ON A QUESTION OF OLSEN CONCERNING COMPACT PERTURBATIONS OF OPERATORS

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1. INTRODUCTION

Let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on a complex separable infinite-dimensional Hilbert space $\mathscr{H}$, and $\mathscr{C}(\mathscr{H})$ the algebra of all compact linear operators on $\mathscr{H}$. The *essential norm* of T in $\mathscr{B}(\mathscr{H})$ is defined to be $\|T\|_e = \inf\{\|T+K\|: K \in \mathscr{C}(\mathscr{H})\}$. The main purpose of this paper is to answer a question posed by C. L. Olsen in her talk at the meeting of the American Mathematical Society at Washington, D. C. in January 1975. We show that *for each* $T \in \mathscr{B}(\mathscr{H})$ *there exists a* $K \in \mathscr{C}(\mathscr{H})$ *such that*

$$\|T+\lambda+K\| = \|T+\lambda\|_e \text{ for every complex number } \lambda.$$

A few words seem appropriate to motivate the consideration of this problem. Recently much interest has been centered about the following unsolved problems.

(a) *Given* $T \in \mathscr{B}(\mathscr{H})$, *does there exist a* $K \in \mathscr{C}(\mathscr{H})$ *such that for any complex polynomial* p, $\|p(T+K)\| = \|p(T)\|_e$? An affirmative answer to (a) would imply the following results of Olsen and West.

1. (Olsen [8]) *Let* $\nu(T)$ *be a coset in the Calkin algebra. For a Hilbert space operator* T, *if* $p(\nu(T)) = 0$ *for some polynomial* p, *then there is a* $K \in \mathscr{C}(\mathscr{H})$ *with* $p(T+K) = 0$.

2. (West [13]) *If* $\lim_n \|\nu(T)^n\|^{1/n} = 0$, *then there is a* $K \in \mathscr{C}(\mathscr{H})$ *such that* $\lim_n \|(T+K)^n\|^{1/n} = 0$.

An affirmative solution to (a) would also answer a question raised by Arveson [2]: *If* $\nu(T)$ *is quasialgebraic, must there exist a* $K \in \mathscr{C}(\mathscr{H})$ *such that*

$$\|p_n(T+K)\|^{1/d(n)} \to 0\,?$$

(An element T of a Banach algebra is *quasialgebraic* if there is a sequence $\{p_n\}$ of monic polynomials of degree d(n) such that $\lim_n \|p_n(T)\|^{1/d(n)} = 0$.)

Question (a) seems to be very interesting, and the particular results of Olsen and West are nontrivial. However, it is far from being settled. For subnormal and essentially normal operators, positive results have been obtained in [10]. Even if we allow the compact operator to depend on the given polynomial, this problem is still open; that is,

(b) *Given* $T \in \mathscr{B}(\mathscr{H})$ *and a polynomial* p, *does there exist a* $K \in \mathscr{C}(\mathscr{H})$ *with* $\|p(T+K)\| = \|p(T)\|_e$?

Results of Olsen [9] along this line are:

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1. *For any* $T \in \mathscr{B}(\mathscr{H})$, *there exists a* $K \in \mathscr{C}(\mathscr{H})$ *with* $\|T + K\| = \|T\|_e$ *and* $\|(T + K)^2\| = \|T^2\|_e$.

2. *For any* $T \in \mathscr{B}(\mathscr{H})$, *there exists a* $K \in \mathscr{C}(\mathscr{H})$ *with* $\|(T + K)^2\| = \|T^2\|_e$ *and* $\|(T + K)^3\| = \|T^3\|_e$.

It is now clear that our result answers (a) for all linear polynomials $p(t) = \alpha t + \beta$.

We would like to thank Professor C. L. Olsen for her many helpful comments concerning the background of this problem. In addition, we would like to thank Professor I. D. Berg for his many helpful comments concerning this paper. He suggested that the paper would be much more readable if a sketch of the proof using operator theory techniques were added. We have included such a sketch, using his ideas, at the end of the paper. However, since the approach to the solution of the problem was suggested by techniques from M-ideal theory, we have retained our original methods for the actual proof.

2. DEFINITIONS AND NOTATION

Throughout the paper, $\mathscr{A}$ will denote the Calkin algebra $\mathscr{B}(\mathscr{H})/\mathscr{C}(\mathscr{H})$ and $\nu: \mathscr{B}(\mathscr{H}) \to \mathscr{A}$ the quotient map. The spectra of T and $\nu(T)$ are denoted by $\sigma(T)$ and $\sigma(\nu(T))$ respectively; the latter is called the *essential spectrum* of T. By $\sigma_\infty(T)$, we mean the set of isolated eigenvalues of finite multiplicity in $\sigma(T)$; by $\sigma_w(T)$, the *Weyl spectrum* of T, we mean $\bigcap \{\sigma(T + K): K \in \mathscr{C}(\mathscr{H})\}$. For an element b of an arbitrary complex Banach algebra G with unit I, the *numerical range* of b, w(b), is given by $w(b) = \{\phi(b) \in \mathbb{C}: \|\phi\| = 1 = \phi(I),\ \phi \in G^*\}$. *The essential numerical range* of $T \in \mathscr{B}(\mathscr{H})$ is defined to be the numerical range of $\nu(T) \in \mathscr{A}$ and will be denoted by $W_e(T)$. The numerical radius of $W_e(T)$ will be denoted by $r(W_e(T))$.

The notion of M-ideal in a Banach space has been formulated and studied in an important paper of Alfsen and Effros [1]. According to these authors, a closed subspace M of a (real) Banach space X is an M-*ideal* in X if its annihilator $M^\perp$ is an L-summand of the dual space X^*. This in turn means that $M^\perp$ is the range of an L-projection defined on X^*; *i.e.*, there exists a projection $Q: X^* \to M^\perp$ with the property that $\|\phi\| = \|Q\phi\| + \|\phi - Q\phi\|$ for all $\phi \in X^*$. It is well known that the ideal $\mathscr{C}(\mathscr{H})$ is a complex M-ideal in $\mathscr{B}(\mathscr{H})$ [12]. In this case, when applying any theorems from [1], we always consider that we are working in the real restriction of X with corresponding dual space consisting of real parts of functionals in X^*. It is straightforward to verify that an M-ideal in a complex Banach space X remains an M-ideal in the real restriction of X.

3. THE MAIN THEOREM

We are now ready to prove our main result. Clearly, the following two questions are equivalent.

(i) *For* $T \in \mathscr{B}(\mathscr{H})$, *does there exist a compact operator* K *with*

$$\|(T + \lambda) + K\| = \|T + \lambda\|_e$$

for any complex number λ?

(ii) *Let* $\mathscr{P}(T+\lambda)$ *be the set of all compact operators of minimum distance from* $T+\lambda$; *i.e., the compact best approximants of* $T+\lambda$. *Is* $\bigcap_{\lambda\in\mathbb{C}}\mathscr{P}(T+\lambda)\neq\emptyset$?

Since the methods to be utilized are Banach space techniques, it seems appropriate to state our theorem in terms of (ii).

THEOREM 1. *Let* $T\in\mathscr{B}(\mathscr{H})$. *Then* $\bigcap_{\lambda\in\mathbb{C}}\mathscr{P}(T+\lambda)\neq\emptyset$.

Proof. We first treat the case in which $W_e(T)$ has no interior point. Since $W_e(T)$ is convex [3], this implies that $W_e(T)$ is a (possibly degenerate) line segment in $\mathbb{C}$. It is easily seen upon translation by a scalar multiple of the identity and a suitable rotation that we may assume T to be essentially self-adjoint. It is a well known fact that such a T may be written as $S+K_1$, where S is a self-adjoint operator and K_1 is a compact operator. Next perturb S with another compact operator K_2 by pulling isolated eigenvalues of finite multiplicity back to the nearest point in the essential spectrum while preserving the eigenvectors. It is readily checked that K_1+K_2 is a best compact approximant to $T+\lambda$ for all $\lambda\in\mathbb{C}$.

Now suppose that $W_e(T)$ contains three noncollinear points. Since $W_e(T)$ is convex, this means that $W_e(T)$ contains some ball $B(\alpha,\varepsilon)$ relative to $\mathbb{C}$. To complete the proof of Theorem 1, we need three lemmas which we state and prove below.

LEMMA 1. *Let* $\rho(T-\lambda)=d(T-\lambda,\mathscr{C}(\mathscr{H}))$. *Then* $\bigcap_{\lambda\in\mathbb{C}}B(T-\lambda,\rho(T-\lambda))$ *has nonempty interior.*

Proof. We will show that $B(T-\alpha,\varepsilon/2)\subset\bigcap_{\lambda\in\mathbb{C}}B(T-\lambda,\rho(T-\lambda))$, where $B(\alpha,\varepsilon)\subset W_e(T)$. Let $\lambda\in\mathbb{C}$. It suffices to show that there exists a $\delta>0$ independent of λ so that $|\alpha-\lambda|+\delta\leq\rho(T-\lambda)$. Now,

$$|\alpha-\lambda|+\varepsilon/2\leq\sup_{\mu\in B(\alpha,\varepsilon)}|\mu-\lambda|\leq\sup_{\mu\in W_e(T)}|\mu-\lambda|=r(W_e(T-\lambda))$$

$$\leq\|\nu(T-\lambda)\|=d(T-\lambda,\mathscr{C}(\mathscr{H}))=\rho(T-\lambda)$$

Thus our assertion holds with $\delta=\varepsilon/2$.

Alfsen and Effros [1, p. 120] discovered the following two equivalent conditions.

THEOREM A. *Suppose that* J *is a closed subspace of a Banach space* X. *Then the following statements are equivalent.*

i) J *is an* M-*ideal;*

ii) *If* $D_1,\cdots,D_n$ *are closed balls with* $\operatorname{int}(D_1\cap\cdots\cap D_n)\neq\emptyset$ *and* $D_i\cap J\neq\emptyset$ *for all* i, *then* $D_1\cap\cdots\cap D_n\cap J\neq\emptyset$.

Our aim is to modify the above result for our particular case; namely, we wish to show: *Given the* M-*ideal* $\mathscr{C}(\mathscr{H})$, *and an operator* T *satisfying*

a) $\operatorname{int}\bigcap_{\lambda\in\mathbb{C}}B(T+\lambda,\rho(T+\lambda))\neq\emptyset$ *and*

b) $B(T+\lambda,\rho(T+\lambda))\cap\mathscr{C}(\mathscr{H})\neq\emptyset$ *for all* λ,

then

$$\bigcap_{\lambda\in\mathbb{C}}B(T+\lambda,\rho(T+\lambda))\cap\mathscr{C}(\mathscr{H})\neq\emptyset\ .$$

The fact that the x-axis is an M-ideal in $\mathbb{R}^2$ endowed with the ℓ^∞ norm helps one to visualize the Alfsen-Effros result. In order to motivate what follows, we sketch their argument of how (i) implies (ii). Let $v_1, \cdots, v_n$ denote the centers of the balls $D_1, \cdots, D_n$, and $r_1, \cdots, r_n$ the corresponding radii. Where occasion demands, we make the usual identification of an element of a Banach space X with a w^*-continuous linear functional in X^{**}. In [1], a vector $v \in D_1 \cap \cdots \cap D_n \cap J$ was constructed via a w^*-continuous Hahn-Banach extension of a certain linear functional dominated by the w^*-lower-semicontinuous concave function $g(\phi) = \inf_i (v_i + r_i)\phi$, where, for each i, $v_i + r_i$ is now an affine functional on the unit ball $U(X^*)$ of X^*. Since g is the infimum of a *finite* number of w^*-continuous affine functionals, it is automatically w^*-lower-semicontinuous. This is the basic Alfsen-Effros idea.

Before returning to the proof of Theorem 1, observe that the same argument as in [1, p. 120, Theorem 5.8] goes through except for the verification that the functional

$$g(\phi) = \inf_{\lambda \in \mathbb{C}} \operatorname{Re} (\phi(T + \lambda) + \rho(T + \lambda))$$

is w^*-lower-semicontinuous on $U(\mathscr{B}(\mathscr{H})^*) \equiv \{\phi \in \mathscr{B}(\mathscr{H})^*: \|\phi\| \leq 1\}$. As a pointwise infimum of affine w^*-continuous functionals, g is automatically a concave w^*-upper-semicontinuous function. To prove that g is w^*-lower-semicontinuous, and hence continuous, we need the following.

LEMMA 2. *Let $\varepsilon > 0$ be given. Then there exists an $M < \infty$ such that for each θ, $0 \leq \theta \leq 2\pi$, there is some real number ℓ_θ so that $|\rho(T+\lambda) - |\lambda| - \ell_\theta| < \varepsilon$ whenever $\arg \lambda = \theta$ and $|\lambda| > M$.*

Proof. From [6], we know that $\rho(T + \lambda) = \sup_{\{e_i\}} \overline{\lim}_i \|(T + \lambda) e_i\|$, where $\{e_i\}$ is any normalized sequence converging weakly to zero. Hence,

$$\begin{aligned}\rho(T + \lambda) &= \sup \overline{\lim}_i \left(|\lambda|^2 + 2 \operatorname{Re} \lambda \langle e_i, Te_i\rangle + \|Te_i\|^2\right)^{1/2} \\ &= \sup \overline{\lim}_i \left[(|\lambda| + \operatorname{Re} e^{i \arg \lambda} \langle e_i, Te_i \rangle)^2 \right. \\ &\quad \left. + \|Te_i\|^2 - (\operatorname{Re} e^{i \arg \lambda} \langle e_i, Te_i \rangle)^2\right]^{1/2}.\end{aligned}$$

On the other hand, there exists an $M > 0$ such that $|\lambda| > M$ implies that

$$-\varepsilon < [(|\lambda| + \operatorname{Re} e^{i \arg \lambda} \langle e_i, Te_i \rangle)^2 + \|Te_i\|^2$$
$$-(\operatorname{Re} e^{i \arg \lambda} \langle e_i, Te_i \rangle)^2]^{1/2} - |\lambda| - \operatorname{Re} e^{i \arg \lambda} \langle e_i, Te_i \rangle < \varepsilon$$

for any sequence $\{e_i\}$ with $\|e_i\| = 1$. Thus, by taking

$$\ell_\theta = \sup_{\{e_i\}} \overline{\lim}_i \operatorname{Re} e^{i\theta} \langle e_i, Te_i \rangle,$$

where $\theta = \arg \lambda$, we complete the proof of the lemma.

LEMMA 3. *Let*

$$g(\phi) = \inf_{\lambda \in \mathbb{C}} \{\operatorname{Re} \phi(T) + \operatorname{Re} \lambda \phi(I) + \rho(T + \lambda)\},$$

$$g_M(\phi) = \inf_{|\lambda| \leq M} \{\operatorname{Re} \phi(T) + \operatorname{Re} \lambda\phi(I) + \rho(T + \lambda)\},$$

and

$$p(\lambda) = \mathrm{Re}\ \phi(T) + \mathrm{Re}\ \lambda\phi(I) + \rho(T + \lambda).$$

Let $\varepsilon > 0$ *be given. Then there is an* $M > 0$ *such that* $g_M(\phi) - g(\phi) < \varepsilon$ *for all* $\phi \in U(\mathscr{B}(\mathscr{H})^*)$.

Proof. Let M be chosen so large that

$$|\rho(T + \lambda) - |\lambda| - \ell_\theta| < \varepsilon/2 \text{ for all } \lambda \text{ with } |\lambda| \geq M.$$

Then

$$\begin{aligned} p(\lambda) &= \mathrm{Re}\ \phi(T) + \mathrm{Re}\ \lambda\phi(I) + \rho(T + \lambda) \\ &= \mathrm{Re}\ \phi(T) + |\lambda|\ (1 + \mathrm{Re}\ e^{i \arg \lambda}\ \phi(I)) + \ell_\theta + f(\theta), \end{aligned}$$

where $|f(\theta)| < \varepsilon/2$. Note that if $|\lambda_1| = M$, $|\lambda_2| > M$, and $\arg \lambda_1 = \arg \lambda_2$, then $p(\lambda_2) > p(\lambda_1) - \varepsilon$. Hence, $g_M(\phi) - g(\phi) < \varepsilon$.

We will now prove that g is w*-continuous, and hence complete the proof of Theorem 1. Let $\{M_n\}$ be an increasing sequence diverging to infinity. By Lemma 3, g_{M_n} converges uniformly to g on $U(\mathscr{B}(\mathscr{H})^*)$. Since it is easily seen that g_{M_n} is w*-continuous for each M_n, g, as a uniform limit of continuous functions, is itself a w*-continuous function. This completes the proof of the theorem.

4. AN OPERATOR THEORY INTERPRETATION

As was noted in the introduction, the purpose of this section is to sketch the proof of Theorem 1 in operator-theoretic terms. We again wish to thank Professor I. D. Berg, on whose ideas we rely heavily.

The following lemma allows us to reduce the proof of Theorem 1 to operators which are tri-block-diagonal. In the remainder of this section, P_V will denote the orthogonal projection onto V.

LEMMA 5. *Suppose* $\mathrm{int}\ W_e(T) \neq \emptyset$. *Then there exists a compact perturbation* $\hat{T}$ *of* T *such that* $\hat{T} + \lambda$ *is simultaneously tri-block-diagonal for all* λ *and* $\mathrm{int} \bigcap_\lambda B(\hat{T} + \lambda, \rho(\hat{T} + \lambda)) \neq \emptyset$.

Proof. By Lemma 1, there exist $\alpha \in \mathbb{C}$ and $\varepsilon > 0$ such that $B(T + \alpha, \varepsilon)$ is contained in $\mathrm{int} \bigcap_\lambda B(T + \lambda, \rho(T + \lambda))$. Set $\delta = \varepsilon/2$ and let $\{\delta_i\}$ be a monotone decreasing sequence of positive numbers with $\sum_{i=1}^\infty \delta_i = \delta$. Split the basis $\{\phi_n\}$ into a sequence of adjacent finite blocks increasing in length so rapidly that the spaces $H_1, H_2, \cdots$ spanned by these successive blocks of ϕ_n satisfy $\|P_L T P_{H_i}\| < \delta_i$ and $\|P_{H_i} T P_L\| < \delta_i$ whenever the space L is perpendicular to P_{H_i}, $P_{H_{i-1}}$, and $P_{H_{i+1}}$. These estimates clearly hold for all $T + \lambda$, since λ affects only the diagonal blocks of the operator matrix. Now one notes that T minus the tri-diagonal part of the operator matrix is a compact operator of norm less than or equal to δ, and is independent of λ. By defining $\hat{T}$ to be the tri-diagonal part of T, it is easily seen that

$$d(\hat{T} + \lambda, \mathscr{C}(\mathscr{H})) = d(T + \lambda, \mathscr{C}(\mathscr{H}))$$

and

$$\operatorname{int} \bigcap_{\lambda} B(\hat{T} + \lambda, \rho(\hat{T} + \lambda)) \supset B(\hat{T} + \alpha, \varepsilon/2).$$

This completes the proof of the lemma.

The next lemma will be used extensively in the remainder of the proof. In what follows, E_m will denote the subspace spanned by $\phi_1, \cdots, \phi_m$, and $S: E_m^{\perp} \to E_m^{\perp}$ will be an operator on $\mathscr{H}$ invariant on $E_m^{\perp}$ and identically equal to zero on E_m.

LEMMA 6. *Let* $\{\phi_n\}$ *be an orthonormal basis of* $\mathscr{H}$ *and* $\|T\| = 1$. *Suppose that* $\|TP_{E_m^{\perp}}\| \leq \rho < 1$ *for large enough* m. *Then for any* $\varepsilon > 0$, *there exists an* n *so large that if* $S: E_n^{\perp} \to E_n^{\perp}$ *is of norm less than or equal to* $1 - \rho$, *then* $\|T + S\| < 1 + \varepsilon$.

Proof. By Lemma 5, it may be assumed that T is tri-block-diagonal and that the ϕ_n's represent whole blocks. For each $\varepsilon > 0$, choose n so large that if ψ is any unit vector, then each of the components ϕ_ℓ and $\phi_{\ell+1}$ of ψ is less than ε for some ℓ, $m < \ell < n - 1$. One can do this because if $\sum_{i=1}^{k} \alpha_i = 1$ and the α_i's are all nonnegative, then some consecutive pair of the α_i's can not exceed 2/k. Of course, ℓ varies with ψ although m and n remain unchanged. We now have $\psi = \psi_1 + \psi_2 + \psi_3$, where ψ_1 is a vector contained in blocks whose indices are less than ℓ, ψ_2 is a vector contained in blocks whose indices are greater than $\ell + 1$, and $\|\psi_3\| < 2\varepsilon$. Also,

$$S\psi_1 = 0, \quad \|S\psi_2\| \leq (1 - \rho)\|\psi_2\|, \quad T\psi_1 \perp T\psi_2, \quad \text{and} \quad \|(T + S)\psi_1\| = \|T\psi_1\| \leq 1.$$

Since $(T + S)\psi_1 \perp (T + S)\psi_2$, it follows that

$$\|(T + S)(\psi_1 + \psi_2 + \psi_3)\| \leq 1 + 4\varepsilon.$$

This completes the proof of the lemma.

As noted earlier, if $\operatorname{int} W_e(T) \neq \emptyset$, then $S \equiv T - \alpha \in \operatorname{int} \bigcap_\lambda B(T + \lambda, \rho(T + \lambda))$ for some $\alpha \in \mathbb{C}$. A construction will now be given showing that $T - \alpha$ can be altered in such a fashion that the resulting operator $\hat{S}$ is compact and remains inside $B(T, \rho(T))$. A modification of this argument shows that $\hat{S}$ can be constructed to remain in $\bigcap_{i=1}^{n} B(T + \lambda_i, \rho(T + \lambda_i))$ for any fixed λ_i, $i = 1, \cdots, n$. Lemma 7 extends the argument so that $\hat{S}$ remains in $\bigcap_{\lambda \in Q} B(T + \lambda, \rho(T + \lambda))$ for any prescribed compact set Q.

The operator $\hat{S}$, which will be seen to be compact and to satisfy

$$\|T - \hat{S}\| = d(T, \mathscr{C}(\mathscr{H})),$$

is constructed by induction. In the first step, note that

$$\|T - S\| = |\alpha| \leq d(T, \mathscr{C}(\mathscr{H})) - \varepsilon \text{ for some } \varepsilon > 0.$$

Pick $\delta = \varepsilon/2$ and note that $\|(T - S)P_{E^\perp}\| = |\alpha|$ for any subspace E. By Lemma 6, there exists an n (in this case any n is suitable) such that for some $\gamma > 0$, the operator $\gamma P_n^\perp SP_n^\perp$ may be added to $T - S$ with the resulting operator having norm at most $|\alpha| + \delta$. Here and throughout, $P_n^\perp$ denotes the orthogonal projection onto $E_n^\perp$. Now assume that N steps in the induction have been completed. The modified operator has the form

$$\hat{S}_N \equiv T - S + \sum_{i=1}^{N} \gamma_i P_i^\perp SP_i^\perp .$$

For sufficiently large k, it is easily seen that

$$\hat{S}_N\Big|_{E_k^\perp} = \left(T - \left(1 - \sum_{i=1}^{N} \gamma_i\right)S\right)\Big|_{E_k^\perp} .$$

From the above construction, $\|\hat{S}_N\| < d(T, \mathscr{C}(\mathscr{H}))$. Thus, we may select a $\delta_{N+1} > 0$ so that

$$\|\hat{S}_N\| + \delta_{N+1} < d(T, \mathscr{C}(\mathscr{H})) .$$

For any $\beta > 0$, one may choose E_{N+1} so that

$$\begin{aligned}\left\|\hat{S}_N\Big|_{E_{N+1}^\perp}\right\| &\le \left(1 - \sum_{i=1}^{N} \gamma_i\right)\left\|(T - S)\Big|_{E_{N+1}^\perp}\right\| + \left\|\left(\sum_{i=1}^{N} \gamma_i\right)T\Big|_{E_{N+1}^\perp}\right\| \\ &\le \left(1 - \sum_{i=1}^{N} \gamma_i\right)|\alpha| + \sum_{i=1}^{N} \gamma_i\left\|T\Big|_{E_{N+1}^\perp}\right\| < d(T, \mathscr{C}(\mathscr{H})) .\end{aligned}$$

The above inequality, together with Lemma 6, assures that one can continue to add on the terms $\gamma P_{E^\perp} SP_{E^\perp}$ in such a fashion that $\sum_{i=1}^{N} \gamma_i$ increases to one, and as N gets large, $\|T - S + \sum_{i=1}^{N} \gamma_i P_n^\perp SP_n^\perp\|$ increases to $d(T, \mathscr{C}(\mathscr{H}))$. Thus, the operator $T - S + \sum_{i=1}^{\infty} \gamma_i P_i^\perp SP_i^\perp$ satisfies

$$\left\|T - S + \sum_{i=1}^{\infty} \gamma_i P_i^\perp SP_i^\perp\right\| = d(T, \mathscr{C}(\mathscr{H})) .$$

Furthermore, since $\sum_{i=1}^{\infty} \gamma_i = 1$, $S - \sum_{i=1}^{\infty} \gamma_i P_i^\perp SP_i^\perp$ is compact. This completes the construction.

LEMMA 7. *Let* $\{B_\lambda\}$ *be a collection of closed balls in* $\mathscr{B}(\mathscr{H})$ *having the form* $B(T + \lambda, \rho(T + \lambda))$ *with the properties that*

a) int $\bigcap_\lambda B_\lambda \neq \emptyset$; *and*

b) *there exists a conditionally compact subset* $\mathscr{K}$ *of* $\mathscr{C}(\mathscr{H})$ *such that* $B_\lambda \cap \mathscr{K} \neq \emptyset$ *for each* λ. *Then* $\left(\bigcap_\lambda B_\lambda\right) \cap \mathscr{C}(\mathscr{H}) \neq \emptyset$.

Proof. The proof of this lemma is quite similar to the previous construction. Since $\mathscr{K}$ is conditionally compact, for each $\varepsilon > 0$ there is a finite dimensional subspace V such that $\|\mathscr{K}|_{V^\perp}\| < \varepsilon$. The following is one step in the induction proof. Suppose $\sum_{i=1}^n \gamma_i P_i^\perp SP_i^\perp$ has been constructed, and assume for the sake of clarity that $\sum_{i=1}^n \gamma_i = 1/2$. Pick E_{n+1} so that $\|\mathscr{K}|_{E_{n+1}^\perp}\| \leq \delta$. In particular, this means that $\|(T+\lambda)|_{E_{n+1}^\perp}\| \leq d(T+\lambda, \mathscr{C}(\mathscr{H})) + \delta$ for all λ, and

$$\left\|\left(T + \lambda - S + \sum_{i=1}^n \gamma_i P_i^\perp SP_i^\perp\right)\Big|_{E_{n+1}^\perp}\right\|$$

$$\leq 1/2(\rho(T+\lambda) - \varepsilon) + 1/2(\rho(T+\lambda) + \delta) = \rho(T+\lambda) - 1/2(\varepsilon - \delta).$$

Having picked δ sufficiently small, one may add on the term $\gamma_{n+1} P_i^\perp SP_i^\perp$ for all $T + \gamma I$ simultaneously. This completes the proof.

We are now ready to give another proof of Theorem 1. It remains to define a conditionally compact set $\mathscr{K}$ of compact operators so that the conditions of Lemma 7 are satisfied. First observe that for any compact set Q of complex numbers λ there is no difficulty in constructing $\mathscr{K}$, because if $\|T + K_{\lambda'} + \lambda\| - \|T+\lambda\|_e < \varepsilon$, then there is a K_λ such that $\|K_{\lambda'} - K_\lambda\| < 2\varepsilon$ and $\|T + K_\lambda + \lambda\| = \|T+\lambda\|_e$. Indeed, note that $B(K_{\lambda'}, 2\varepsilon) \cap B(T+\lambda, \rho(T+\lambda))$ has nonempty interior. Pick an element in the interior and taper it to a compact operator in a manner similar to the construction outlined previously. The compactness of Q, together with our above observation, assures us that a finite set F of compact operators may be chosen which is as close as desired to satisfying the conditions of Lemma 7. Choose a decreasing sequence of positive ε's and a corresponding increasing sequence of finite sets. The closure of the union of these finite sets produces the desired $\mathscr{K}$.

To extend the argument to unbounded sets, we may appeal to Lemma 2. This shows that a $K_{\lambda'}$ which satisfies $\|T + K_{\lambda'} + \lambda'\| = \|T+\lambda'\|_e$ for large enough λ' also satisfies $\|T + K_{\lambda'} + \lambda\| - \|T+\lambda\|_e < \varepsilon$ for all larger λ of the same argument. By a process similar to the above, we may construct a set $\mathscr{K}$ of compact operators so that $\mathscr{K} \cap B(T+\lambda, \rho(T+\lambda)) \neq \emptyset$ for all λ. Therefore, the conditions of Lemma 7 are satisfied, and one may construct a K which satisfies $\|T+\lambda+K\| = \|T+\lambda\|_e$ for all λ. This idea completes the proof of Theorem 1.

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