STRUCTURE OF CERTAIN POLYNOMIAL HULLS

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1. INTRODUCTION

Let X be a compact subset of $\mathbb{C}^n$ with $\hat{X}$ its polynomially convex hull. Conditions on X are known which ensure that $\hat{X}\setminus X$ is a (possibly empty) complex analytic subvariety of $\mathbb{C}^n\setminus X$ of pure dimension one. The case for X being a smooth curve was treated by Stolzenberg [11] and for X a connected set of finite linear measure, by the author [1]. The basic ideas in the subject were introduced by Wermer [12] and Bishop [4]. Whenever the set $\hat{X}\setminus X$ is an analytic set of pure dimension one, it has locally finite "area". This is because pure k-dimensional subvarieties of $\mathbb{C}^n$ have locally finite $\mathscr{H}^{2k}$ measure [10], where $\mathscr{H}^s$ is s-dimensional Hausdorff measure. Our first result is a converse. It is also contained in the work of N. Sibony ([9], Theorem 17, p. 158). We have included a proof as a steppingstone for the generalization in Theorem 2 below.

THEOREM 1. *Let* X *be a compact subset of* $\mathbb{C}^n$. *If* $\mathscr{H}^2(\hat{X}\setminus X) < \infty$, *then* $\hat{X}\setminus X$ *is an analytic subvariety of* $\mathbb{C}^n\setminus X$ *of dimension one. More generally, if a point of* $\hat{X}\setminus X$ *has a neighborhood of finite* $\mathscr{H}^2$ *measure, then* $\hat{X}$ *is locally a pure one-dimensional variety in a neighborhood of the point.*

When the hypothesis of this theorem fails, it may occur that $\hat{X}\setminus X$ can be a countable union of varieties, without being a subvariety of $\mathbb{C}^n\setminus X$. For example, let A be a countable compact subset of $\mathbb{C}$ which is *not* discrete in its relative topology. Put $X = T \times A \subseteq \mathbb{C}^2$, where T is the unit circle. Then $\hat{X}\setminus X = U \times A$ (where U is the open unit disc) fails locally to be a variety at each point (z, α) with $|z| < 1$ for which α is a cluster point of A. Notice that the set $\hat{X}\setminus X$ has σ-finite $\mathscr{H}^2$ measure. (We will take this to mean that $\hat{X}\setminus X$ is a countable union of compact subsets each of which is of finite $\mathscr{H}^2$ measure.) The next result describes the structure of a hull with σ-finite $\mathscr{H}^2$ measure.

THEOREM 2. *Let* X *be a compact subset of* $\mathbb{C}^n$ *such that* $\hat{X}\setminus X$ *has* σ*-finite* $\mathscr{H}^2$ *measure. Then there exist a countable ordinal* μ *(possibly an integer) and a family of compact sets* K_α *with* $X \subseteq K_\alpha \subseteq \hat{X}$, *defined for each ordinal* $1 \leq \alpha \leq \mu$, *such that*

(i) $K_1 = \hat{X}$ *and* $K_\mu = X$;

(ii) $K_\alpha \supsetneq K_\beta$ *for* $1 \leq \alpha < \beta \leq \mu$;

(iii) *The set* $W_\alpha \equiv K_\alpha \setminus K_{\alpha+1}$ *is a relatively open dense subset of* $K_\alpha\setminus X$ *for* $1 \leq \alpha < \mu$;

(iv) W_α *is a nonempty pure one-dimensional analytic subvariety of* $\mathbb{C}^n\setminus K_{\alpha+1}$ *for* $1 \leq \alpha < \mu$;

Received August 23, 1976.
This work was supported in part by the National Science Foundation.

Michigan Math. J. 24 (1977).

(v) $\hat{X}\setminus X = \bigcup \{W_\alpha: 1 \le \alpha < \mu\}$ *is a countable disjoint union of local varieties;*

(vi) *If* V *is any branch* (= *analytic component*) *of some* W_α, *then* $V \subseteq (\overline{V} \cap X)^\wedge$ *and, in particular, branches of the* W_α *are never relatively compact in* $\hat{X}\setminus X$.

Theorem 1 raises the question of when $\mathscr{H}^2(\hat{X}) < \infty$. We conjecture, for an arbitrary compact subset X of $\mathbb{C}^n$, that

$$(*) \qquad \mathscr{H}^2(\hat{X}) \le \frac{1}{4\pi}\{\mathscr{H}^1(X)\}^2.$$

For absolutely area minimizing minimal surfaces (*i.e.*, when X is a real one-dimensional $\mathscr{C}^1$ manifold in $\mathbb{R}^n$ and $\hat{X}$ is the minimal surface of smallest area having X as boundary), (*) is the known [2] "isoperimetric inequality". When X is a finite union of smooth curves, then $\hat{X}\setminus X$ is an analytic 1-variety with boundary contained in X; by a result of Federer [6], $\hat{X}\setminus X$ is absolutely area minimizing, and so (*) is valid in this case. For an arbitrary X, the conjecture would imply that $\mathscr{H}^2(\hat{X}) < \infty$ whenever $\mathscr{H}^1(X) < \infty$, and so $\hat{X}\setminus X$ would be analytic by Theorem 1. Whether or not this is the case is unknown unless X is contained in a *connected* set of finite $\mathscr{H}^1$ measure; see [1]. For example, it is not known whether or not a totally disconnected set E in $\mathbb{C}^n$ with $\mathscr{H}^1(E) < \infty$ must be polynomially convex. The convexity does follow whenever $\hat{E}\setminus E$ is analytic (and consequently empty), but, as indicated above, this is known only when E lies in a connected set of finite linear measure, which is not always the case, even if E is countable.

2. PROOF OF THEOREM 1

Let $p \in \hat{X}\setminus X$ be chosen to have a neighborhood $\mathscr{U}$ in $\mathbb{C}^n$ such that $\mathscr{H}^2(\mathscr{U} \cap \hat{X}) < \infty$. According to a result of Bishop ([5], Lemma 8), we may assume, after a complex affine change of coordinates, that p is the origin and that $\mathscr{U} \cap \hat{X} \cap \{z: z_1 = 0\}$ is totally disconnected. Thus there is an (n - 1)-dimensional neighborhood $\mathscr{N}$ of the origin in the hyperplane $\{z: z_1 = 0\}$ such that $\mathscr{N}$ is a relatively compact subset of $\mathscr{U}$ and $\partial\mathscr{N}$ is disjoint from $\hat{X}$. Therefore, there is a $\delta > 0$ such that $\overline{U}_\delta \times \partial\mathscr{N}$ is disjoint from $\hat{X}$ and $\Delta \equiv U_\delta \times \mathscr{N}$ satisfies $\overline{\Delta} \subseteq \mathscr{U}$, where $U_\delta = \{\lambda \in \mathbb{C}: |\lambda| < \delta\}$. Then the boundary Y of $\overline{\Delta} \cap \hat{X}$ in $\hat{X}$ is contained in $\{z: |z_1| = \delta\}$. By the local maximum modulus principle, $\hat{Y} = \hat{X} \cap \overline{\Delta}$. We shall call Δ a *good neighborhood* of p.

Let $N(\lambda)$ be the number of points in $\overline{\Delta} \cap \hat{X} \cap z_1^{-1}\{\lambda\}$ for $\lambda \in \overline{U}_\delta$. Because $\mathscr{H}^2(\overline{\Delta} \cap \hat{X}) < \infty$, $N(\lambda)$ is integrable with respect to planar Lebesgue measure on $\overline{U}_\delta$ (see [7], [1]); it follows that $N(\lambda)$ is finite a.e. on $\overline{U}_\delta$. By decreasing δ if necessary, we may assume, without loss of generality, that there exist a set $E \subseteq \partial U_\delta$ of positive linear measure and an integer s such that $N(\lambda) = s$ for $\lambda \in E$. By a result of Bishop ([4], p. 497; *cf.* [1]), $\hat{Y} \cap \Delta = \hat{X} \cap \Delta$ is an analytic subvariety of Δ of pure dimension one. This completes the proof of Theorem 1.

3. PROOF OF THEOREM 2

Recall that $\hat{X}\setminus X = \bigcup \{C_k: 1 \le k < \infty\}$, where the C_k are compact with $\mathscr{H}^2(C_k) < \infty$.

LEMMA 1. *Let* L *be a compact set with* $X \subseteq L \subseteq \hat{X}$. *Define*

$$\theta(L) = \{p \in L\backslash X: \textit{some neighborhood of } p \textit{ in } L \textit{ has finite } \mathscr{H}^2 \textit{ measure}\}.$$

Then $\theta(L)$ *is a relatively open dense subset of* $L\backslash X$.

Proof. Let N be an arbitrary open relatively compact subset of $L\backslash X$. Then $N = \bigcup (N \cap C_k)$. The Baire category theorem applied to the locally compact space N implies that some $N \cap C_j$ contains an open subset of N; this open set is contained in $\theta(L)$. Thus, $\theta(L)$ is dense in L.

Remark. When $X \subsetneq L \subseteq \hat{X}$, then $L\backslash \theta(L)$ is compact and $X \subseteq L\backslash \theta(L) \subsetneq L$.

Next we define K_α inductively for all ordinals. Put $K_1 = \hat{X}$. Suppose K_α is defined for all $\alpha < \gamma$. In defining K_γ, there are two possibilities: (a) If $K_\alpha = X$ for some $\alpha < \gamma$, put $K_\gamma = X$; (b) If $K_\alpha \supsetneq X$ for all $\alpha < \gamma$, then there are two subcases: (b1) If γ has a predecessor α (*i.e.*, $\gamma = \alpha + 1$), put $K_\gamma = K_\alpha \backslash \theta(K_\alpha)$ ($\subsetneq K_\alpha$); (b2) If γ is a limit ordinal, put $K_\gamma = \bigcap \{K_\alpha: \alpha < \gamma\}$.

Observe that if $\alpha < \beta$ and $K_\alpha \neq X$, then $K_\alpha \supsetneq K_\beta$. Let

$$\mu = \text{minimum}\{\alpha: K_\alpha = X\}.$$

We shall show below that μ is a countable ordinal. Henceforth, we shall only consider K_α with $\alpha \leq \mu$. It is clear that (i) and (ii) are valid. Define $W_\alpha = \theta(K_\alpha)$ for $\alpha < \mu$. Lemma 1 implies (iii).

To verify (iv), we shall prove by induction the following more general statement $\mathscr{P}(\alpha)$, for $1 \leq \alpha < \mu$:

$$\mathscr{P}(\alpha) \begin{cases} \text{(a) } W_\alpha \equiv K_\alpha \backslash K_{\alpha+1} \text{ is a nonempty subvariety of } \mathbb{C}^n \backslash K_{\alpha+1} \text{ of dimension one.} \\ \text{(b) } K_\alpha \backslash X \text{ satisfies the local maximum modulus principle.} \end{cases}$$

Remark. We shall say that a closed subset L of $\mathbb{C}^n\backslash X$ satisfies the *local maximum modulus principle* (LMMP) if, for an open subset $\mathscr{U}$ of $\mathbb{C}^n$ with $\overline{\mathscr{U}} \subseteq \mathbb{C}^n\backslash X$, we have $\mathscr{U} \cap L \subseteq (\partial\mathscr{U} \cap L)^\wedge$; *i.e.*, for $x \in \mathscr{U} \cap L$ and f a polynomial,

$$|f(x)| \leq \|f\|_{\partial\mathscr{U}\cap L}.$$

LEMMA 2. *If* $\mathscr{U}_1 \cap L \subseteq (\partial\mathscr{U}_1 \cap L)^\wedge$ *for every open subset* $\mathscr{U}_1$ *of* $\mathbb{C}^n$ *with* $\overline{\mathscr{U}}_1 \subseteq \mathbb{C}^n\backslash X$ *for which* $\partial\mathscr{U}_1$ *is real analytic, then* L *satisfies the* LMMP.

Proof. Given an arbitrary $\mathscr{U}$ and a polynomial f, we argue by contradiction and suppose that $\|f\|_{\mathscr{U}\cap L} > \|f\|_{\partial\mathscr{U}\cap L}$. Let $E = \{x \in \mathscr{U} \cap L: |f(x)| = \|f\|_{\mathscr{U}\cap L}\}$. Then E is compact and disjoint from $\partial\mathscr{U} \cap L$. Choose an open set $\mathscr{U}_1$ such that $E \subseteq \mathscr{U}_1 \subseteq \overline{\mathscr{U}}_1 \subseteq \mathscr{U}$ with $\partial\mathscr{U}_1$ real analytic. By hypothesis, we have

$$\|f\|_E \leq \|f\|_{\partial\mathscr{U}_1\cap L}.$$

This is a contradiction, as $\partial\mathscr{U}_1 \cap L \subseteq \mathscr{U}\backslash E$.

As for $\mathscr{P}(1)$, (a) is Theorem 1 and (b) is the usual LMMP due to Rossi [8]. Now we assume $\mathscr{P}(\beta)$ for $\beta < \alpha < \mu$ and prove $\mathscr{P}(\alpha)$.

We consider two cases in verifying $\mathscr{P}(\alpha)$ (b): (b1) Suppose α is a limit ordinal. Then $K_\beta \downarrow K_\alpha$ as $\beta \uparrow \alpha$. Since the LMMP holds for K_β for $\beta < \alpha$, we get the LMMP for K_α in the limit. Case (b2): Suppose $\alpha = \sigma + 1$. Then $K_\sigma = W_\sigma \cup K_\alpha$. Let $\mathscr{U}$ be open in $\mathbb{C}^n$ such that $\overline{\mathscr{U}} \cap X$ is empty and $\partial\mathscr{U}$ is real analytic. We must verify that $\mathscr{U} \cap K_\alpha \subseteq (\partial\mathscr{U} \cap K_\alpha)\hat{}$. Fix x in $\mathscr{U} \cap K_\alpha$ and suppose, by way of contradiction, that $x \notin (\partial\mathscr{U} \cap K_\alpha)\hat{}$. By $\mathscr{P}(\sigma)$ (b), $x \in \mathscr{U} \cap K_\sigma \subseteq (\partial\mathscr{U} \cap K_\sigma)\hat{}$. But $\partial\mathscr{U} \cap K_\sigma = (\partial\mathscr{U} \cap K_\alpha) \cup Y$, where $Y \equiv \partial\mathscr{U} \cap W_\sigma$ is locally a real analytic curve, since, by $\mathscr{P}(\sigma)$ (a), W_σ is a variety. There exists a finite union Γ of subarcs of Y such that $\partial\mathscr{U} \cap K_\sigma = L \cup \Gamma$, where $L \supseteq \partial\mathscr{U} \cap K_\alpha$ and $x \notin \hat{L}$ ("fatten" $\partial\mathscr{U} \cap K_\alpha$ to get L). By a theorem of Stolzenberg [11], $(\partial\mathscr{U} \cap K_\sigma)\hat{} = (\hat{L} \cup \Gamma)\hat{}$ is an analytic variety of dimension one near x. Hence, K_σ has finite $\mathscr{H}^2$ measure in a neighborhood of x; *i.e.*, $x \notin K_{\sigma+1} = K_\alpha$. This is a contradiction.

Next we prove $\mathscr{P}(\alpha)$ (a) using $\mathscr{P}(\alpha)$ (b). Fix $x \in W_\alpha$. Since there is by definition a neighborhood of x in K_α which has finite $\mathscr{H}^2$ measure, we can apply the construction of a "good neighborhood" in Theorem 1 so that, after an affine change of coordinates, we may assume the following (in the notation of the proof of Theorem 1): x is the origin; there exist a neighborhood $\mathscr{N}$ of the origin in $\mathbb{C}^{n-1}$ and $\delta > 0$ such that, if $\Delta = U_\delta \times \mathscr{N}$, then $\overline{\Delta} \cap K_\alpha \subseteq W_\alpha$ and $K_\alpha \cap \partial\Delta \subseteq \partial U_\delta \times \mathscr{N}$; there exists a set $E \subseteq \partial U_\delta$ of positive linear measure such that $z_1^{-1}\{\lambda\} \cap Y$ is finite for each $\lambda \in E$, where $Y = K_\alpha \cap \partial\Delta$. Then, as above, we get that $\hat{Y} \cap \Delta$ is a subvariety V of Δ of dimension one and, by $\mathscr{P}(\alpha)$ (b), $\hat{Y} \supseteq K_\alpha \cap \overline{\Delta} = W_\alpha \cap \overline{\Delta}$.

Let V^* be the set of regular points of V at which z_1 is regular; *i.e.*, locally one-to-one. Then V^* is obtained from V by deleting a discrete subset. We claim that $V^* \cap K_\alpha$ is a (relatively) closed and open subset of V^*. It is closed because K_α is compact. To prove openness, fix $x \in V^* \cap K_\alpha$. There exists a neighborhood N of x in V^* such that z_1 maps N biholomorphically onto a disc

$$\{\lambda \in \mathbb{C}: |\lambda - \alpha| < \delta_0\}.$$

By the LMMP on K_α, $x \in (\partial N \cap K_\alpha)\hat{}$. As any proper closed subset of the "circle" ∂N is polynomially convex, we conclude that $K_\alpha \supseteq \partial N$. By repeating this argument for all δ with $0 < \delta < \delta_0$, we see that $K_\alpha \supseteq N$. This is the desired openness. Now, as $K_\alpha \cap \Delta = W_\alpha \cap \Delta$ has no isolated points, we conclude that $W_\alpha \cap \Delta$ equals the union of those analytic components C of V for which $C \cap V^* \cap K_\alpha$ is nonempty. This proves (iv).

For (v), take $x \in \hat{X}\setminus X$; we shall show that $x \in W_\tau$ for some τ. Let $Q = \{\beta: x \notin K_\beta\}$. Then Q is nonempty, as $\mu \in Q$. Therefore, Q has a least element α. We claim that α is not a limit ordinal. Otherwise, $K_\alpha = \bigcap \{K_\beta: \beta < \alpha\}$ and $x \in K_\beta$ for $\beta < \alpha$ imply $x \in K_\alpha$, contradicting the fact that $\alpha \in Q$. Hence, $\alpha = \tau + 1$ for some τ. Then $x \in K_\tau$, as $\tau < \alpha$. Consequently,

$$x \in K_\tau \setminus K_\alpha = K_\tau \setminus K_{\tau+1} \equiv W_\tau,$$

as desired.

From (v), we deduce that μ is countable; for $\hat{X}\setminus X$ is a disjoint union of W_α for $\alpha < \mu$ and each W_α has positive $\mathscr{H}^2$ measure, while $\hat{X}\setminus X$ has σ-finite $\mathscr{H}^2$ measure, by assumption.

Finally we verify (vi). Let V be an analytic component of some W_α. We first show

LEMMA 3. $\overline{V}\setminus X$ *satisfies the* LMMP.

Proof. Let $\mathscr{U}$ be an open subset of $\mathbb{C}^n$ with $\overline{\mathscr{U}} \subseteq \mathbb{C}^n\setminus X$. We must show that $\overline{V} \cap \mathscr{U} \subseteq (\overline{V} \cap \partial\mathscr{U})\hat{}$. By Lemma 2, we may assume that $\partial\mathscr{U}$ is real analytic. Arguing by contradiction, we suppose that there is an $x \in V \cap \mathscr{U}$ with $x \notin (\overline{V} \cap \partial\mathscr{U})\hat{}$. Now $V \cap \mathscr{U}$ is a subvariety V' of $\mathscr{U}\setminus K_{\alpha+1}$ with "boundary" contained in $(\overline{V} \cap \partial\mathscr{U}) \cup (K_{\alpha+1} \cap \overline{\mathscr{U}})$. By the LMMP for $K_{\alpha+1}$, $K_{\alpha+1} \cap \overline{\mathscr{U}} \subseteq (K_{\alpha+1} \cap \partial\mathscr{U})\hat{}$. Hence, by the maximum principle, for each $x \in V'$, $x \in [(\overline{V} \cap \partial\mathscr{U}) \cup (K_{\alpha+1} \cap \partial\mathscr{U})]\hat{}$. Let L be a minimal compact set with $\overline{V} \cap \partial\mathscr{U} \subseteq L \subseteq (\overline{V} \cup K_{\alpha+1}) \cap \partial\mathscr{U}$ such that $x \in \hat{L}$. Then $L\setminus\overline{V}$ is nonempty and $L\setminus\overline{V} \subseteq K_{\alpha+1}\setminus X = \bigcup\{W_\gamma: \gamma \geq \alpha + 1\}$. The set $Q = \{\gamma: \gamma \geq \alpha + 1 \text{ and } (L\setminus\overline{V}) \cap W_\gamma \text{ is nonempty}\}$ has a minimal element σ. As a consequence, $L\setminus\overline{V} \subseteq K_\sigma$. As W_σ is relatively open in K_σ, it follows that $(L\setminus\overline{V}) \cap W_\sigma$ contains a relatively open subset of $L\setminus\overline{V}$. Therefore, in a neighborhood of some point of W_σ, L is (locally) an open real analytic Jordan arc Γ. Let $L_1 = L\setminus\Gamma$. By minimality, $\hat{L}_1$ does not contain x. By Stolzenberg's theorem [11], $\hat{L}\setminus(\hat{L}_1 \cup \Gamma) = (\hat{L}_1 \cup \Gamma)\hat{}\setminus(\hat{L}_1 \cup \Gamma)$ is an analytic subvariety of $\mathbb{C}^n\setminus(\hat{L}_1 \cup \Gamma)$ containing x.

Let A be an irreducible analytic component of $\hat{L}\setminus(\hat{L}_1 \cup \Gamma)$ containing x. Since $\overline{A}\setminus A \subseteq \hat{L}_1 \cup \Gamma$ and $x \notin \hat{L}_1$, it follows that $\overline{A}$ contains an open subset of Γ as a relatively open subset of $\overline{A}\setminus A$. From this we deduce that $W_\sigma \cap A$ contains a nonempty open subset of A.

Since $\hat{X}\setminus K_{\alpha+1}$ is a neighborhood of x in $\hat{X}$, $x \in A \subseteq \hat{X}$, and

$$\hat{X}\setminus K_{\alpha+1} = \bigcup \{W_\beta: \beta \leq \alpha\},$$

the above argument implies that $W_\beta \cap A$ contains a nonempty open subset of A for some $\beta \leq \alpha$.

Now write $A = \bigcup \{A \cap W_\tau: 1 \leq \tau < \mu\}$ and note that $A \cap W_\tau = U_\tau \cup D_\tau$, where U_τ is open in A and D_τ is countable. Then $D = \bigcup D_\tau$ is countable and so $A\setminus D$ is connected. But $A\setminus D = \bigcup U_\tau$ is a disjoint union and we know that U_σ and U_β are nonempty, with $\beta \leq \alpha < \sigma$. This contradicts the connectedness of $A\setminus D$.

Now (vi) follows by applying the LMMP to a sequence of $\mathscr{U}_n$ with $\overline{\mathscr{U}}_n \cap X = \emptyset$ and $\mathscr{U}_n \cap \overline{V} \uparrow \overline{V}\setminus X$.

Remark 1. Can we say that $\hat{X}\setminus X$ is a countable, not necessarily disjoint, union of *subvarieties* of $\mathbb{C}^n\setminus X$? The answer is no! Put

$$V_1 = \{(z, w) \in \mathbb{C}^2: |w| < 1,\ \Re w > 0, \text{ and } z = e^{-1/w}\}.$$

Then V_1 is a subvariety of $U^2\setminus V_2$, where $V_2 = \{(z, w) \in U^2: w = 0\}$. Let $X = \overline{V}_1 \cap \partial U^2$ and note that X contains the circle $\{(z, w): w = 0, |z| = 1\}$, that $\hat{X}\setminus X = V_1 \cup V_2$, and that $V_2 \subseteq \overline{V}_1$. In the notation of Theorem 2: $\mu = 3$, $W_1 = V_1$, $W_2 = V_2$. It is not true that $\hat{X}\setminus X$ is a union of subvarieties of $\mathbb{C}^2\setminus X$. For if $V \subseteq \hat{X}\setminus X$ were a subvariety of $\mathbb{C}^2\setminus X$ containing a point of V_1, then V would contain all of V_1; hence, as V is closed in $\mathbb{C}^2\setminus X$, V would contain

$\overline{V}_1 \cap (\mathbb{C}^2 \setminus X) \supseteq V_2$; *i.e.*, $V = \hat{X} \setminus X$. But this set is clearly not locally a variety at points of V_2.

Remark 2. Our results carry over to a somewhat different context considered by Basener [3]. Let A be a uniform algebra with maximal ideal space M and Shilov boundary X. Let $f \in A$ and let $\mathcal{U}$ be a component of $\mathbb{C} \setminus f(X)$ such that $f^{-1}\{\lambda\}$ is countable for each $\lambda \in \mathcal{U}$, where we view f as a continuous function on M. For a closed subset L of $f^{-1}(\mathcal{U})$, define

$$\theta(L) = \{x \in L\text{: for some neighborhood } N \text{ of } x \text{ in } L, \; f^{-1}\{\lambda\} \cap N \text{ is finite for each } \lambda \in \mathcal{U}\}.$$

Basener showed that $W_1 \equiv \theta(f^{-1}(\mathcal{U}))$ is a relatively open dense subvariety of $f^{-1}(\mathcal{U})$. Arguing as in Theorem 2, one shows that $K_2 \equiv f^{-1}(\mathcal{U}) \setminus W_1$ satisfies the LMMP. Again by Basener's argument, $W_2 \equiv \theta(K_2)$ is a relatively open dense subvariety of K_2. Proceeding inductively in this way, one gets a description of $f^{-1}(\mathcal{U})$ analogous to that of $\hat{X} \setminus X$ in Theorem 2. In particular, $f^{-1}(\mathcal{U})$ is a countable disjoint union of local varieties. The example of Remark 1 can be adapted to this setting to show that $f^{-1}(\mathcal{U})$ is not always a countable union of subvarieties of $f^{-1}(\mathcal{U})$.

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