A GENERALIZATION OF MERGELYAN'S UNIQUENESS THEOREM

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In [4, Chapter 2, inequality (21.3)], S. N. Mergelyan used the Phragmén-Lindelöf Principle [6] to prove the following uniqueness theorem.

If f *is holomorphic in* $H = \{z: \Re z > 0\}$, *and if there exist positive numbers* K *and* A *such that* $|f(z)| \leq Ke^{-A|z|}$ *for each* $z \in H$, *then* $f(z) \equiv 0$.

The essential condition in this theorem is the requirement that the inequality holds throughout the half-plane H. Naturally, we may ask whether instead of the whole half-plane we might consider only a sequence of arcs in H. The purpose of this paper is to answer this question. Our results are similar to those of A. L. Šaginjan [8], V. I. Gavrilov [1], and D. C. Rung [7].

We use methods based on the notion of harmonic measure, the Carleman-Milloux problem, and the two-constants theorem of F. and R. Nevanlinna [5, p. 42].

Definition 1. Let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence of disjoint Jordan arcs in the right half-plane $H = \{z: \Re z > 0\}$. Write

$$z = re^{i\theta}, \quad \ell_n = \min_{z \in \gamma_n} |z|, \quad L_n = \max_{z \in \gamma_n} |z|, \quad \lambda_n = \ell_n / L_n, \quad \theta_n = \min_{z \in \gamma_n} \operatorname{Arg} z$$

(the capital A indicates the principal branch), and let α_n denote the angle subtended by γ_n at the origin. We call $\{\gamma_n\}$ *an arc-like sequence* if

$$\lim_{n \to \infty} \ell_n = \lim_{n \to \infty} L_n = \infty, \quad \liminf_{n \to \infty} \lambda_n > 0, \quad \liminf_{n \to \infty} \alpha_n > 0.$$

Definition 2. Corresponding to each are-like sequence $\{\gamma_n\}$ with associated parameters L_n, θ_n, and α_n, we define the sequence of circular sectors

$$F_n = \{z: 0 < |z| < L_n, \ \theta_n < \arg z < \theta_n + \alpha_n\}.$$

Definition 3. Let F be a domain in H, and let f be a complex-valued function in H. By M(f, F) we denote the supremum of $\operatorname{Max}\{\log|f(z)|, 1\}$ in F.

THEOREM. *Suppose*

(i) f *is holomorphic in* H,

(ii) $\{\gamma_n\}$ *is an arc-like sequence in* H, *with associated parameters* ℓ_n, L_n, *and* α_n,

(iii) $\{A_n\}$ *and* $\{R_n\}$ *are sequences of positive numbers such that* $\ell_n \leq R_n \leq L_n$ *and such that, for some constants* α_0 *and* p $(0 < \alpha_0 \leq \pi, \ p \geq 1)$,

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$$0 < \liminf_{n\to\infty} \frac{A_n^{1/p}}{R_n^{\pi/\alpha_n - \pi/\alpha_0}} \leq \limsup_{n\to\infty} \frac{A_n}{R_n^{\pi/\alpha_n - \pi/\alpha_0}} = \infty,$$

(iv) $$\limsup_{n\to\infty} \frac{M(f, F_n)}{A_n^{1/q}} < \infty \quad (1/p + 1/q = 1),$$

(v) $$|f(z)| \leq \exp(-A_n |z|^{\pi/\alpha_0}) \quad \text{on } \gamma_n \ (n = 1, 2, \cdots).$$

Then $f(z) \equiv 0$.

The proof of this theorem is based on the notion of harmonic measure [2, p. 408]. We divide it into three lemmas. The first of these is almost the same as the lemma in [3].

LEMMA 1. *Let* D_ρ *denote the half-disk* $\{w: \Re w > 0, |w| < \rho\}$, *let* Γ *be the semicircle on the boundary of* D_ρ, *and let* $\omega(w, \Gamma)$ *denote the harmonic measure of the arc* Γ *at the point* w, *relative to the domain* D_ρ. *Then*

$$\omega(w, \Gamma) = \frac{1}{\rho}\left[\frac{4}{\pi}\Re w + o(\rho^{-1})\right] \quad \text{as } \rho \to \infty.$$

Proof. We first map D_ρ conformally onto the first quadrant by means of the formula $z = \frac{i\rho - w}{i\rho + w}$. Then the image $z(\Gamma)$ is the upper half of the imaginary axis. Thus, by [2, p. 407, Exercise 8],

$$\omega(z, z(\Gamma)) = \frac{2}{\pi} \arg z.$$

Since harmonic measure is invariant under conformal mappings [5, p. 38], we have the relations

$$\omega(w, \Gamma) = \omega(z, z(\Gamma)) = \frac{2}{\pi} \arg \frac{i\rho - w}{i\rho + w} = \frac{2}{\pi} \Im \log \frac{i\rho - w}{i\rho + w} = \frac{1}{\rho}\frac{4}{\pi} \Re w + O(\rho^{-3})$$

$$= \frac{1}{\rho}\left[\frac{4}{\pi}\Re w + o(\rho^{-1})\right] \quad \text{as } \rho \to \infty.$$

LEMMA 2. *Let* f *satisfy the conditions* (i), (ii), (iv), *and* (v) *of the theorem, and suppose that in addition there exists a constant* α_0 $(0 < \alpha_0 \leq \pi)$ *such that*

(iii') $$0 < \liminf_{n\to\infty} \frac{A_n^{1/p}}{L_n^{\pi/\alpha_n - \pi/\alpha_0}} \leq \limsup_{n\to\infty} \frac{A_n}{L_n^{\pi/\alpha_n - \pi/\alpha_0}} = \infty.$$

Then $f(z) \equiv 0$.

Proof. By Definition 1 and conditions (iii') and (iv), the arc-like sequence $\{\gamma_n\}$ has a subsequence (which we again denote by $\{\gamma_n\}$) such that

(1) $$0 < G_p = \lim_{n\to\infty} \frac{A_n^{1/p}}{L_n^{\pi/\alpha_n - \pi/\alpha_0}} \leq \lim_{n\to\infty} \frac{A_n}{L_n^{\pi/\alpha_n - \pi/\alpha_0}} = \infty,$$

(2)
$$\lim_{n\to\infty} \frac{M(f, F_n)}{A_n^{1/q}} = M < \infty,$$

(3)
$$\lim_{n\to\infty} \lambda_n = \lambda > 0.$$

Without loss of generality, we may assume that for each n, the arc γ_n meets the rectilinear portions of the boundary of F_n only in its endpoints

$$\ell_n' e^{i\theta_n} \quad \text{and} \quad \ell_n'' e^{i(\theta_n+\alpha_n)}.$$

Also, extracting an appropriate subsequence if necessary, we may suppose that

(4)
$$\lim_{n\to\infty} \alpha_n = \alpha \quad \text{and} \quad \lim_{n\to\infty} \theta_n = \theta,$$

where $0 < \alpha \leq \pi$ and $-\pi/2 \leq \theta < \pi/2$.

By condition (v), Definition 3, and the two-constants theorem [5, p. 42], we have for each z in F_n the inequality

(5)
$$\begin{cases} \log|f(z)| \leq \omega(z, \gamma_n)(-A_n \ell_n^{\pi/\alpha_0}) + (1 - \omega(z, \gamma_n)) M(f, F_n) \\ \qquad \leq -A_n \ell_n^{\pi/\alpha_0} \omega(z, \gamma_n) + M(f, F_n). \end{cases}$$

To estimate the harmonic measure of γ_n at a point z in F_n, we denote by C_n the circular portion of the boundary of F_n, and we observe that Carleman's principle of monotoneity [5, p. 69] yields the inequality

(6)
$$\omega(z, \gamma_n) \geq \omega(z, C_n).$$

In order to apply Lemma 1, we map F_n conformally onto the half-disk $D_n = D_{\rho_n}$ by the mapping

(7)
$$w_n(z) = \{z \exp[-i(\theta_n + \alpha_n/2)]\}^{\pi/\alpha_n}.$$

Clearly, we have the relation

(8)
$$\rho_n = L_n^{\pi/\alpha_n}.$$

Let Γ_n denote the semicircular part of the boundary of D_n, and let $\omega(w, \Gamma_n)$ denote its harmonic measure at the point w in D_n. Lemma 1 implies that

(9)
$$\omega(w, \Gamma_n) = \frac{1}{\rho_n}\left[\frac{4}{\pi}\Re w + o(\rho_n^{-1})\right].$$

By virtue of the conformal invariance of harmonic measure, equations (6), (7), (8), and (9) allow us to write, for each point $z = |z| e^{i\phi}$ in F_n,

$$(10)\quad \begin{cases} \omega(z,\gamma_n) \geq \omega(z, C_n) = \omega(w, \Gamma_n) \\ \quad = L_n^{-\pi/\alpha_n}\left\{\frac{4}{\pi}\,\Re\{z\exp[-i(\theta_n+\alpha_n/2)]\}^{\pi/\alpha_n} + o(L_n^{-\pi/\alpha_n})\right\} \\ \quad = L_n^{-\pi/\alpha_n}\left\{\frac{4}{\pi}|z|^{\pi/\alpha_n}\cos\frac{\pi(\phi-\theta_n-\alpha_n/2)}{\alpha_n} + o(L_n^{-\pi/\alpha_n})\right\}. \end{cases}$$

Combining equations (5) and (10) with condition (iv), we obtain the estimate

$$(11)\quad \begin{cases} \log|f(z)| \\ \leq -A_n\left\{\frac{\ell_n^{\pi/\alpha_0}}{L_n^{\pi/\alpha_n}}\left[\frac{4}{\pi}|z|^{\pi/\alpha_n}\cos\frac{\pi(\phi-\theta_n-\alpha_n/2)}{\alpha_n} + o(L_n^{-\pi/\alpha_n})\right] - \frac{M(f, F_n)}{A_n}\right\} \\ = \frac{-A_n}{L_n^{\pi/\alpha_n-\pi/\alpha_0}}\left\{\lambda_n^{\pi/\alpha_0}\left[\frac{4}{\pi}|z|^{\pi/\alpha_n}\cos\frac{\pi(\phi-\theta_n-\alpha_n/2)}{\alpha_n} + o(L_n^{-\pi/\alpha_n})\right]\right. \\ \qquad\left. - \frac{L_n^{\pi/\alpha_n-\pi/\alpha_0}M(f, F_n)}{A_n^{1/p}A_n^{1/q}}\right\}. \end{cases}$$

For each ε with $0 < \varepsilon < \lambda$, it follows from equations (1), (2), (3), and Definition 1 that there exists a positive integer N_1 such that, for all $n \geq N_1$,

$$\frac{1}{G_p} - \varepsilon < \frac{L_n^{\pi/\alpha_n-\pi/\alpha_0}}{A_n^{1/p}} < \frac{1}{G_p} + \varepsilon,$$

$$M - \varepsilon < \frac{M(f, F_n)}{A_n^{1/q}} < M + \varepsilon,$$

$$0 < \lambda - \varepsilon < \lambda_n < \lambda + \varepsilon,$$

and the term $o(L_n^{-\pi/\alpha_n})$ is less than ε. For sufficiently large n, we can write the inequality (11) in the form

$$(12)\quad \begin{cases} \log|f(z)| \\ \leq -\frac{A_n}{L_n^{\pi/\alpha_n-\pi/\alpha_0}}\left\{(\lambda-\varepsilon)^{\pi/\alpha_0}\left[\frac{4}{\pi}|z|^{\pi/\alpha_n}\cos\frac{\pi(\phi-\theta_n-\alpha_n/2)}{\alpha_n} - \varepsilon\right]\right. \\ \qquad\left. - \left(\frac{1}{G_p}+\varepsilon\right)(M+\varepsilon)\right\}. \end{cases}$$

Condition (4) implies that as $n \to \infty$, the sequence $\{F_n\}$ converges to the sector

$$F_\infty = \{z: 0 < |z| < \infty,\ \theta < \arg z < \theta + \alpha\}.$$

By (4), there exists an integer N_2 such that

$$\cos \frac{\pi(\theta + \alpha/2 - \theta_n - \alpha_n/2)}{\alpha_n} > 0$$

for $\phi = \theta + \alpha/2$, whenever $n \geq N_2$. Now let $R(\theta + \alpha/2)$ denote the ray that bisects the domain F_∞. There exists a number N_3 $(N_3 \geq N_2)$ such that, for all z on $R(\theta + \alpha/2)$ with $|z| \geq N_3$,

$$(13) \quad \begin{cases} (\lambda - \varepsilon)^{\pi/\alpha_0} \left[\frac{4}{\pi} |z|^{\pi/\alpha_n} \cos \frac{\pi(\theta + \alpha/2 - \theta_n - \alpha_n/2)}{\alpha_n} - \varepsilon \right] \\ \qquad - \left(\frac{1}{G_p} + \varepsilon \right) (M + \varepsilon) \geq G > 0. \end{cases}$$

Let $N = \max(N_1, N_3)$. For each $n \geq N$ and each $z \in R(\theta + \alpha/2)$ with $|z| \geq N$, the relations (12) and (13) imply that

$$(14) \quad \log |f(z)| \leq - \frac{A_n G}{L_n^{\pi/\alpha_n - \pi/\alpha_0}}.$$

Let $n \to \infty$; from (14) and the second inequality in (1) it follows that $\log |f(z)| = -\infty$, in other words, that $f(z) = 0$ on the ray $R(\theta + \alpha/2)$, for $|z| \geq N$. By the uniqueness theorem for holomorphic functions, we can conclude that $f(z) \equiv 0$.

LEMMA 3. *Let* f *satisfy conditions* (i), (ii), (iv), *and* (v) *of the theorem; suppose that in addition there exists a constant* α_0 $(0 < \alpha_0 \leq \pi)$ *such that*

$$(\text{iii}'') \quad 0 < \liminf_{n \to \infty} \frac{A_n^{1/p}}{\ell_n^{\pi/\alpha_n - \pi/\alpha_0}} \leq \limsup_{n \to \infty} \frac{A_n}{\ell_n^{\pi/\alpha_n - \pi/\alpha_0}} = \infty.$$

Then $f(z) \equiv 0$.

Proof. Using the technique that lead to the estimate (11), we obtain the inequality

$$(11') \quad \begin{cases} \log |f(z)| \\ \quad \leq \frac{-A_n}{\ell_n^{\pi/\alpha_n - \pi/\alpha_0}} \left\{ \lambda_n^{\pi/\alpha_n} \left[\frac{4}{\pi} |z|^{\pi/\alpha_n} \cos \frac{\pi(\phi - \theta_n - \alpha_n/2)}{\alpha_n} + o(L_n^{-\pi/\alpha_n}) \right] \right. \\ \qquad \left. - \frac{\ell_n^{\pi/\alpha_n - \pi/\alpha_0} M(f, F_n)}{A_n^{1/p} A_n^{1/q}} \right\}. \end{cases}$$

By the argument in the proof of Lemma 2, $\lim_{n \to \infty} \lambda_n^{\pi/\alpha_n} = \lambda^{\pi/\alpha}$, and therefore we see that

$$(12')\quad \left\{\begin{aligned} &\log|f(z)| \\ &\quad \leq \frac{-A_n}{\ell_n^{\pi/\alpha_n - \pi/\alpha_0}} \left\{ (\lambda - \varepsilon)^{\pi/\alpha + \varepsilon} \left[\frac{4}{\pi} |z|^{\pi/\alpha_n} \cos \frac{\pi(\phi - \theta_n - \alpha_n/2)}{\alpha_n} - \varepsilon \right] \right. \\ &\qquad\qquad \left. - \left(\frac{1}{G_p} + \varepsilon \right)(M + \varepsilon) \right\}. \end{aligned}\right.$$

The remainder of the proof is the same as that of Lemma 2.

In the proof of our theorem, we distinguish two cases.

Case 1. If there exist infinitely many indices n for which the subtended angle α_n is at least α_0, then condition (iii) implies condition (iii'), and the theorem follows from Lemma 2.

Case 2. If there exist infinitely many indices n for which $\alpha_n \leq \alpha_0$, then condition (iii) implies (iii"), and the theorem follows from Lemma 3. This concludes the proof.

We shall now discuss possible choices of the angle α_0. If (after extraction of a subsequence) the arcs γ_n satisfy the condition $\alpha_n \geq \alpha_0$, we can write (11) in the form

$$\begin{aligned} \log|f(z)| \leq -L_n^{\pi/\alpha_0 - \pi/\alpha_n} &\left\{ A_n \lambda_n^{\pi/\alpha_0} \left[\frac{4}{\pi} |z|^{\pi/\alpha_n} \cos \frac{\pi(\phi - \theta_n - \alpha_n/2)}{\alpha_n} + o(L_n^{-\pi/\alpha_n}) \right] \right. \\ &\left. - \frac{M(f, F_n)}{L_n^{\pi/\alpha_0 - \pi/\alpha_n}} \right\}. \end{aligned}$$

In this case, condition (iii) in the theorem becomes trivial, provided $\limsup_{n \to \infty} A_n > 0$. We can then replace condition (iv) with the condition

$$\limsup_{n \to \infty} \frac{M(f, F_n)}{L_n^{\pi/\alpha_0 - \pi/\alpha_n}} < \infty .$$

Instead of assuming that the sequence $\{A_n\}$ tends to infinity, we need only assume that it is bounded away from 0. Observing that $R_n \leq L_n$, we then obtain the following corollary.

COROLLARY 1. *If*

(i) f(z) *is holomorphic in* H,

(ii) $\{\gamma_n\}$ *is an arc-like sequence such that* $\alpha_n \geq \alpha_0$ $(0 < \alpha_0 < \pi)$,

(iii) $$\limsup_{n \to \infty} A_n > 0 \quad \textit{and} \quad \limsup_{n \to \infty} R_n^{\pi/\alpha_0 - \pi/\alpha_n} = \infty,$$

(iv) $$\limsup_{n \to \infty} \frac{M(f, F_n)}{R_n^{\pi/\alpha_0 - \pi/\alpha_n}} < \infty,$$

(v) $$|f(z)| \leq \exp(-A_n |z|^{\pi/\alpha_0}) \quad \textit{on } \gamma_n \ (n = 1, 2, \cdots),$$

then $f(z) \equiv 0$.

If in this corollary we assume that f(z) is bounded in F_n (or in H), we can omit condition (iv). On the other hand, condition (ii) enables us to choose for each n a subarc γ_n^* of γ_n such that the subtended angles α_n^* are all equal to α_0. Thus, if $\alpha_n \geq \alpha_0$, we need consider only the special case $\alpha_n = \alpha_0$.

We have the following general result, whose proof we omit.

COROLLARY 2. *Let conditions* (i) *and* (ii) *be the same as in the theorem. If in addition*

(iii) $$\limsup_{n \to \infty} A_n = \infty,$$

(iv) $$\limsup_{n \to \infty} \frac{M(f, F_n)}{A_n} < \infty,$$

(v) $$|f(z)| \leq \exp(-A_n |z|^{\pi/\alpha_n}) \quad \textit{on } \gamma_n \ (n = 1, 2, \cdots),$$

then $f(z) \equiv 0$.

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