ACTIONS OF ELEMENTARY p-GROUPS ON $S^n \times S^m$

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1. INTRODUCTION

By an *elementary* p-*group of rank* k we shall mean a group isomorphic to the direct product of k copies of Z_p (the additive group of integers modulo a prime p). Smith [7] has shown that, for an effective action of such a group on the n-sphere S^n, it is necessary that $k \leq (n+1)/2$ if $p \neq 2$ and $k \leq n+1$ if $p = 2$. In this paper we extend Smith's result by showing that if an elementary p-group of rank k acts effectively upon $S^n \times S^m$, then

$$k \leq [(n+1)/2] + [(m+1)/2] \qquad \text{for } p \neq 2,$$

$$k \leq n + m + 2 \qquad \text{for } p = 2;$$

here [x] denotes, as usual, the largest integer not exceeding x. Conner [2] and Heller [4] have investigated the free actions of such groups on $S^n \times S^m$.

Following Smith, we deal with cohomology manifolds rather than with manifolds, and our final result will be stated for a generalized cohomology product sphere rather than for the actual product $S^n \times S^m$. The techniques of this paper will be those used in [5].

2. PRELIMINARIES

An action of a transformation group G on a space X is said to be *effective* if the identity of G is the only element of G that leaves X pointwise fixed; an action is said to be *free* if every element of G, except the identity, moves each point of X. The fixed point set of G on X is denoted by F(G). If H is a normal subgroup of G, there exists a natural action, not necessarily effective, of the quotient group G/H on F(H).

All spaces considered will be compact Hausdorff spaces, and $H^i(X)$ will denote the ith Čech cohomology group of X with coefficient group Z_p (p prime). Our definition of a *cohomology* n-*manifold over* Z_p, denoted by n-cm mod p, will be that given in [1]. Roughly speaking, an n-cm mod p is a connected compact Hausdorff space that has a local cohomology structure (coefficient group Z_p) resembling that of euclidean n-space.

We shall need the following results from [5] dealing with the actions of elementary p-groups on n-cm's mod p. Lemmas 2.1 and 2.2 correspond to Theorem 2.2 and Lemma 3.4, respectively, of [5].

LEMMA 2.1. *Let* G *be an elementary* p-*group of rank* k *acting effectively on an* n-cm X *mod* p *satisfying the first axiom of countability. If* F(G) *is not empty, then*

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$$k \leq \begin{cases} n/2 & \textit{for } p \neq 2, \\ n & \textit{for } p = 2. \end{cases}$$

LEMMA 2.2. *Let* G *be an elementary* p-*group acting effectively, but not freely, on an* n-cm X *mod* p *satisfying the first axiom of countability. Consider the set of components*

$$\mathscr{C} = \{C \mid C \textit{ is a component of } F(K), \textit{ where } K \textit{ runs through the cyclic subgroups of } G\}.$$

Suppose C_M *is a component in* $\mathscr{C}$ *that is not properly contained by any member of* $\mathscr{C}$, *and let* K_M *be a cyclic subgroup such that* $F(K_M) \supset C_M$. *Finally, suppose that* T *is a subgroup of* G/K_M *leaving* C_M *invariant. Then* T *is effective on* C_M.

3. MAIN RESULTS

The i^{th} *modulo* p *Betti number* of X, denoted by $\beta_i(X)$, is defined as the dimension of the vector space $H^i(X)$. As in [1], $\dim_p X$ will denote the *cohomology dimension* of X modulo the coefficient group Z_p. If $\dim_p X = n$, we shall say that X satisfies *modulo* p *Poincaré duality* provided $\beta_i(X) = \beta_{n-i}(X)$ for all $i \geq 0$. An n-cm X mod p is said to be *orientable* if $\beta_n(X) = 1$. An orientable n-cm mod p satisfies modulo p Poincaré duality in the above sense [1].

LEMMA 3.1. *Suppose* X *is a connected space such that* $\dim_p X = n$, X *satisfies modulo* p *Poincaré duality, and* $\Sigma_{i=0}^n \beta_i(X) \leq 4$. *If* G *is an elementary* p-*group of rank* k *acting freely on* X, *then* $k \leq 2$.

Proof. If $\Sigma_{i=0}^n \beta_i(X) = 1$, then $n = 0$, and X is a point; hence, $k = 0$. If $\Sigma_{i=0}^n \beta_i(X) = 2$, then X is a modulo p cohomology n-sphere, and $k \leq 1$ by a well-known result of Smith [7]. If $\Sigma_{i=0}^n \beta_i(X) = 4$, then X has the modulo p Betti numbers of a product of two spheres, and $k \leq 2$ by a result of Heller [4]. (Strictly speaking, Heller's results are in the setting of singular homology theory, and one should probably refer to [5, Theorem 2.3] for a proof of the above statement in the present Čech cohomology setting.) Finally, if $\Sigma_{i=0}^n \beta_i(X) = 3$, the techniques of [4] or [5] may be used to show that $k \leq 1$. To be precise, one can employ formula (3) of Theorem 2.3 in [5] and obtain a contradiction by letting $k = 2$ and $s = n/2$.

THEOREM 3.2. *Let* X *be an orientable* n-cm *mod* p, *satisfying the first axiom of countability, and with* $\Sigma_{i=0}^n \beta_i(X) \leq 4$. *Then, if* G *is an elementary* p-*group of rank* k *acting effectively on* X,

$$k \leq \begin{cases} \frac{n+2}{2} & \textit{for } p \neq 2, \\ n+2 & \textit{for } p = 2. \end{cases}$$

Proof. We consider first the case where $p \neq 2$ and proceed by induction on n. If $n = 1$, X is a circle and $k \leq 1$. Suppose then that $n > 1$. By Lemmas 3.1 and 2.1, we may suppose that the action of G is not free and has no fixed points.

Let C_M be a component maximal in the sense of Lemma 2.2, and let K_M be an associated cyclic subgroup, that is, let $F(K_M) \supset C_M$. It will be assumed that C_M is chosen from among the components of largest modulo p cohomology dimension.

By Floyd [3], $\Sigma \beta_i(F(K_M)) \leq \Sigma \beta_i(X) \leq 4$, and hence $F(K_M)$ consists of at most four components. Moreover, G/K_M leaves $F(K_M)$ invariant. We proceed to verify that either G/K_M leaves C_M invariant or C_M is a point.

By Smith [6], each component of $F(K_M)$ is an orientable mod p cm. If C_M is not a point and if G/K_M does not leave C_M invariant, there must be another component of $F(K_M)$ of the same modulo p cohomology dimension as C_M. In this case $F(K_M)$ would consist of precisely two components. Since G/K_M contains no elements of even order, no element of G/K_M could permute the two components of $F(K_M)$; hence, G/K_M would have to leave C_M invariant. We consider now the two cases.

(i) G/K_M *leaves* C_M *invariant.* Since C_M is *maximal,* G/K_M must be effective on C_M by Lemma 2.2. Moreover, C_M is an orientable mod p cm, $\Sigma \beta_i(C_M) \leq 4$, and

$$\dim_p C_M \leq \dim_p X - 2 = n - 2.$$

We may therefore apply our induction hypothesis to the action of G/K_M on C_M. Since this action might possibly be free,

$$k - 1 = \text{rank } G/K_M \leq \max\left(2, \frac{\dim_p C_M + 2}{2}\right).$$

Since $\dim_p C_M \leq n - 2$, we finally conclude that

$$k \leq \max\left(3, \frac{n+2}{2}\right).$$

Now $\frac{n+2}{2} \geq 3$ for $n \geq 4$. Hence we need to consider separately the cases $n = 2, 3$. If $n = 3$, C_M is a circle, and hence

$$k - 1 = \text{rank } G/K_M \leq 1.$$

Therefore $k \leq 2$. If $n = 2$, C_M is a point, $k - 1 = 0$, and $k = 1$.

(ii) C_M *is a point.* Since $F(K_M)$ now consists of at most four points, it is easy to verify that there must exist a subgroup T of G/K_M of rank $k - 2$ that leaves C_M invariant. In fact, if $p \neq 3$, then G/K_M itself must leave C_M invariant. By Lemma 2.2, T is effective on the point C_M. Hence, $k - 2 = \text{rank } T = 0$, and $k = 2$.

If $p = 2$, we proceed as above, after first noting that $k \leq 2$ for the circle. Again consider a maximal component C_M and associated cyclic subgroup K_M. We recall that $\Sigma \beta_i(F(K_M)) \leq 4$ and consequently $F(K_M)$ consists of at most four components, each an orientable mod 2 cm. If $F(K_M)$ consists of four components, each component must be a point; if $F(K_M) = C_M$, G/K_M leaves C_M invariant; if $F(K_M)$ consists of three components, either G/K_M leaves C_M invariant or each component is a point. Finally, if $F(K_M)$ consists of two components, either G/K_M leaves C_M invariant, C_M is a point, or $F(K_M)$ consists of two cohomology spheres. Therefore, we have three cases to consider.

(i) G/K_M *leaves* C_M *invariant.* By Lemma 2.2, G/K_M is effective on the orientable mod 2 cm C_M. Moreover, $\Sigma\beta_i(C_M) \leq 4$ and $\dim_2 C_M \leq n - 1$. By induction on n,

$$k - 1 = \text{rank } G/K_M \leq \max(2, \dim_2 C_M + 2).$$

Hence, $k \leq \max(3, n+2) \leq n+2$.

(ii) C_M *is a point.* $F(K_M)$ consists of at most four points, and consequently there exists a subgroup T of G/K_M of rank at least $k-3$ that leaves C_M invariant [5, Lemma 2.4]. Since T is effective on C_M,

$$k - 3 \leq \operatorname{rank} T = 0.$$

Hence $k \leq 3$.

(iii) $F(K_M)$ *consists of two cohomology spheres.* In this case there exists a subgroup T of G/K_M of rank at least $k-2$ that leaves C_M invariant. Since C_M is a cohomology sphere,

$$k - 2 \leq \operatorname{rank} T \leq \dim_2 C_M + 1 \leq (n-1) + 1.$$

Hence $k \leq n+2$.

COROLLARY 3.3. *Let* X *be an* $(n+m)$-cm *mod* p, *satisfying the first axiom of countability, that has the same modulo* p *Betti numbers as* $S^n \times S^m$. *If* G *is an elementary* p-*group of rank* k *acting effectively on* X, *then*

$$k \leq \begin{cases} \left[\frac{n+1}{2}\right] + \left[\frac{m+1}{2}\right] & \text{for } p \neq 2, \\ n + m + 2 & \text{for } p = 2. \end{cases}$$

Proof. The case $p = 2$ follows directly from (3.2).

Suppose now $p \neq 2$. If n or m is odd,

$$\left[\frac{n+1}{2}\right] + \left[\frac{m+1}{2}\right] = \left[\frac{m+n+2}{2}\right],$$

and the corollary follows from Theorem 3.2. If n and m are both even, we proceed to show that the action must have a fixed point, in which case it would follow from Lemma 2.1 that

$$k \leq \frac{n+m}{2} = \left[\frac{n+1}{2}\right] + \left[\frac{m+1}{2}\right].$$

Letting $\chi(X) = \Sigma (-1)^i \beta_i(X)$ (that is, taking $\chi(X)$ to be the modulo p Euler characteristic of X) and using a result of Floyd [3], we conclude that

$$\chi(X) \equiv \chi(F(G)) \bmod p.$$

But $\chi(X) = 4$, and p is an odd prime. Therefore $\chi(F(G)) \neq 0$, and $F(G)$ is not empty.

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