly pseudoconvex Reinhardt domain with real analytic boundary, then in ξ-coordinates, ∂D becomes a strictly convex curve $\Gamma \subset \mathbf{R}^2$. Let us assume that in ξ-coordinates E is a finite union of closed sub-arcs $\gamma_1, \ldots, \gamma_p$ of Γ. We let D_0 be the convex hull of $\Gamma \setminus \left(\bigcup_{j=1}^{p} \gamma_j \right)$, and we set $u(\xi) = 0$ for $\xi \in \bar{D}_0$. Since Γ is strictly convex, $\bar{D}_0 \cap \Gamma = \Gamma \setminus \left(\bigcup_{j=1}^{p} \operatorname{int} \gamma_j \right)$. Thus $D \setminus \bar{D}_0$ consists of disjoint connected regions $D_1, \ldots, D_p$ with $\bar{D}_j \cap \Gamma = \gamma_j$.

For $P \in \gamma$, we let $\nu(P)$ be the outward normal to γ at P. The arc γ_j has endpoints A_j and B_j, and we let C_j be the point on γ_j such that $\nu(C_j)$ is orthogonal to $B_j - A_j$. We define u to be linear on the triangle $A_j B_j C_j$ such that

$$u(A_j) = u(B_j) = 0, \; u(C_j) = 1.$$

Finally, $\gamma_j \setminus \{C_j\}$ consists of two components. For $P \neq A_j$ in the component containing A_j we make u linear on the segment $A_j P$ with $u(A_j) = 0$, $u(P) = 1$. For P in the other component, we make u linear on the segment $B_j P$.

After defining $u(\xi)$ in this fashion, we see that u is real analytic on

$$\bar{D} \setminus \bigcup_{j=1}^{p} \partial(A_j B_j C_j).$$

Further, $u \in \mathrm{Lip}^1(D)$ and $u \in C^{1,1}(\bar{D} \setminus \partial D_0)$.

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