A CONNECTION BETWEEN THE CESARI AND LERAY-SCHAUDER METHODS

Stephen A. Williams

1. INTRODUCTION

A method that L. Cesari, J. K. Hale, and R. A. Gambill [2], [9], [12], [13], [14], [15] used to solve perturbation problems was generalized in 1963 and 1964 by Cesari [3], [5] so as to apply to strictly nonlinear problems. We shall call the method of [3] and [5] the *Cesari method*. Cesari, Hale, and H. W. Knobloch [8], [16], [17], [18] have since applied this method to boundary-value problems for ordinary and partial differential equations. S. Bancroft, J. K. Hale, and D. Sweet [1] and J. Locker [20] have extended the Cesari method in ways that we shall not consider here. Cesari [4], [6], [7] has proved the existence of periodic solutions of certain hyperbolic partial differential equations, solving his determining equation by use of the Tychonoff theorem in an infinite-dimensional space. In the present paper, however, we use only finite-dimensional methods (degree theory) to solve our determining equation.

By the term *Leray-Schauder method* we mean the method introduced in 1934 by J. Leray and J. Schauder [19].

Theorem 1 describes a theoretical link between the Cesari method and the Leray-Schauder method. Theorem 2 asserts the existence of a certain invariance property of an index (see the next section) associated with the Cesari method.

2. AN ABSTRACT DEFINITION OF THE CESARI INDEX

In defining the Cesari index below, we make several assumptions. Some of these assumptions made in [5]; the others are propositions proved in [5] as the results of assumptions of a more analytical nature. The reader may refer to Section 4 of the present paper for a comparison of the notation used in this paper with the notation in [5].

Let B be a Banach space, and let S be a finite-dimensional subspace of B. Let $P: B \to S$ be a projection, that is, let P be continuous and linear, with $P^2 = P$. Suppose that $\Gamma \subset B$, that $P\Gamma$ is compact, and that $(P^{-1}x) \cap \Gamma$ is closed for every x in $P\Gamma$. Let W be a continuous map from Γ into B. The Cesari method—after a suitable change in notation—gives sufficient conditions for W to have a fixed point in Γ.

Let I be the identity map in B, and let $T: \Gamma \to B$ be defined by $T = P + (I - P)W$. For each $x \in P\Gamma$, the restriction of T to $(P^{-1}x) \cap \Gamma$ is a map from $(P^{-1}x) \cap \Gamma$ into $P^{-1}x$. We shall assume that for each $x \in P\Gamma$ this restriction is a contraction from $(P^{-1}x) \cap \Gamma$ into itself, and we shall denote the resulting unique fixed point by y(x) to indicate the dependence on x (see Remark 1 below). We shall assume that

Received April 26, 1968.
This work was undertaken as a thesis project at the California Institute of Technology, under a National Science Foundation fellowship, and it was continued at the University of Michigan under partial support of AF-OSR grant 942-65. The author wishes to thank F. B. Fuller and L. Cesari for many helpful suggestions.

$y\colon P\Gamma \to B$ is continuous and that $PWy(x) \neq x$, for each x on the boundary of $P\Gamma$. This assures the existence of the finite-dimensional *fixed-point index* $i(PWy, \text{int } P\Gamma)$, that is, the *Brouwer topological degree* $d(I - PWy, \text{int } P\Gamma, 0)$ (see Remarks 1 and 2 below). Under these assumptions, the Cesari index $n(\Gamma, W, P)$ for the mapping W is said to be defined; it is given by

$$n(\Gamma, W, P) = i(PWy, \text{int } P\Gamma) = d(I - PWy, \text{int } P\Gamma, 0).$$

If the Cesari index is defined and is not zero, then W has a fixed point in Γ; for if $n(\Gamma, W, P) = i(PWy, \text{int } P\Gamma) \neq 0$, then there exists an x in int $P\Gamma$ such that $PWy(x) = x$, and therefore (by Remark 1 below) $y(x) = Wy(x)$ with $y(x)$ in Γ.

Remark 1. Clearly, s is a fixed point of W in Γ (that is, $s = Ws$) if and only if

$$Ps = PWs \quad \text{and} \quad s = (P + (I - P)W)s = Ts.$$

Suppose now that s is a fixed point of W. The relations $s \in (P^{-1}(Ps)) \cap \Gamma$ and $s = Ts$ imply that $s = y(Ps)$. Also, $Ps = PWs = PWy(Ps)$, so that $x = Ps$ satisfies the equation $x = PWy(x)$. On the other hand, if any x in $P\Gamma$ satisfies $PWy(x) = x$, then, since

$$PWy(x) = Py(x) \quad \text{and} \quad y(x) = Ty(x),$$

$y(x)$ is a fixed point of W in Γ. For this reason we shall call the equation $x = PWy(x)$ the *determining equation* for the fixed points of W in Γ.

Remark 2. If $\Omega \subseteq S$ is a bounded open set and $f\colon \Omega \to S$ is continuous, with $f(x) \neq x$ for each x on the boundary of Ω, then $i(f, \Omega)$ is the *fixed-point index* of f with respect to Ω. Leray and Schauder [19] call $i(\, , \,)$ the finite-dimensional *total index*. $i(f, \Omega) = d(I - f, \Omega, 0)$, where $d(\, , \, , \,)$ is the topological (Brouwer) degree. We shall use $i_{LS}(\, , \,)$ and $d_{LS}(\, , \, , \,)$ to represent the corresponding *total index* and *topological degree* defined by Leray and Schauder in [19] for infinite-dimensional Banach spaces.

Remark 3. Theorems 1 and 2 assume that Γ is the closure of a bounded open set Ω and that $y(x)$ is in Ω for every x in int $P\Gamma$. Both of these assumptions hold when the Cesari method is applied (see Section 4 and Cesari [5]).

3. TWO THEOREMS

THEOREM 1. *Let* $n(\Gamma, W, P)$ *be defined as above, where* Γ *is the closure of a bounded open set* Ω. *For* x *in* int $P\Gamma$, *assume that* $y(x)$ *is in* Ω. *Then there exists a function* $W'\colon \overline{\Omega} \to B$, *having the same set of fixed points as* W, *for which both the Cesari index* $n(\Gamma, W', P)$ *and the Leray-Schauder fixed-point index* $i_{LS}(W', \Omega)$ *are defined, and* $n(\Gamma, W', P) = n(\Gamma, W, P) = i_{LS}(W', \Omega)$.

Proof. Define $W'\colon \overline{\Omega} \to B$ by setting $W'(x) = Wy(Px)$ for every x in Γ. If $x = W'x$, then $x = Wy(Px)$; therefore $Px = PWy(Px)$, and thus (by Remark 1) $y(Px) = Wy(Px) = x$, hence $x = Wx$. On the other hand, if $x = Wx$, then (again by Remark 1) $x = y(Px)$, so that $x = Wy(Px) = W'x$. Thus W and W' have the same set of fixed points in Γ.

Note that all points z in $(P^{-1}x) \cap \Gamma$ have projection $Pz = Px$, that hence they have the same image under $W' = WyP$, and that therefore they have the same image under $P + (I - P)W'$. Thus, for z in $(P^{-1}x) \cap \Gamma$, we have the relation

$$(P + (I - P)W')z = (P + (I - P)W')y(x) = Py(x) + (I - P)Wy(x) = Ty(x) = y(x).$$

Since the map $P + (I - P)W'$ carries each set $(P^{-1}x) \cap \Gamma$ into the set consisting of the single point $y(x)$, this map is a contraction on each $(P^{-1}x) \cap \Gamma$. The fixed point $y'(x)$ of $P + (I - P)W'$ in $(P^{-1}x) \cap \Gamma$ is $y(x)$. Therefore the functions y and y' are equal. Since $PW'y' = PWyPy = PWy$, the index $n(\Gamma, W', P) = i(PW'y', \text{int } P\Gamma)$ exists and is equal to $n(\Gamma, W, P) = i(PWy, \text{int } P\Gamma)$.

$W' = WyP$ is the composite of continuous functions and hence is continuous. The range $Wy(P\Gamma)$ of the function W' is compact. Therefore W' is completely continuous.

Now let us prove that W' has no fixed points on the boundary of Ω. If x is a fixed point of W' and $Px \in \text{int } P\Gamma$, then $x = Wx$; by Remark 1, $x = y(Px)$, and this point is in Ω. If on the other hand x is a fixed point of W' with Px on the boundary of $P\Gamma$, then $x = Wx$ and (by Remark 1) $Px = PWy(Px)$, so that Px is a fixed point of PWy on the boundary of $P\Gamma$; this contradicts the existence of

$$n(\Gamma, W, P) = i(PWy, \text{int } P\Gamma).$$

W' is completely continuous and has no fixed points on the boundary of Ω. Therefore the Leray-Schauder fixed-point index $i_{LS}(W', \Omega) = d_{LS}(I - W', \Omega, 0)$ exists. It remains only to show that $n(\Gamma, W', P) = i_{LS}(W', \Omega)$.

Let S be the finite-dimensional space that is the range of P. For every t in $[0, 1]$, define u_t: $P^{-1}(P\overline{\Omega}) \to P^{-1}(P\overline{\Omega})$ by

$$u_t(z) = z - t(I - P)y'(Pz).$$

Since $u_1(y'(Pz)) = Pz$, the homotopy u_t "flattens" the range of y' as it moves it into S. Note that $Pu_t(z) = Pz$. For fixed t, u_t is one-to-one and u_t^{-1} is continuous, since $u_t^{-1}(z) = z + t(I - P)y'(Pz)$. Thus each $u_t(\Omega)$ is open in B, and $u_t(\overline{\Omega}) = \overline{u_t(\Omega)}$. For each t in $[0, 1]$, define W_t: $u_t(\overline{\Omega}) \to B$ by

$$W_t(z) = z + (W'u_t^{-1}(z) - u_t^{-1}(z)).$$

Since $W_t(z) - z = W'u_t^{-1}(z) - u_t^{-1}(z)$, each point $z = W_t(z)$ on the boundary of $u_t(\Omega)$ must have a preimage $u_t^{-1}(z)$ on the boundary of Ω that is a fixed point of W', a contradiction. Hence this homotopy introduces no fixed points of W_t on the boundary of $u_t(\Omega)$.

If $z \in \overline{\Omega}$ and $t \in [0, 1]$, then $W_t(u_t(z)) = W'(z) - t(I - P)y'(Pz)$, and thus

$$W_t(u_t(\overline{\Omega})) \subset W'(\overline{\Omega}) - t(I - P)y'(P\overline{\Omega});$$

the right-hand side is a compact set, since it is the difference of compact sets. Thus W_t is completely continuous for each t. Throughout the homotopy W_t: $u_t(\overline{\Omega}) \to B$, the Leray-Schauder index is preserved. Thus

$$i_{LS}(W', \Omega) = i_{LS}(W_0, u_0(\Omega)) = i_{LS}(W_1, u_1(\Omega)).$$

But since $(I - P)y' = (I - P)W'y'$, we have the relations

$$W_1(u_1(z)) = W'(z) - (I - P)y'(Pz) = W'y'(Pz) - (I - P)W'y'(Pz) = PW'y'(Pz),$$

and thus $W_1(u_1(\overline{\Omega})) \subset S$. Also, $u_1(\Omega) \cap S = \text{int } P\Gamma$, and $W_1(x) = PW'y'(x)$ for all x in $P\Gamma$. Using the definition of the Leray-Schauder index, we see that

$$i_{LS}(W', \Omega) = i_{LS}(W_1, u_1(\Omega)) = i(PW'y', \text{int } P\Gamma) = n(\Gamma, W', P).$$

This completes the proof of Theorem 1.

When giving an application to boundary value problems for ordinary differential equations, Cesari [5, Section 14] uses a sequence $\{P_i\}$ of orthogonal projections (B in his example is a Hilbert space) whose ranges S_i have the property that $S_i \subseteq S_{i+1}$. He shows that for each particular problem there is an integer M such that whenever $i \geq M$, his method applies with the projection P_i. It is the content of the next theorem that in this situation the Cesari numbers associated with the P_i for $i \geq M$ are equal.

THEOREM 2. *Let* Γ_1 *and* Γ_2 *be the closures of the bounded open sets* Ω_1 *and* Ω_2, *respectively, and suppose that* $W: \Gamma_1 \cup \Gamma_2 \to B$ *has no fixed points outside* $\Gamma_1 \cap \Gamma_2$. *Let* P_1 *and* P_2 *be projections with* $P_1 P_2 = P_2 P_1 = P_1$. *Suppose (for* $i = 1, 2$) *that* $n(\Gamma_i, W, P_i)$ *is defined and* $y_i(x)$ *is in* Ω_i *whenever* x *is in* $\text{int } P_i \Gamma_i$. *Finally, assume that* $\|b - P_2 b\| \leq \|b - z\|$ *for every* z *in the range of* P_2 *and for every* b *in* B. *Then* $n(\Gamma_1, W, P_1) = n(\Gamma_2, W, P_2)$.

Proof. Let $\Gamma_2' = \{x \in \Gamma_2;\ y_1(P_1 x) \text{ is in } \Gamma_2 \text{ and } y_2(P_2 x) \text{ is in } \Gamma_1\}$. Notice that if x is in Γ_2', then each point of $(P_2^{-1}(P_2 x)) \cap \Gamma_2$ is in Γ_2'. First we shall prove that $n(\Gamma_2', W, P_2)$ exists and is equal to $n(\Gamma_2, W, P_2)$. Since Γ_2' is closed and $y_2(P_2 \Gamma_2)$ is compact, $y_2(P_2 \Gamma_2) \cap \Gamma_2'$ is compact, so that

$$P_2 \Gamma_2' = P_2(y_2(P_2 \Gamma_2) \cap \Gamma_2')$$

is compact. For every x in $P_2 \Gamma_2'$, x is in $P_2 \Gamma_2$ and $(P_2^{-1} x) \cap \Gamma_2' = (P_2^{-1} x) \cap \Gamma_2$ is closed. Moreover, $T_2 = P_2 + (I - P_2)W$ is a contraction from $(P_2^{-1} x) \cap \Gamma_2'$ into itself with fixed point $y_2(x)$.

Assume now, if possible, that x is a fixed point of $P_2 W y_2$ on the boundary of $P_2 \Gamma_2'$. Then (by Remark 1 above) $y_2(x) = Wy_2(x)$ and thus $y_2(x)$ is in $\Gamma_1 \cap \Gamma_2$. Since $P_2 y_2(x)$ is on the boundary of $P_2 \Gamma_2'$, it follows from the interior mapping theorem applied to P_2 that $y_2(x)$ is on the boundary of Γ_2'. In the proof of Theorem 1, we saw that W has no fixed points on the boundary of Γ. Similarly, we see here that $y_2(x)$ is in $\Omega_1 \cap \Omega_2$. But $y_2(x) \in \Omega_1 \cap \Omega_2$ implies that $y_2(x)$ is not on the boundary of Γ_2', since it is in the interior of each of the sets Γ_2, $(y_1 P_1)^{-1} \Gamma_2$, and $(y_2 P_2)^{-1} \Gamma_1$. This contradiction proves that the assumption at the beginning of this paragraph is not tenable. Hence $i(P_2 W y_2, \text{int } P_2 \Gamma_2')$ exists. Now, if $P_2 W y_2$ had a fixed point x outside of $P_2 \Gamma_2'$, then (by Remark 1 above) $y_2(x) = y_1(P_1 x)$ would be a fixed point of W outside of Γ_2', which is impossible. Thus

$$n(\Gamma_2', W, P_2) = i(P_2 W y_2, \text{int } P_2 \Gamma_2') = i(P_2 W y_2, \text{int } P_2 \Gamma_2) = n(\Gamma_2, W, P_2).$$

For $i = 1, 2$, let S_i be the finite-dimensional space that is the range of P_i. The assumption $P_1 P_2 = P_2 P_1 = P_1$ implies that $S_1 \subset S_2$. Let $\bar{x}: P_1 \Gamma_2' \to P_2 \Gamma_2'$ be defined by $\bar{x}(x) = P_2 y_1(x)$. Set $\Gamma_1' = \Gamma_1$. Then $y_2(\bar{x}(x)) = y_1(x)$ for all x in $P_1 \Gamma_2'$, since (for $i = 1, 2$) $y_i(x)$ is the only point of $(P_i^{-1} x) \cap \Gamma_i'$ whose displacement $Wy_i(x) - y_i(x)$ belongs to S_i, and since $S_1 \subset S_2$.

For every t in $[0, 1]$, define $u_t: S_2 \cap P_1^{-1}(P_1 \Gamma_2') \to S_2$ by the formula $u_t(z) = z - t(I - P_1)\bar{x}(P_1 z)$. Each u_t is one-to-one, and each u_t^{-1} is continuous, by reasoning similar to that in Theorem 1.

For every t in $[0, 1]$, let T_t: $u_t(P_2 \Gamma_2') \to S_2$ be defined by

$$T_t(z) = z + (P_2 Wy_2 u_t^{-1}(z) - u_t^{-1}(z)).$$

The homotopy carries the range of $\bar{x}$ into S_1, and it introduces no fixed points of T_t on the boundary of $u_t(P_2 \Gamma_2')$. Let $\theta_2 = \text{int } P_2 \Gamma_2'$. Then

$$i(P_2 Wy_2, \theta_2) = i(T_0, u_0(\theta_2)) = i(T_1, u_1(\theta_2)).$$

For every t in $[1, 2]$, let T_t: $u_1(P_2 \Gamma_2') \to S_2$ be defined by

$$T_t(z) = (1 - (t - 1)) T_1(z) + (t - 1) P_1 T_1(z). \tag{1}$$

In the next paragraph, we shall prove that this homotopy introduces no fixed points on the boundary of $u_1(P_2 \Gamma_2')$; if we assume this for the moment, it follows that

$$i(P_2 Wy_2, \theta_2) = i(T_1, u_1(\theta_2)) = i(T_2, u_1(\theta_2)).$$

But for x in $\overline{u_1(\theta_2)} \cap S_1$,

$$T_2(x) = P_1 T_1(x) = P_1 [x + (P_2 Wy_2 \bar{x}(x) - \bar{x}(x))] = P_1 P_2 Wy_2(\bar{x}(x)) = P_1 Wy_1(x).$$

Hence, by the reduction theorem,

$$i(P_2 Wy_2, \theta_2) = i(T_2, u_1(\theta_2)) = i(P_1 Wy_1, u_1(\theta_2) \cap S_1).$$

But if $P_1 Wy_1(x) = x$, then $y_1(x) = Wy_1(x)$ (by Remark 1 above); hence $y_1(x)$ is in $\Gamma_1 \cap \Gamma_2$ and therefore in Γ_2'. Therefore $P_2 Wy_2(\bar{x}(x)) = \bar{x}(x)$ (by Remark 1), and thus (since $i(P_2 Wy_2, \theta_2)$ is defined), $\bar{x}(x)$ is in θ_2, so that x is in $u_1(\theta_2) \cap S_1$. Therefore

$$i(P_2 Wy_2, \theta_2) = i(P_1 Wy_1, u_1(\theta_2) \cap S_1) = i(P_1 Wy_1, \text{int } P_1 \Gamma_1).$$

It remains only to prove that homotopy (1) above introduces no fixed point z on the boundary. This is equivalent to the assertion that no point $r = u_1^{-1}(z)$ on the boundary of $P_2 \Gamma_2'$ lies on the line segment joining $P_2 Wy_2(r)$ and $\bar{x}(P_1 r)$, each of the three points having the same P_1-projection. Suppose, if possible, that such an r exists. Let $x = P_1 r$. We see that $r = \bar{x}(x)$ implies that $P_2 Wy_2(r) = r$, and this contradicts the fact that $i(P_2 Wy_2, \text{int } P_2 \Gamma_2')$ is defined. Therefore $r \neq \bar{x}(x)$.

Because $P_1 + (I - P_1)W$ is a contraction mapping on $(P_1^{-1} x) \cap \Gamma_1$, with fixed point $y_1(x) = y_2(\bar{x}(x))$, and since $y_2(r) \neq y_2(\bar{x}(x))$, the point

$$A_1 = P_1 y_2(r) + (I - P_1) Wy_2(r)$$

is closer to

$$A_2 = y_1(x) = P_1 y_1(x) + (I - P_1) Wy_1(x)$$

than is the point

$$A_3 = y_2(r) = r + (I - P_2) Wy_2(r).$$

Now, for $i = 1, 2, 3$, let $A_i = A_{i1} + A_{i2} + A_{i3}$, where

$$P_1 A_i = A_{i1}, \qquad (P_2 - P_1) A_i = A_{i2}, \qquad A_i - P_2 A_i = A_{i3}.$$

Clearly,

$$A_{11} = A_{21} = A_{31} = x, \qquad A_{13} = A_{33} = (I - P_2)Wy_2(r), \qquad \|A_1 - A_2\| < \|A_3 - A_2\|.$$

But $r = A_{31} + A_{32}$ is on the line segment joining $P_2 Wy_2(r) = A_{11} + A_{12}$ and $\bar{x}(P_1 r) = A_{21} + A_{22}$, so that A_{32} is on the line segment joining A_{12} and A_{22}. Let $A_{32} = \lambda_0 A_{12} + (1 - \lambda_0)A_{22}$ with $0 \leq \lambda_0 \leq 1$. Then $\|A_1 - A_2\| < \|A_3 - A_2\|$ implies that

$$\|A_{12} + A_{13} - A_{22} - A_{23}\| < \|\lambda_0 A_{12} + (1 - \lambda_0)A_{22} + A_{33} - A_{22} - A_{23}\|$$
$$= \|\lambda_0(A_{12} - A_{22}) + A_{13} - A_{23}\|.$$

Let q be a real number between the two norms above. Then the set

$$S = \{b \in B;\ \|b - (A_{23} - A_{13})\| < q\}$$

contains $A_{12} - A_{22}$ in S_2, and hence it must also contain $P_2(A_{23} - A_{13}) = 0$. Since S is convex, it must contain $\lambda_0(A_{12} - A_{22})$, contrary to the definitions of q and S. This contradiction completes the proof of the theorem.

4. COMPARISON OF NOTATIONS

In this section, we shall compare the notation of this paper with that of Cesari [5]. The notations I, P, and T are common to both. The Banach space B and its finite-dimensional subspace S in this paper are Cesari's S and S_0, respectively. In [5], Cesari studies the equation $Kx = 0$, where $K = E - N$, E is linear (but not necessarily bounded), and N is nonlinear (E and N are not necessarily everywhere defined). The operator E is assumed to have a "partial inverse" H, and the mapping T is then defined by

$$T = P + H(I - P)N. \tag{2}$$

It is then shown [5, Theorem (ii)] that under suitable assumptions, T is for each x^* in V a contraction of $S_0^*(x^*)$ into itself (the symbol S_0^* is used in [5] when x^* is understood), with fixed point $\mathfrak{T}(x^*)$. V, T, $S_0^*(x^*)$, and $\mathfrak{T}(x^*)$ in [5] correspond to $P\Gamma$, T, $(P^{-1}x^*) \cap \Gamma$, and $y(x^*)$ in this paper. $\mathfrak{T}(x^*)$ satisfies the condition $K\mathfrak{T}(x^*) = PK\mathfrak{T}(x^*)$, so that $K\mathfrak{T}(x^*) = 0$ if and only if

$$PK\mathfrak{T}(x^*) = 0. \tag{3}$$

This is Cesari's determining equation.

In [5], the Γ of this paper would be denoted by $\bigcup S_0^*(x^*)$, where the union is to be taken over all x^* in V. If we write

$$W(z) = Pz - PK\mathfrak{T}(Pz) + H(I - P)N(z) \tag{4}$$

for z in Γ, then, since $PH = 0$ and $(I - P)P = 0$,

$$Tz = Pz + (I - P)Wz = Pz + (I - P)H(I - P)Nz = Pz + H(I - P)Nz, \tag{5}$$

and this coincides with (2). On the other hand, if we use y(x) for $\mathfrak{X}(x)$ in (4), then

$$W(y(x)) = Py(x) + H(I - P)Ny(x) - PKy(Py(x));$$

since (5) implies that $y(x) = Ty(x) = Py(x) + H(I - P)Ny(x)$, it follows that $y(x) - W(y(x)) = PKy(Py(x))$, and finally, by applying P to both sides, we find that

$$x - PWy(x) = PK\mathfrak{X}(x).$$

Thus the determining equation of the present paper (Section 2) coincides with the determining equation of Cesari [5, (8)]. Moreover, if μ denotes the degree used by Cesari in [5], then

$$n(\Gamma, W, P) = i(PWy, \text{int } P\Gamma) = d(I - PWy, \text{int } P\Gamma, 0) = d(PK\mathfrak{X}, \text{int } V, 0) = \mu.$$

5. CONCLUDING REMARKS

In 1950, J. Cronin [10] introduced a "multiplicity" which is this paper's $n(\Gamma, W, P)$, and she proved that if W is completely continuous and "differentiably close" to a completely continuous linear operator, then this number is the same as the Leray-Schauder index of W. Theorem 1 of this paper generalizes Cronin's result; its proof requires no "differentiability" hypothesis.

The theoretical connection demonstrated in Theorem 1 depends on the construction of a map W' satisfying the Leray-Schauder hypothesis; but in order to construct the map W', we must know Cesari's H and $\mathfrak{X}$ as well as the given maps E, N, and P. Therefore the connection established by Theorem 1 is not a reduction of the Cesari method to the Leray-Schauder method.

REFERENCES

1. S. Bancroft, J. Hale, and D. Sweet, *Alternative problems for nonlinear functional equations* (to appear).

2. L. Cesari, *Existence theorems for periodic solutions of nonlinear Lipschitzian differential systems and fixed point theorems.* Contributions to the Theory of Nonlinear Oscillations, Vol. 5, pp. 115-172. Princeton Univ. Press, Princeton, N. J., 1960.

3. ———, *Functional analysis and periodic solutions of nonlinear differential equations.* Contributions to Differential Equations, 1 (1963), 149-187.

4. ———, *Periodic solutions of hyperbolic partial differential equations.* Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics, pp. 33-57. Academic Press, New York, 1963.

5. ———, *Functional analysis and Galerkin's method.* Michigan Math. J. 11 (1964), 385-414.

6. ———, *A criterion for the existence in a strip of periodic solutions of hyperbolic partial differential equations.* Rend. Circ. Mat. Palermo (2) 14 (1965), 95-118.

7. ———, *Existence in the large of periodic solutions of hyperbolic partial differential equations.* Arch. Rational Mech. Anal. 20 (1965), 170-190.

8. L. Cesari, *A nonlinear problem in potential theory* (in preparation).

9. L. Cesari and J. K. Hale, *A new sufficient condition for periodic solutions of weakly nonlinear differential systems.* Proc. Amer. Math. Soc. 8 (1957), 757-764.

10. J. Cronin, *Branch points of solutions of equations in Banach space.* Trans. Amer. Math. Soc. 69 (1950), 208-231.

11. ———, *Fixed points and topological degree in nonlinear analysis.* Mathematical Surveys, No. 11. Amer. Math. Soc., Providence, R. I. (1964).

12. R. A. Gambill and J. K. Hale, *Subharmonic and ultraharmonic solutions for weakly non-linear systems.* J. Rational Mech. Anal. 5 (1956), 353-394.

13. J. K. Hale, *On the behavior of the solutions of linear periodic differential systems near resonance points.* Contributions to the Theory of Nonlinear Oscillations, Vol. 5, pp. 55-89. Princeton Univ. Press, Princeton, N. J., 1960.

14. ———, *On the stability of periodic solutions of weakly nonlinear periodic and autonomous differential systems.* Contributions to the Theory of Nonlinear Oscillations, Vol. 5, pp. 91-113. Princeton Univ. Press, Princeton, N. J., 1960.

15. ———, *Oscillations in nonlinear systems.* McGraw-Hill, New York-Toronto-London, 1963.

16. ———, *Periodic solutions of a class of hyperbolic equations containing a small parameter.* Arch. Rational Mech. Anal. 23 (1966), 380-398.

17. H. W. Knobloch, *Eine neue Methode zur Approximation periodischer Lösungen nicht-linearer Differentialgleichungen zweiter Ornung.* Math. Z. 82 (1963), 177-197.

18. ———, *Remarks on a paper of L. Cesari on functional analysis and nonlinear differential equations.* Michigan Math. J. 10 (1963), 417-430.

19. J. Leray and J. Schauder, *Topologie et équations fonctionelles.* Ann. Sci. École Norm. Sup. (3) 51 (1934), 45-78.

20. J. Locker, *An existence analysis for nonlinear non-selfadjoint boundary value problems of ordinary differential equations.* Thesis, Univ. of Michigan, 1965.

21. S. Williams, *On the Cesari fixed point method in a Banach space.* Thesis, California Institute of Technology, 1967.

The University of Michigan
Ann Arbor, Michigan 48104