COMPLETE THEORIES OF ALGEBRAICALLY CLOSED FIELDS WITH DISTINGUISHED SUBFIELDS

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Two models M_1 and M_2 for an applied first order predicate logic L are said to be *elementarily equivalent*, in symbols $M_1 \equiv M_2$ (see [13]), if every sentence of L that holds in one of the models holds in the other. It is well known that *any two algebraically closed fields of the same characteristic are elementarily equivalent.* (For fields of characteristic zero, Tarski proved this result in [14; note 16]; see also [12]. In [14], he also stated the result without proof for fields of prime characteristic. The details of his proof have not been published, but can easily be reconstructed from his remarks. For a proof of the theorem for fields of arbitrary characteristic, see Robinson [8] or Vaught [17]. Tarski also proved that any two real closed fields are elementarily equivalent [14].)

Tarski [14] mentioned several problems which arise when a new unary predicate symbol P is added to the formal system for field theory so that one can study couples (A, B) of fields, where B is a subfield of A and the set of elements of B corresponds to the predicate P. The following results concerning couples (A_1, B_1) and (A_2, B_2) of fields are known from the literature. In each case A_1 and A_2 are assumed to be algebraically closed fields of the same characteristic.

I. *If* $A_1 = B_1$ *and* $A_2 = B_2$, *then* $(A_1, B_1) \equiv (A_2, B_2)$. (This follows at once from the fact that $A_1 \equiv A_2$.)

II. *If* B_1, B_2 *are algebraically closed fields and* $A_1 \neq B_1$, $A_2 \neq B_2$, *then* $(A_1, B_1) \equiv (A_2, B_2)$ (A. Robinson [9; Th. 5.3]).

III. *If* B_1, B_2 *are real closed fields and* A_1, A_2 *are finite algebraic extensions, then* $(A_1, B_1) \equiv (A_2, B_2)$ (Tarski [14; p. 43]).

IV. *If* B_1, B_2 *are real closed fields and* A_1, A_2 *are infinite extensions, then* $(A_1, B_1) \equiv (A_2, B_2)$. (This is a consequence of Robinson's Theorem 4.9 in [9].)

V. *If* B_1, B_2 *are elementarily equivalent to the field of rational numbers, then* $(A_1, B_1) \equiv (A_2, B_2)$. (Robinson proved this in [10], and pointed out that the result remains true if in place of "the field of rational numbers" we read either "the semiring of natural numbers" or "a finite algebraic extension of the field of rational numbers".)

D. Scott asked the following more general question:

Suppose B_1, B_2 *are fields that are elementarily equivalent to each other, and* A_1, A_2 *are algebraically closed extensions of* B_1, B_2. *Under what conditions are* (A_1, B_1) *and* (A_2, B_2) *elementarily equivalent?*

The purpose of this paper is to prove the following theorem, which, in conjunction with the results I-IV above, answers Scott's question.

THEOREM A. *Suppose* B_1, B_2 *are fields that are elementarily equivalent to each other, and suppose* A_1, A_2 *are algebraically closed extensions of* B_1, B_2. *Suppose further that the field* B_1 *is neither algebraically closed nor real closed. Then the couples* (A_1, B_1) *and* (A_2, B_2) *are elementarily equivalent.*

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Notice that Theorem A is a generalization of the result V above of Robinson. It will be seen that the argument that we shall use to prove Theorem A can also be used to prove Robinson's Theorems II and IV above; however, our methods of proof appear to be quite different from those of Robinson in [9], [10]. Our theorem was first announced in [3], where it was indicated that the limit ultrapower construction was used in the proof. (Limit ultrapowers are discussed, for example, in [4] and [6].) The proof that we shall give here uses the more convenient notion of a special model, due to Morley and Vaught [7], in place of limit ultrapowers. The special models, together with some classical results of Steinitz [11] in the theory of fields, are our chief tools. The theorems of Steinitz have previously been used by Robinson [8] and Vaught [17] to prove that any two algebraically closed fields of the same characteristic are elementarily equivalent.

We mention in passing the natural generalization of Scott's question that arises where A_1, A_2 are only elementarily equivalent to each other and not necessarily algebraically closed. Robinson in [9] has proved the following special theorem: *if all of* A_1, A_2, B_1, B_2 *are real closed fields and* B_1, B_2 *are dense proper subfields of* A_1, A_2, *then* $(A_1, B_1) \equiv (A_2, B_2)$. The general question of Scott seems to be extremely difficult, but an attempt to obtain further special results is likely to be fruitful.

In Section 1 we introduce our notation, and in Section 2 we state the definition of a special model and some known results that we shall need. Some known results from field theory are stated in Section 3. The proof of Theorem A is given in Section 4. Some generalizations of Theorem A and some results that are analogous to Theorem A but do not generalize it are given in Section 5. The proofs of the results in Section 5 are easy modifications of the proof of Theorem A. Finally, in Section 6 we shall give a restatement of our results in terms of complete theories.

1. PRELIMINARIES

We shall work with a first order predicate logic L that is similar to the formal system developed in [14]. L has individual variables

$$u, u_0, u_1, \cdots, v, v_0, v_1, \cdots, w, w_0, w_1, \cdots,$$

an identity symbol =, logical connectives $\wedge, \vee, \neg$ (and, or, not), quantifiers $\forall, \exists$ (for all, there exists), individual constants 1, 0, binary operation symbols $+, \cdot$, and a unary predicate symbol P. We refer to [14, pp. 6-15] for the definitions of *term*, *atomic formula, formula, free variable, sentence, polynomial*, and abbreviations such as $u - v$, nu, u^n, $u_1 + \cdots + u_n$. The letter n will always be used for an arbitrary natural number.

For a detailed exposition of the theory of models we refer to [13]. The discussion below, however, should enable one with sufficient background in algebra to read this paper.

By a *model* (*for* L) we mean a 6-tuple

$$\langle M, N, +, \cdot, 0, 1 \rangle$$

where M is a non-empty set, $N \subset M$ (inclusion in the wide sense), where + and are binary operations on $M \times M$ into M, and where 1, 0 are elements of M. No confusion will arise from our convention of using the same symbols $+, \cdot, 1, 0$ as names for the operation symbols and constants of L and as names for their interpretations in a model. We shall also adopt the convention of using the same symbol for the set of elements of a model as for the model itself. By the *power* (or *cardinality*) of a model M we shall mean the power of the set M of all its elements.

If ϕ is a sentence, then the following three statements all have the same meaning: M is a model of ϕ; M satisfies ϕ; ϕ holds in M. By a *theory* (*in* L) we simply mean a set of sentences (of L). If M is a model of every sentence ϕ in a theory Γ, we say that M is a model of Γ. A theory Γ will be said to be *consistent* if it has a model. The following *compactness theorem* is a consequence of Gödel's completeness theorem; see [2], for example.

If every finite subset of a theory Γ *is consistent, then* Γ *is consistent.*

We shall write $M \equiv M'$ if M and M' are elementarily equivalent to each other. By the *theory of* M, in symbols Th(M), we mean the set of all sentences that hold in M. Thus $M \equiv M'$ if and only if Th(M) = Th(M'). It is easily seen that if $M \equiv M'$ and M is finite, then M' is also finite and, in fact M and M' are isomorphic (see [13], for example).

We shall also consider the formal system L_0 that has all the symbols of L with the exception of the predicate symbol P, and the formal systems L(C) that are obtained from L by adding a set C of new individual constants c. The notions introduced above will be applied to models for L_0 and L(C) as well as to models for L. Notice that any formula of L_0 is also a formula of L, and any formula of L is also a formula of L(C).

By a field A we shall understand a model

$$A = \langle A, +, \cdot, 1, 0\rangle$$

for L_0 that satisfies the field axioms (see [16], for example). If A is a field and B is a subalgebra of A, then by the couple (A, B) we shall mean the model

$$(A, B) = \langle A, B, +, \cdot, 1, 0\rangle$$

for L. Thus the couples (A, B) are models for L which satisfy the field axioms and also the axioms which state that B is closed under the operations of A. It is not difficult to see that if $(A_1, B_1) \equiv (A_2, B_2)$, then $A_1 \equiv A_2$ and $B_1 \equiv B_2$. On the other hand, it could well be that $A_1 \equiv A_2$ and $B_1 \equiv B_2$ but not $(A_1, B_1) \equiv (A_2, B_2)$, for the embeddings of B_1 in A_1 and of B_2 in A_2 may differ. We assume a familiarity with the theory of fields, but we shall state some standard definitions and results as we need them.

If M is a model for L, we denote by $(M, m_c)_{c\in C}$ the model

$$(M, m_c)_{c\in C} = \langle M, N, +, \cdot, 0, 1, m_c\rangle_{c\in C}$$

for L(C).

2. SPECIAL MODELS

In this section we introduce the important notion of a special model, which is due to Morley and Vaught [7]. We shall also state without proof those theorems from [7] concerning special models that we shall need. First we develop some set-theoretic notation.

The letter γ will always denote an arbitrary ordinal number, and α, β will denote arbitrary cardinals. We identify cardinals with initial ordinals; thus ω is both the first infinite cardinal and the first infinite ordinal. The confinality of α is written $\mathrm{cf}(\alpha)$ and is defined as the least cardinal β such that α can be represented in the form $\alpha = \Sigma_{\xi<\beta}\gamma_\xi$, where each γ_ξ is a cardinal less than α. Let

$$\alpha^* = \sum_{\beta<\alpha} 2^\beta .$$

The following lemma is well known (see [7]).

LEMMA 2.1. *There exist arbitrarily large cardinals* α *such that* $\alpha = \alpha^*$.

It is easily seen that the statement "$\alpha = \alpha^*$ for all infinite α" is equivalent to the generalized continuum hypothesis. In this paper, however, we shall not assume the continuum hypothesis. Two cardinals α such that $\alpha = \alpha^*$ are $\alpha = \omega$ and

$$\alpha = \omega + 2^\omega + 2^{2^\omega} + \cdots .$$

A set I of subsets of a set X is said to be an ideal of subsets of X if $x, y \in I$ and $z \subset x \cup y$ implies $z \in I$; I is said to be α-complete if, wherever $J \subset I$ and J has power less than α, then $\bigcup J \in I$.

In Definition 2.2 below we shall define the notion of a special model. The reader will notice that Definition 2.2 is somewhat more involved than the properties of special models stated in Lemmas 2.3 to 2.5. Furthermore, we shall never have occasion in this paper to use the definition of special models, but shall require only the properties 2.3 to 2.5. Thus the reader may skip the statement of Definition 2.2 and proceed at once to Lemmas 2.3 to 2.5.

DEFINITION 2.2. *A model* M *for* L *is said to be special if* M *is of infinite power* α, *and there exists an* I *such that:*

(i) I *is a* $\mathrm{cf}(\alpha)$*-complete ideal of subsets of* M;

(ii) *each member* X *of* I *has power less than* α;

(iii) M *is the union of an ascending chain of members of* I; *and*

(iv) *whenever* $X \in I$, M' *is another model for* L *such that*

$$(M, m)_{m \in X} \equiv (M', m)_{m \in X},$$

$X \subset Y \subset M'$, *and* Y *has power less than* α, *then there exists a function* f *on* Y *onto a member of* I *such that* $fm = m$ *for all* $m \in X$, *and*

$$(M, fm)_{m\in Y} \equiv (M', m)_{m\in Y} .$$

The following result of Morley and Vaught was not explicitly stated in [7] but follows easily from the definition. For a proof see [5, Lemma 1].

LEMMA 2.3. *Suppose that* M *is a special model of power* α, *that* $M' \equiv M$, *and that* C *is a subset of* M' *of power at most* α. *Then there exists a function* f *on* C *into* M *such that*

$$(M, fc)_{c\in C} \equiv (M', c)_{c\in C} .$$

The main theorem of [7] (its Th. 3.5) is the following.

LEMMA 2.4. (a) *If* M *is an infinite model for* L *and if* $\omega < \alpha = \alpha^*$, *then there exists a special model* $M' \equiv M$ *of power* α.

(b) *Any two elementarily equivalent special models of the same power are isomorphic.*

Definition 2.2 and Lemmas 2.3 and 2.4 apply not only to models for L, but also to any logic L' which is like L but has a different collection of predicate and operation symbols (for example, L_0 or L(C)); in case the logic L' has an uncountable number β of symbols, Lemma 2.4 (a) requires the additional hypothesis that $\alpha > \beta$. We also need the following result, which is a special case of Theorem 3.7 of [7].

LEMMA 2.5. (a) *If* (A, B) *is a special model of power* α, *then* B *is either finite or is also a special model of power* α.

(b) *If* $(M, m_c)_{c \in C}$ *is a special model of power* α, *then so is* M.

3. LEMMAS FROM FIELD THEORY

Let us first recall some algebraic facts, which may be found, for example, in [11, pp. 113-125]. It follows from Zorn's lemma that if A is a field and B is a subfield of A, then there exists a maximal set C of elements of A that are algebraically independent over B; such a set C is called an *algebraic basis of* A *over* B. It is known that any two algebraic bases of A over B have the same power, and that power is called the *degree of transcendence of* A *over* B. The field A is an algebraic extension of B if and only if A is an extension of transcendence degree zero over B. We need the following classical theorem of Steinitz.

Let A_1, A_2 *be algebraically closed extensions of the fields* B_1, B_2 *such that the degree of transcendence of* A_1 *over* B_1 *is equal to that of* A_2 *over* B_2, *and let* g *be an isomorphism of* B_1 *onto* B_2. *Then* g *can be extended to an isomorphism* f *of* A_1 *onto* A_2.

If, moreover, h *is a one-to-one function on an algebraic basis* C_1 *of* A_1 *over* B_1 *onto an algebraic basis* C_2 *of* A_2 *over* B_2, *then* f *may be chosen so that* $h \subset f$.

A special case of the above result is that, for any two fields B_1 and B_2, each isomorphism g of B_1 onto B_2 can be extended to an isomorphism f of the algebraic closure of B_1 onto that of B_2.

The following lemma, which was pointed out to the author by Bjarni Jónsson, is a corollary of the theorem of Artin and Schreier in [1].

LEMMA 3.1. *Let* B *be a field which is neither real closed nor algebraically closed, and let* A *be an algebraically closed extension of* B. *Then for each natural number* n *there exists an element* $a \in A$ *which is not algebraic of degree less than or equal to* n *over* B.

Proof. We may assume that A is an algebraic extension of B. By the theorem of Artin and Schreier in [1], A cannot be a finite algebraic extension of B. Suppose first that B is a perfect field [16, pp. 121-125]. Then for any element $a \in A$, there exists an $a' \in A - B(a)$; since B is perfect there exists an $a'' \in A$ such that $B(a'') = B(a, a')$. The degree of a'' over B must be greater than the degree of a, and the desired result follows. On the other hand, if B is not perfect then B has prime characteristic p, and for some $b \in B$ the p-th root of b is not in B; it follows that, for each n, the p^n-th root of b has degree p^n over B.

In the lemma below we collect some useful facts that follow easily from the definitions involved.

LEMMA 3.2. (i) *If* A_1 *is a field and* $A_2 \equiv A_1$, *then* A_2 *is a field.*

(ii) *If* A_1 *is an algebraically closed field and* $A_2 \equiv A_1$, *then* A_2 *is algebraically closed.*

(iii) *If* A_1 *is a real closed field and* $A_2 \equiv A_1$, *then* A_2 *is real closed.*

4. THE MAIN THEOREM

LEMMA 4.1. *Suppose that*

(i) A *is a field and* B *is a subfield of* A;

(ii) (A, B) *is a special model of power* α;

(iii) *for every natural number* n, *there exists an element* $a \in A$ *which is not algebraic of degree at most* n *over* B.

Then the degree of transcendence of A *over* B *is* α.

Proof. Since A is of power α, the degree of transcendence of A over B is at most α.

Let us consider a set C of α new individual constants. For each pair of natural numbers n, k, let $\theta_{n,k}(v_0, \cdots, v_k)$ be the formula of L which states that "no polynomial of degree at most n and of $k+1$ variables, with coefficients in P, has $v_0, \cdots, v_k$ as a root." We may now form the set $\triangle(C)$ of all sentences

$$\theta_{n,k}(c_0, \cdots, c_k)$$

of $L(C)$ such that $c_0, \cdots, c_k$ are distinct elements of C. Then $(A', B', a_c)_{c\in C}$ is a model of $\triangle(C)$ if and only if the elements a_c, $c \in C$, are algebraically independent over B'.

We shall show that the theory

$$\Gamma = \mathrm{Th}((A, B)) \cup \triangle(C)$$

is consistent. If $\theta_1, \theta_2 \in \triangle(C)$, then it is easily seen that $\theta_1 \wedge \theta_2$ is a consequence of some member of $\triangle(C)$. Thus to show that Γ is consistent, it suffices to prove that, for each n and k, there exists a k-tuple $\langle a_0, \cdots, a_k\rangle \in A^k$ which satisfies the formula $\theta_{n,k}(v_0, \cdots, v_k)$ in the model (A, B), that is, which is a root of no polynomial of degree less than or equal to n with coefficients in B. For this purpose, let $m = (n+1)^{kn}$. By (iii) there is an element $a \in A$ which is not algebraic of degree less than or equal to m over B. It follows that no polynomial of $k+1$ variables, of degree at most n, and with coefficients in B, has the $(k+1)$-tuple

$$\langle a, a^{n+1}, a^{(n+1)^2}, \cdots, a^{(n+1)^k}\rangle$$

as a root. Thus we may take $a_j = a^{(n+1)^j}$ for $j = 0, 1, \cdots, k$, and we have shown that Γ is consistent.

By the Compactness Theorem, Γ has a model $(A', B', c)_{c\in C}$. Since

$$\mathrm{Th}((A, B)) \subset \Gamma,$$

it follows that $(A', B') \equiv (A, B)$. Hence by Lemma 2.3, there exists a function f on C into A such that

$$(A, B, fc)_{c \in C} \equiv (A', B', c)_{c \in C}.$$

Then, since $\triangle(C) \subset \Gamma$, $(A, B, fc)_{c \in C}$ is a model of $\triangle(C)$, and the elements fc, $c \in C$ are algebraically independent over B. Recalling that C is of power α, we conclude that the degree of transcendence of A over B is at least α.

LEMMA 4.2. *Suppose that, for* $i = 1, 2$, B_i *is a field,* A_i *is an algebraically closed extension of* B_i, *and there exists an element* $a \in A_i$ *that is not algebraic of degree at most* 2 *over* B_i. *If* $B_1 \equiv B_2$, *then* $(A_1, B_1) \equiv (A_2, B_2)$.

Proof. We first observe that, for $i = 1, 2$ and for all n, there exist elements $a \in A_i$ that are not algebraic of degree at most n over B_i. In case B_i is real closed or algebraically closed, this follows because A_i must be a transcendental extension; otherwise we apply Lemma 3.1. By Lemma 2.1, there is a cardinal $\alpha > \omega$ such that $\alpha = \alpha^*$. Then by Lemma 2.4 (a), there exist special models $(A_1', B_1') \equiv (A_1, B_1)$ and $(A_2', B_2') \equiv (A_2, B_2)$ of power α. Then the hypotheses (i) to (iii) of Lemma 4.1 are satisfied when $(A, B) = (A_1', B_1')$ and when

$$(A, B) = (A_2', B_2').$$

By Lemma 4.1, the degrees of transcendence of A_1' over B_1' and of A_2' over B_2' are both α. Applying Lemma 2.5 (a), we see that B_1' and B_2' are either finite fields or special fields of power α. Since $B_1 \equiv B_2$, $B_1 \equiv B_1'$, and $B_2 \equiv B_2'$, it follows that $B_1' \equiv B_2'$. Thus B_1', B_2' are either both finite or both infinite. In either case, there is an isomorphism f on B_1' onto B_2'; in the infinite case we use Lemma 2.4 (b). By Lemma 3.1 (b), A_1' and A_2' are both algebraically closed. Therefore we may apply the theorem of Steinitz to show that f may be extended to an isomorphism g on A_1' onto A_2'. It follows that (A_1', B_1') and (A_2', B_2') are isomorphic, and hence elementarily equivalent, so that $(A_1, B_1) \equiv (A_2, B_2)$.

Theorem A now follows at once from Lemmas 4.2 and 3.1.

5. SOME GENERALIZATIONS

In this section we shall briefly indicate three directions in which Theorem A can be modified. The first result is analogous to Theorem A, while the next two results generalize Theorem A.

THEOREM 5.1. *Suppose that* B_1, B_2 *are elementarily equivalent fields with algebraically closed extension fields* A_1, A_2 *respectively. Let* C *be a non-empty set of new individual constants, and suppose that for* $i = 1, 2$ *the elements* a_{ic}, $c \in C$, *of* A_i *are algebraically independent over* B_i. *Then*

$$(A_1, B_1, a_{1c})_{c \in C} \equiv (A_2, B_2, a_{2c})_{c \in C}.$$

Proof. By Lemma 2.1 there exists a cardinal $\alpha > \omega$ such that $\alpha = \alpha^*$ and $\overline{\overline{C}} < \alpha$. By Lemma 2.4 (a), there exist special models

$$(A_i', B_i', a_{ic})_{c \in C} \equiv (A_i, B_i, a_{ic})_{c \in C}$$

of power α for $i = 1, 2$. In the models $(A_i', B_i', a_{ic})_{c \in C}$, the elements a_{ic}, $c \in C$ are algebraically independent over B_i'. By Lemma 2.5 (b), the models (A_i', B_i')

are special; since C is non-empty, it follows from Lemma 4.1 that A_i' has degree of transcendence α over B_i'. By Lemma 2.5 (a), B_1' and B_2' are either special fields of power α or finite. Since $B_1' \equiv B_2'$, it follows by Lemma 2.4 (b) that there is an isomorphism g on B_1' onto B_2'. Since $\overline{\overline{C}} < \alpha$, the function $a_{1c} \to a_{2c}$ on C onto C may be extended to a one-to-one function h on an algebraic basis on A_1' over B_1' onto an algebraic basis on A_2' over B_2'. Then, by the theorem of Steinitz stated in Section 3, g can be extended to an isomorphism f on A_1' onto A_2' such that $fa_{1c} = a_{2c}$ for all $c \in C$. Then f is an isomorphism on the model

$$(A_1', B_1', a_{1c})_{c\in C}$$

onto $(A_2', B_2', a_{2c})_{c\in C}$. It follows that

$$(A_1', B_1', a_{1c})_{c\in C} \equiv (A_2', B_2', a_{2c})_{c\in C},$$

and hence

$$(A_1, B_1, a_{1c})_{c\in C} \equiv (A_2, B_2, a_{2c})_{c\in C}.$$

Our proof is complete.

Recalling that any subalgebra of a field A generates a unique subfield of A, we may state the following theorem.

THEOREM 5.2. *Suppose that, for* $i = 1, 2$, B_i *is a model for* L_0, A_i *is an algebraically closed extension field of* B_i, *and there is an* $a \in A_i$ *that is not algebraic of degree at most* 2 *over the subfield generated by* B_i. *If* $B_1 \equiv B_2$, *then* $(A_1, B_1) \equiv (A_2, B_2)$.

The proof of Theorem 5.2 is a straightforward modification of the proof of Theorem A. It depends on the fact that, since B_i is closed under $+$ and $\cdot$, all elements of the field generated by B_i are of the form $(a - b)/(c - d)$, where $a, b, c, d \in B_i$ and $c \neq d$. As we mentioned in the introduction, the special case of Theorem 5.2 in which B_1, B_2 are elementarily equivalent to the semiring of natural numbers was proved by Robinson in [10].

Let us now consider an arbitrary β-termed sequence $\langle P_\gamma \rangle_{\gamma<\beta}$ of new finitary predicate symbols, and let L_0', L' be the logics obtained from L_0, L by adjoining these new symbols. For each $\gamma < \beta$, let P_γ have $t(\gamma)$ argument places. If A is a field with a subfield B and each S_γ is a $t(\gamma)$-ary relation on B, then we may form the model

$$(B, S_\gamma)_{\gamma<\beta} = \langle B, +, \cdot, 1, 0, S_\gamma \rangle_{\gamma<\beta}$$

for L_0' and the model

$$(A, B, S_\gamma)_{\gamma<\beta} = \langle A, B, +, \cdot, 1, 0, S_\gamma \rangle_{\gamma<\beta}$$

for L'. The following result may be proved by an easy modification of the proof of Theorem A.

THEOREM 5.3. *Assume the hypotheses of Lemma* 4.2, *and suppose that* $S_{i\gamma}$ *is a* $t(\gamma)$*-ary relation on* B_i *for* $\gamma < \beta$, $i = 1, 2$. *If*

$$(B_1, S_{1\gamma})_{\gamma<\beta} \equiv (B_2, S_{2\gamma})_{\gamma<\beta},$$

then

$$(A_1, B_1, S_{1\gamma})_{\gamma<\beta} \equiv (A_2, B_2, S_{2\gamma})_{\gamma<\beta}.$$

The ideas involved in Theorems 5.1 to 5.3 may be combined in the obvious way to obtain the following more general results.

THEOREM 5.4. *Theorem* 5.3 *remains true when we assume the hypothesis of Theorem* 5.2 *in place of the hypotheses of Lemma* 4.2.

THEOREM 5.5. *Suppose that, for* $i = 1, 2$, $(B_i, S_{i\gamma})_{\gamma<\beta}$ *is a model for* L_0', A_i *is an algebraically closed extension field of* B_i, C *is non-empty, and the elements* a_{ic}, $c \in C$, *of* A_i *are algebraically independent over the subfield generated by* B_i. *If*

$$(B_1, S_{1\gamma})_{\gamma<\beta} \equiv (B_2, S_{2\gamma})_{\gamma<\beta},$$

then

$$(A_1, B_1, S_{1\gamma}, a_{1c})_{\gamma<\beta, c\in C} \equiv (A_2, B_2, S_{2\gamma}, a_{2c})_{\gamma<\beta, c\in C}.$$

6. COMPLETE THEORIES

We can state Theorem A and its modifications in a different way, by saying that certain theories are complete instead of that certain models are elementarily equivalent. Indeed, in Robinson's papers [9] and [10], the results were usually stated in terms of complete theories. A theory Γ is said to be *complete* in L if Γ is consistent and any two models (for L) of Γ are elementarily equivalent. Thus, for example, $\mathrm{Th}(M)$ is a complete theory for every model M. A theory Γ is *decidable* if the set of all logical consequences of Γ is recursive. It is well known (for example, see [15], Th. 1) that any complete recursive (or even recursively enumerable) theory is decidable.

We shall denote by Σ the theory consisting of the field axioms together with, for each $n > 0$, the formula

$$\forall u_0 u_1 \cdots u_n \exists v(u_0 + u_1 \cdot v + \cdots + u_n \cdot v^n = 0 \vee u_n = 0).$$

Thus Σ is a recursive theory in L_0, and the class of models of Σ (for L_0) is exactly the class of algebraically closed fields.

The *relativization* $R(\phi)$ of the formula ϕ of L, or $L(C)$, to the unary predicate P is defined recursively as follows (for example, see [15], [9]):

$$\text{if } \phi \text{ is atomic, then } R(\phi) = \phi;$$

$$R(\neg\phi) = \neg R(\phi);$$

$$R(\phi \wedge \psi) = R(\phi) \wedge R(\psi);$$

$$R(\phi \vee \psi) = R(\phi) \vee R(\psi);$$

$$R(\exists x\, \phi) = \exists x(P(x) \wedge R(\phi));$$

$$R(\forall x\, \phi) = \forall x(\neg P(x) \vee R(\phi)).$$

Roughly speaking, $R(\phi)$ is obtained from ϕ by restricting all bound variables of ϕ to P. We let $R(\Gamma)$ be the set of all $R(\phi)$, $\phi \in \Gamma$.

Notice that if B is a field, then

$$\Sigma \cup R(\mathrm{Th}(B))$$

is a theory in L; furthermore, (A', B') is a model of that theory if and only if $B' \equiv B$ and A' is an algebraically closed extension of B'.

Theorem 6.1 is a restatement of Theorem A.

THEOREM 6.1. *If* B *is a field that is neither algebraically closed nor real closed, then the theory*

$$\Sigma \cup R(\mathrm{Th}(B))$$

is complete in L.

COROLLARY 6.2. *If* B *is a field and* Th(B) *is decidable, then the theory*

$$\Sigma \cup R(\mathrm{Th}(B))$$

is decidable.

Proof. Let $\Gamma = \Sigma \cup R(\mathrm{Th}(B))$. Since Th(B) is decidable, it is a recursive set of sentences. It follows that R(Th(B)) is recursive. Since Σ is also recursive, Γ is recursive. If B is neither real closed nor algebraically closed, then Γ is complete and hence is decidable. In the case that B is algebraically closed, the fact that Γ is decidable is stated by Robinson and follows from the results I and II in the introduction. If B is real closed, the decidability of Γ follows from the results III and IV in the introduction.

Although many fields B are known to have undecidable theories Th(B), the only fields that are known to have decidable theories (and thus satisfy the hypotheses of Corollary 6.2) are the finite fields, the real closed fields, and the algebraically closed fields. Moreover, the special cases of Corollary 6.2 that arise when B is either finite, real closed, or algebraically closed were known to Robinson in [9]. Tarski in [14; note 16], has raised the interesting question whether a simple mathematical characterization of those fields whose theories are decidable can be given, and this question is still open.

Theorems 5.1 to 5.5, like Theorem A, may be restated in a form analogous to Theorem 6.1, and they have corollaries analogous to Corollary 6.2. We give such a restatement only for Theorem 5.1.

THEOREM 6.3. *Let* B *be a field, and let* C *be a non-empty set of new individual constants. Let* $\triangle(C)$ *be the set of sentences of* L(C) *which states that the elements of* C *are algebraically independent over* P. *Then the theory*

$$\Sigma \cup R(\mathrm{Th}(B)) \cup \triangle(C)$$

is complete in L(C).

Note that the set $\triangle(C)$ of sentences introduced in the above theorem is exactly the same as the set $\triangle(C)$ introduced in the proof of Lemma 4.1. For the following corollary let us assume that C is denumerable and that we have a suitable Gödel numbering of the symbols and formulas of the logic L(C).

COROLLARY 6.4. *If* B *is a field such that* Th(B) *is decidable, and if* $\triangle(C)$ *is as in Theorem* 6.3, *then the theory*

$$\Sigma \cup R(Th(B)) \cup \triangle(C)$$

is decidable.

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