PUSHING A 2-SPHERE INTO ITS COMPLEMENT

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1. INTRODUCTION

We investigate the extent to which a closed set in E^3 can be slightly pushed to one side of a surface. One of the main results (Theorem 2.1) states that if U is a complementary domain of a 2-sphere S (possibly wild) in E^3, then $\overline{U}$ can be slightly shoved into U plus a Cantor set. We show in Theorem 5.1 that for each $\varepsilon > 0$ there exists a Cantor set C on S and an ε-map f: $S \to U + C$ that takes S - C homeomorphically into U. This and other related results are given for surfaces in 3-manifolds as well as for 2-spheres in E^3.

Wilder has given a converse for Theorem 5.1 in which he shows [19] that a compact locally connected continuum S in E^3 is a topological 2-sphere if its first homology is trivial, if S separates E^3, and if for each $\varepsilon > 0$ and each component U of E^3 - S there exists a Cantor set C on S and an ε-map f: $S \to U + C$. He gives a corresponding characterization for 2-manifolds, where instead of supposing that the first homology of S is trivial, he supposes it is finitely generated, and he insists furthermore that C does not locally separate S.

For a 2-sphere S in E^3 we use Int S and Ext S to denote respectively the bounded and the unbounded components of E^3 - S. In case D is a cell, we use Bd D to denote the combinatorial boundary of D, and Int D to denote D - Bd D.

The distance function is denoted by D. In case f, g are maps of a set A into a metric space, we use D(f, g) to denote the least upper bound of D(f(a), g(a)), $a \in A$. We call a map f an *ε-map* if $D(f, I) \leq \varepsilon$, where I is the identity map. A *null sequence* of sets is a sequence of sets whose diameters converge to zero. We use $V(X, \varepsilon)$ to denote the set of all points q whose distance from X is less than ε. We call $V(X, \varepsilon)$ the *ε-neighborhood of* X, but we note that this is only a special sort of neighborhood of X. In general, we call any open set containing X a *neighborhood* of X.

A subset X of E^3 (or of a triangulated manifold M) is called *tame* if there exists a homeomorphism h: $E^3 \to E^3$ (or $M \to M$) such that h(X) is a polyhedron. A closed set which is a topological complex but for which there exists no such homeomorphism is called *wild*. They say that X is *locally tame* at a point p of X if there exists a neighborhood N of p and a homeomorphism h of $\overline{N}$ onto a combinatorial cell that takes $\overline{N} \cdot X$ onto a polyhedron. We say that X is *locally tame* mod K if X is locally tame at each point of X - K. A *finite graph* is the sum of a finite number of arcs (topological segments) such that if two of the segments intersect each other, the intersection is an end point of each.

There are several criteria for determining whether or not a 2-sphere in E^3 is tame [1, 3, 4, 8, 9, 11, 13, 14, 16]. One of the most useful of these [4] says that a 2-sphere is tame if its complement is 1-ULC. Using I^m to denote an m-cell, we say that a set Y is n-ULC if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that each map of Bd I^{n+1} into a δ-subset of Y can be extended to map I^{n+1} into an ε-subset of Y.

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We say that a set Y is *locally simply connected* at a point p of $\overline{Y}$ if for each neighborhood N_1 of p there exists a neighborhood N_2 of p such that each map of Bd I^2 into $N_2 \cdot Y$ can be extended to map I^2 into $N_1 \cdot Y$.

Suppose S is a 2-sphere and $D_1, D_2, \cdots$ is a null sequence of mutually exclusive disks in S such that ΣD_i is dense in S. Any set homeomorphic with $S - \Sigma \operatorname{Int} D_i$ is called a *Sierpiński curve* or *universal plane curve*. A Sierpiński curve X in E^3 is called *tame* if there exists a homeomorphism $h: E^3 \to E^3$ such that $h(X)$ lies in a horizontal plane. A point in the image of one of the Bd D_i's is called an *accessible point* of the Sierpiński curve, and an image point of $S - \Sigma D_i$ is called an *inaccessible point.* Accessible points have the property that they lie on arcs in the Sierpiński curve that do not locally separate it, but inaccessible points do not have this property. Hence, the property of being an accessible point is a topological property (Theorem 3.2 of [6]). It has been shown [18] that any two Sierpiński curves are homeomorphic, and that if X_1, X_2 are two Sierpiński curves in the same plane P, then there exists a homeomorphism of P onto itself taking X_1 onto X_2.

In the last section of this paper we prove some theorems about tame Sierpiński curves. It was shown in [5] that for each 2-sphere S (possibly wild) in E^3 there exists a tame Sierpiński curve X in S such that each component of $S - X$ is small. We show in Section 9 that certain sets in S can be buried in such tame Sierpiński curves. This enables one to construct tame arcs in S with great abundance and some precision. Tame arcs may be extended to bigger tame arcs. Tame arcs are accessible at interior points from either side with other tame arcs. Triangulations of S with tame 1-skeletons can be refined by other triangulations with tame 1-skeletons. David Gillman [10] and Joseph Martin have obtained results of this type.

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2. FREEING ALL BUT A CANTOR SET

THEOREM 2.1. *Suppose* S *is a* 2*-sphere in* E^3 *and* U *is a component of* $E^3 - S$. *Then for each* $\varepsilon > 0$ *there exists a Cantor set* C *on* S *and a map* $f: \overline{U} \to U + C$ *such that*

$$D(f, I) < \varepsilon,$$

$$f = I \text{ on } U - V(S, \varepsilon), \text{ and}$$

$$f \text{ is a homeomorphism on } \overline{U} - f^{-1}(C).$$

Proof. We shove $\overline{U}$ "almost" into U with a sequence of shoves. At intermediate stages it is permitted that parts of $\overline{U}$ be moved into $E^3 - \overline{U}$ provided they go into certain controlled neighborhoods of certain open 2-simplexes on S. The theorem will follow from one application of Theorem 3.3 of the next section and then repeated applications of Theorem 3.2.

Let T_1 be a curvilinear triangulation of S of mesh less than $\varepsilon/4$ such that the 1-skeleton K_1 of T_1 is tame. That there exists such a triangulation follows from [5]. Let U_1 be an open set containing $S - K_1$ such that each component of U_1 is of diameter less than $\varepsilon/4$, and such that for each component U_{1i} of U_1 there exists a 2-simplex D_{1i} of T_1 with $\operatorname{Int} D_{1i} \subset U_{1i}$ and $S \cdot \overline{U}_{1i} = D_{1i}$. Note that $U_1 \cdot K_1 = 0$.

We find from Theorem 3.3 that there exists a homeomorphism

$$f_1 : U_1 + U + K_1 \to U_1 + U$$

such that $D(f_1, I) < \varepsilon/2$. In applying Theorem 3.3, we let T_1 be both the T_1 and T_2 of the statement of that theorem, and U_1 both the U_1 and the U_2, and we let U be Ext S.

Let T_2' be a curvilinear triangulation of S of mesh less than $\varepsilon/8$ such that T_2' refines T_1 and the 1-skeleton K_2' of T_2' is tame. That there exists such a refinement follows from Theorem 9.2 of the last section of this paper. Let U_2' be an open set containing $S - K_2'$ such that $U_2' \subset U_1$, each component of U_2' is of diameter less than $\varepsilon/8$, and for each component U_{2i}' of U_2' there exists a 2-simplex D_{2i}' of T_2' with $\operatorname{Int} D_{2i}' \subset U_{2i}'$ and $S \cdot \overline{U}_{2i}' = D_{2i}'$.

It follows from Theorem 3.2 that there exists a homeomorphism

$$f_2 : U_1 + U + K_1 \to U_2' + U + K_1$$

such that $f_2 = I$ on $(U + K_1) - U_1$ and f_2 takes each component of U_1 into itself. Since $f_1(\overline{U}) \subset U_1 + U$, $f_2 f_1(\overline{U}) \subset U_2' + U$. Since each component of U_1 is of diameter less than $\varepsilon/4$, $D(f_2, I) < \varepsilon/4$.

The homeomorphisms f_1, f_2 we have defined are the first two terms of a sequence of homeomorphisms $f_1, f_2, f_3, \cdots$ such that $D(f_i, I) < \varepsilon/2^i$. Hence the sequence $f_1, f_2 f_1, f_3 f_2 f_1, \cdots$ converges to a continuous map f that moves no point as much as ε. We note that since $f_2 f_1(\overline{U}) \subset U_2' + U$, $K_1 \cdot f_2 f_1(\overline{U}) = 0$. We wish to preserve this property for the limit map f, and with this end in mind we subdivide T_2' before reapplying Theorem 3.2. If we were to define f_3 before subdividing, we might pull points of $f_2 f_1(\overline{U})$ near K_1.

Let T_2 be a curvilinear triangulation of S such that T_2 refines T_2', the 1-skeleton K_2 of T_2 is tame, and for each 2-simplex D_{2i}' of T_2' there is a core 2-simplex D_{2i} of T_2 such that

$$D_{2i}' \cdot f_2 f_1(\overline{U}) \subset \operatorname{Int} D_{2i} \subset D_{2i} \subset \operatorname{Int} D_{2i}'.$$

We pick such a core D_{2i} for D_{2i}' even if $D_{2i}' \cdot f_2 f_1(\overline{U}) = 0$. Let U_2 be an open set containing $S - K_2$ such that $U_2 \subset U_2'$, such that $U_{2i}' \cdot f_2 f_1(\overline{U}) \subset U_{2i}$ (where U_{2i} is the component of U_2 associated with the core 2-simplex D_{2i} of T_2 in $\operatorname{Int} D_{2i}'$), and such that for each component U_{2j} of U_2 there exists a 2-simplex D_{2j} of T_2 with $\operatorname{Int} D_{2j} \subset U_{2j}$ and $S \cdot \overline{U}_{2j} = D_{2j}$. Note that $f_2 f_1(\overline{U}) \subset U_2 + U$. When we apply f_3, we shall not pull points of $f_2 f_1(\overline{U})$ near K_1 since f_3 reduces to I on $f_2 f_1(U) - U_2$ and takes each component of U_2 onto itself.

We are now ready to define f_3. Skipping details of how we go from T_3' to T_3, we find the following: a triangulation T_3 of S that refines T_2 so that the 1-skeleton K_3 of T_3 is tame; an open set U_3 containing $S - K_3$ such that $U_3 \subset U_2$, each component of U_3 is of diameter less than $\varepsilon/16$, and for each such component U_{3i} there exists a 2-simplex D_{3i} of T_3 with $\operatorname{Int} D_{3i} \subset U_{3i}$ and $S \cdot \overline{U}_{3i} = D_{3i}$; a homeomorphism $f_3 : f_2 f_1(\overline{U}) \to U_3 + U$ such that

$f_3 = I$ on $f_2 f_1(\overline{U}) - U_2$,

$f_3(U_{2i} \cdot f_2 f_1(\overline{U})) \subset U_{2i}$ for each component U_{2i} of U_2, and

$f_3 f_2 f_1(\overline{U}) \cdot U_{3i} = 0$ if $D_{3i} \cdot K_2 \neq 0$.

Continuing in this fashion, we define T_4, K_4, U_4, f_4, T_5, K_5, U_5, f_5, $\cdots$. As we noted previously, f_1, $f_2 f_1$, $f_3 f_2 f_1$, $\cdots$ converges to a continuous map f. It takes $\overline{U}$ into $\overline{U}$. Since $f(\overline{U})$ misses each K_i, $f(\overline{U}) \cdot S$ is at most a closed 0-dimensional set C' (which in turn lies in a Cantor set C). Since f_i is a homeomorphism and is the identity except very near S, f is a homeomorphism on $\overline{U} - f^{-1}(C)$.

3. FREEING 1-SKELETONS

In this section we take care of some housecleaning details and justify some of the steps used in the proof of Theorem 2.1. For concreteness of treatment we deal with Ext S rather than the arbitrary set U of the previous section.

THEOREM 3.1. *Suppose*

S *is a* 2-*sphere in* E^3,

T_1 *is a curvilinear triangulation of* S *whose* 1-*skeleton* K_1 *is tame, and*

U_1 *is an open set containing* $S - K_1$ *such that for each component* U_{1i} *of* U_1 *there exists a* 2-*simplex* D_{1i} *of* T *so that* $\text{Int } D_{1i} \subset U_{1i}$ *and* $S \cdot \overline{U}_{1i} = D_{1i}$.

Then for each open set V *containing* $S - K_1$, *there exists a homeomorphism*

$$f\colon U_1 + \text{Ext } S + K_1 \to V + \text{Ext } S + K_1$$

such that $f = I$ *on* $\text{Ext } S + K_1 - U_1$.

Proof. Let g be the continuous real-valued function defined on S by the relation

$$g(p) = \text{minimum } (D(p, E^3 - U_1), D(p, E^3 - V)).$$

It follows from Theorem 7 of [2] that there exists a homeomorphism h of S into $K_1 + V$ such that for each point p of $S - K_1$,

$$D(p, h(p)) < g(p) \text{ and}$$

$$h(S) \text{ is locally tame at } h(p).$$

It follows from [9] that h(S) is tame, and from Theorem VI 10 of [15] that

$$V + \text{Ext } S = V + \text{Ext } h(S) \supset \text{Ext } h(S).$$

We now describe a homeomorphism

$$f\colon U_1 + \text{Ext } S + K_1 \to \text{Ext } h(S) + K_1$$

satisfying the conclusion of Theorem 3.1. The homeomorphism f is fixed except on the U_{1i} and takes each of these into itself, so we define only the part f_i of f restricted to U_{1i}.

Let abcd be a solid tetrahedron in E^3 and p_i a point of $h(S) - (K_1 + D_{1i})$. Since h(S) is tame, there exists a homeomorphism $h_i\colon E^3 \to E^3$ such that

$$h_i h(S) = \text{Bd } abcd, \qquad h_i h(D_{1i}) = abc, \qquad h_i h(p_i) = d.$$

By pushing some points radially away from d, one finds that there exists a homeomorphism f_i': $E^3 \to E^3$ such that

$$f_i' = I \text{ outside Int abcd} + h_i(V \cdot U_{1i}) \text{ and}$$

$$f_i' h_i(U_{1i}) \subset E^3 - \text{abcd} = \text{Ext Bd abcd}.$$

Then

$$f_i = h_i^{-1} f_i' h_i \text{ on } U_{1i}.$$

THEOREM 3.2. *Suppose* S, T_1, K_1, U_1 *are as in Theorem* 3.1, *and* T_2 *is a curvilinear triangulation of* S *such that* T_2 *refines* T_1 *and has a tame* 1-*skeleton* K_2.

Then for each open set U_2 *containing* S - K_2 *there exists a homeomorphism*

$$f: U_1 + \text{Ext S} + K_1 \to U_2 + \text{Ext S} + K_1$$

such that

$$f = I \textit{ on } K_1 + \text{Ext S} - U_1 \textit{ and}$$

$$f \textit{ takes each component of } U_1 \textit{ into itself.}$$

Proof. This result follows from the proof of Theorem 3.1 if we insist that $K_2 \subset h(S) \subset U_2 + K_2$; so we suppose this. The homeomorphism f is the homeomorphism f of Theorem 3.1.

The theorem is true without the hypothesis that T_2 refines T_1, but we made the simplifying assumption because it is part of the context in which we used the theorem.

THEOREM 3.3. *Suppose* S, T_1, K_1, U_1, T_2, K_2, U_2 *are as in Theorem* 3.2. *Then for each* $\varepsilon > 0$ *there exists a homeomorphism*

$$f': U_1 + \text{Ext S} + K_1 \to U_2 + \text{Ext S}$$

such that

$$f' = I \textit{ on } \text{Ext S} - V(S, \varepsilon),$$

$$f'(U_{1i}) \subset V(U_{1i}, \varepsilon) \textit{ for each component } U_{1i} \textit{ of } U_1, \textit{ and}$$

$$f' \textit{ moves no point of } (K_1 + \text{Ext S}) - U_1 \textit{ more than } \varepsilon.$$

Proof. The homeomorphism f' is the homeomorphism f of Theorem 2 followed by a homeomorphism that shoves h(S) + Ext h(S) into Ext h(S).

Of course, Theorems 3.1, 3.2, 3.3 hold as well for Int S as for Ext S.

4. SIMPLE CONNECTIVITY OF THE COMPLEMENT OF A 2-SPHERE

The following is an application of Theorem 2.1.

THEOREM 4.1. *Suppose* S *is a* 2-*sphere in* E^3. *For each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *and a sequence of Cantor sets* $C_1, C_2, \cdots$ *on* S *such that if* J *is a simple closed curve in* $E^3 - S$ *of diameter less than* δ *such that* $D(J, S) > 1/i$, *then* J *can be shrunk to a point on an* ε-*subset of* $(E^3 - S) + C_i$.

In addition the following theorem holds.

THEOREM 4.2. *Suppose* S *is a* 2-*sphere in* E^3 *and* U *is a component of* $E^3 - S$. *Then there exists a* 0-*dimensional* F_σ-*set* F *on* S *such that* U + F *is* 1-ULC.

Proof. It follows from repeated applications of Theorem 4.1 that an F_σ-set F exists such that to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that, for each disk D, each map of Bd D into a δ-subset of U can be extended to map D into an ε-subset of U + F. We show that U + F is 1-ULC by showing that each map f of Bd D into a $(< \delta)$-subset of U + F can be extended to map D into a 3ε-subset of U + F.

We suppose that D is a round planar disk and that the closed set $f^{-1}(F)$ is 0-dimensional. Let $p_1 q_1, p_2 q_2, \cdots$ be a null sequence of mutually exclusive secants for D such that the closure of each component of $D - \Sigma\, p_i q_i$ is a disk which misses $f^{-1}(F)$, and such that f can be extended to map $\text{Bd } D + \Sigma\, p_i q_i$ into a subset of U + F with each $f(p_i q_i)$ contained in U. That the $p_i q_i$'s can be chosen so that f can be so extended follows from the fact (Theorem 5.35 in Chapter II of [20]) that U is 0-ULC. Only a few of the $f(p_i q_i)$ will have large diameters, and we discard them so that the diameter of the image of Bd D and the remaining $p_i q_i$'s is less than δ. For convenience in notation we suppose none were thrown away. Since F can be extended to map each component of $D - \Sigma\, p_i q_i$ into a small subset of U + F (in no case requiring an image with a diameter as much as ε but in most cases requiring a much smaller image), f can be extended to map D into a subset of U + F that lies in an ε-neighborhood of $f(\text{Bd } D + \Sigma\, p_i q_i)$.

5. PUSHING A 2-SPHERE "ALMOST" TO THE SIDE

THEOREM 5.1. *Suppose* S *is a* 2-*sphere in* E^3 *and* U *is a component of* $E^3 - S$. *Then for each* $\varepsilon > 0$ *there exists a Cantor set* C *on* S *and a map* g *of* S *into* U + C *such that* $D(g, I) < \varepsilon$ *and* g *takes* S - C *homeomorphically into* U.

Proof. Let δ be a positive number so small that each δ-subset of S lies in an $\varepsilon/2$-disk in S.

It follows from Theorem 2.1 that there exists a Cantor set C_1 on S and a map f of S into $U + C_1$ such that $D(f, I) < \delta/2$ and f is a homeomorphism on $S - f^{-1}(C_1)$. Each component of $f^{-1}(C_1)$ is of diameter less than δ and lies in an $\varepsilon/2$-disk on S.

As pointed out in Theorem 9 of [7], there exists a component Y of $S - f^{-1}(C_1)$ such that each component of S - Y is of diameter less than $\varepsilon/2$. Also, as pointed out there, there exists a map f_1 of $\overline{Y}$ onto S such that $D(f_1, I) < \varepsilon/2$, and such that for each point p of S, $f_1^{-1}(p)$ is either a point of Y or a component of $\overline{Y} - Y$.

Let C_2 be a Cantor set in S that contains $f_1(\overline{Y} - Y)$. Then a C and a g satisfying the conclusion of Theorem 5.1 are

$$C = C_1 + C_2 \quad \text{and} \quad g(p) = ff_1^{-1}(p).$$

Questions. Is a 2-sphere S in E^3 tame if for each $\varepsilon > 0$ and each component U of $E^3 - S$ there exists an ε-map $f: S \to U$? Hempel has shown [14] that S is tame if there exists a homotopy $H_t(s)$ $(0 \leq t \leq 1,\ s \in S)$ of S into S + U such that $H_0 = I$ and $H_t(S) \subset U$ for $t > 0$. Is S tame if for each Sierpiński curve X on S there exists a homotopy H_t of X into X + U such that $H_0 = I$ and $H_t(X) \subset U$ for $t > 0$? Burgess has shown [8] that S is tame if there exists such an H_t and if in addition each H_t is a homeomorphism and $H_{t_1}(X) \cdot H_{t_2}(X) = 0$ if $t_1 \neq t_2$.

6. PUSHING TO THE SIDES OF SURFACES

In the preceding sections we have dealt with 2-spheres in E^3, but since we were dealing with them only locally, our results apply equally well to arbitrary two-sided 2-manifolds embedded in 3-manifolds.

THEOREM 6.1. *Suppose*

M^3 *is a connected* 3-*manifold,*

M^2 *is a connected* 2-*manifold in* M^3 *that separates* M^3,

U *is a component of* $M^3 - M^2$, *and*

g *is a nonnegative continuous real function defined on* M^3 *such that* g *is positive on an open subset* V *of* M^2.

Then there exists a 0-*dimensional subset* C *of* V *and a map* $f: \overline{U} \to (\overline{U} - V) + C$ *such that*

$$D(x, f(x)) \leq g(x),$$

$$f \text{ is a homeomorphism on } \overline{U} - f^{-1}(C).$$

In fact, for each neighborhood N *of* V *there exists such an* f *that is the identity on* $\overline{U} - N$.

Proof. We suppose $g = 0$ on $M^2 - V$.

The only change of approach used here over that used in the proof of Theorem 2.1 is to start with a triangulation T_1 of V such that each 1-simplex of T_1 is tame, each 2-simplex D_i of T_1 lies in an open 3-cell O_i of diameter less than the minimum value of g on O_i, and such that D_i lies in an open subset of M^2 which in turn lies on a 2-sphere in O_i. This is possible by Theorem 5 of [4]. We take small mutually exclusive open subsets U_i in the O_i's about the Int D_i's and shove as before—first freeing the 1-skeleton of T_1 and then in a countable number of steps freeing all but at most a Cantor set on each D_i.

The following result follows from the methods used in the proof of Theorem 4.2.

THEOREM 6.2. *Suppose* M^3, M^2, U *are as in Theorem* 6.1. *Then there exists a* 0-*dimensional* F_σ-*set* F *on* M^2 *such that* U + F *is locally simply connected at each point of* M^2.

Duplicating the arguments used in proving Theorem 5.1, we obtain the following extension of it.

THEOREM 6.3. *Suppose* M^3, M^2, U *are as in Theorem* 6.1. *Then for each* $\varepsilon > 0$ *there exists a closed* 0-*dimensional set* C *on* M^2 *and an* ε-*map* $f: M^2 \to U + C$ *such that* f *takes* $M^2 - C$ *homeomorphically into* U.

7 FREE GROUPS

We now set forth some known results from algebra [12, Chapter 7], one of which we use to prove a topological theorem that will help us in the next section.

Consider a group expressed with generators and relations. An element of the group is an equivalence class of words where the letters of the words are generators and their inverses. If the group is free, two words belong to the same equivalence class if and only if one can be changed to the other with a finite number of operations, where each operation consists either of the insertion of two adjacent letters xx^{-1} somewhere in the word or the cancellation of two such letters. The following result shows that any word that can be reduced to the trivial word can be so reduced with cancellations alone.

THEOREM 7.1. *For a free group, any word that belongs to the equivalence class of the identity element contains two adjacent letters, one of which is the inverse of the other.*

Proof. Suppose w_1 is a word in the equivalence class of the identity. Then there exists a sequence $w_1, w_2, \cdots, w_n$ such that w_n has no letters and w_{i+1} is obtained from w_i by either adjacently inserting two letters xx^{-1} somewhere in the word or cancelling two such letters. We suppose that the operations of cancellation and insertion have been performed so as to minimize the number of steps in going from w_1 to w_n (the word with no letters). We show that in this case, each operation was a cancellation. This will establish the theorem.

Suppose not each operation was a cancellation. There exists a j such that the operation of going from w_j to w_{j+1} was not a cancellation but all the following operations were cancellations. With no loss of generality, we suppose that w_j is the shortest word in the equivalence class of the identity element with the property that

1) w_j cannot be reduced to w_n by cancellation alone, but
2) w_j can be changed, with one insertion, to a word that can be reduced to w_n by cancellation alone.

Suppose w_{j+1} results from w_j by inserting xx^{-1} somewhere, and w_{j+2} results from w_{j+1} by cancelling yy^{-1}. Since w_{j+2} differs as a word from w_j, neither of the letters inserted is one of the letters cancelled. Let w' be the word obtained from w_j by cancelling the same yy^{-1} as was cancelled in going from w_{j+1} to w_{j+2}. This leads to a contradiction because w' is shorter than w_j and has properties (1) and (2) mentioned above. Hence the theorem follows.

We do not need the following two results in this paper, but they are listed since they follow from the same methods and seem of interest.

THEOREM 7.2. *If* w *and* w' *are words in the same equivalence class of a free group, then they can be reduced to the same word by cancellation alone.*

Proof. Theorem 7.2 is an immediate consequence of Theorem 7.1 if we consider that ww'^{-1} is equivalent to the trivial word $w'w'^{-1}$.

Suppose $G = \{a_1, a_2, \cdots / r_1 = r_2 = \cdots = 1\}$ is a group and two words belong to the same equivalence class if one can be changed to the other with a finite sequence of operations of the following types:

1) Insert an $a_i a_i^{-1}$ or an $a_i^{-1} a_i$ somewhere.
2) Insert r_i or r_i^{-1} somewhere.

3) Cancel r_i or r_i^{-1} somewhere.

4) Cancel $a_i a_i^{-1}$ or $a_i^{-1} a_i$ somewhere.

There is no need of operations of type 3, since any such operation can be replaced by one of type 2 followed by a sequence of those of type 4. The following theorem shows that these can be done in a prescribed order.

THEOREM 7.3. *If two words are equivalent, it is possible to change the first to the second by first performing a finite number of operations of type* 1, *then a finite number of type* 2, *and finally a finite number of type* 4.

Proof. We prove the theorem by showing that if w_1 can be changed to w_2 by one operation of type 1, 2, or 4 (say of type j) and w_2 can be changed to w_3 by taking the operations in the prescribed order, then w_1 can be changed to w_3 by operations in the prescribed order.

Case 1. $j = 1$. This order of operations is already as prescribed.

Case 2. $j = 2$. Let $w_1 = v_1 v_2$ and $w_2 = v_1 r v_2$. Change $w_1 = v_1 v_2$ to $v_1 r^{-1} v_1^{-1} v_1 r v_2$ by a sequence of operations of type 1. Since $v_1 r^{-1} v_1^{-1}$ can be changed to 1 by one operation of type 2 and a sequence of type 4 (and by hypothesis $v_1 r v_2$ can be reduced to w_3 by operations in the prescribed order), interspersion of these operations reduces $v_1 r^{-1} v_1^{-1} v_1 r v_2$ to w_3 by a sequence of operations in the prescribed order.

Case 3. $j = 4$. Let $w_1 = v_1 x x^{-1} v_2$ and $w_2 = v_1 v_2$. Change $w_1 = v_1 x x^{-1} v_2$ to $v_1 x x^{-1} v_1^{-1} v_1 v_2$ by operations of type 1. Since each of $v_1 x x^{-1} v_1^{-1}$ and $v_1 v_2$ can be reduced by operations in the prescribed order, their product can also be reduced.

Finally we come to the topological theorem responsible for our excursion into group theory.

THEOREM 7.4. *Suppose* $D_1, D_2, \cdots, D_n$ *is a finite collection of mutually exclusive polyhedral disks in* E^3 *and* J *is a polygonal simple closed curve in general position with respect to* ΣD_i *that can be shrunk to a point in* $E^3 - \Sigma \operatorname{Bd} D_i$. *Then, if* J *intersects* ΣD_i, *there exists an arc* axb *in* J *that intersects* ΣD_i *only at its end points and such that* xa *and* xb *abut on the same* D_j *from the same side.*

Proof. We suppose, without loss of generality, that each D_i lies in the same horizontal plane. Theorem 7.4 then follows from Theorem 7.1, since the fundamental group of $E^3 - \Sigma \operatorname{Bd} D_i$ is a free group on n generators. A loop corresponds to a word, where a letter x of the word is represented by the loop crossing a D_i in one direction and the inverse x^{-1} of the letter is a crossing in the other direction. The letters in the word are ordered as the crossings of the D's by the loop. We note in the following paragraph why a trivial loop corresponds to a word in the fundamental group of $E^3 - \Sigma \operatorname{Bd} D_i$, which is in the equivalence class of the identity element.

Suppose E is an oriented 2-simplex, p_0 is a vertex of E, and f is a map of Bd E into $E^3 - \operatorname{Bd} D_i$ such that $f(p_0)$ goes to the starting point used in computing the fundamental group of $E^3 - \Sigma \operatorname{Bd} D_i$. The map under f of the positively oriented path around Bd E starting and ending at p_0 represents a loop. For each loop there corresponds a word whose letters are ordered in the order that the loop crosses the D_i's and the letter a_i or a_i^{-1} is assigned to the crossing according to whether the crossing is from below or above. If the loop can be shrunk to a point in $E^3 - \Sigma \operatorname{Bd} D_i$, then the map f can be extended to map E into $E^3 - \Sigma \operatorname{Bd} D_i$. We call this extended map f also, and we suppose that it is so nice that there exists a triangulation T of E such that f takes each 2-simplex of T homeomorphically onto a 2-simplex in E^3 without vertices on any of the D_i's. By shelling T, we get a sequence of disks

$E = E_1, E_2, E_3, \cdots, E_m$ such that E_m is a single 2-simplex of T and each E_{i-1} is the sum of E_i and such a single 2-simplex. We suppose that E_m misses each D_i. The word corresponding to f(Bd E_i) belongs to the same equivalence class as the word corresponding to f(Bd E_{i+1}), since each the words are the same or one word can be changed to the other by the single insertion or deletion of two letters of the sort xx^{-1}. This is because the boundary of each 2-simplex of T either misses a D_i or crosses it twice in opposite directions. Since f(Bd E_m) corresponds to the word with no letters, f(E) corresponds to a word in the equivalence class of the identity.

8. SPHERES LOCALLY TAME MODULO SIERPIŃSKI CURVES

In a certain sense the theorems of this section are generalizations of the result of Doyle and Hocking [9] that a 2-sphere is tame if it is tame modulo a tame finite graph. We use the results of this section only mildly in the rest of this paper (to help establish Theorem 9.2, which in turn was used in the proofs of Theorems 2.1 and 6.1), but the results have outside interest and will be used extensively in another paper to give a simplified proof of the result that any open subset of any two-sided surface in any 3-manifold can be "almost" approximated from either side.

THEOREM 8.1. *Suppose* X *is a tame Sierpiński curve that lies in a* 2*-sphere* S *(possibly wild) in* E^3. *Then for each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that each simple closed curve of diameter less than* δ *in* $E^3 - S$ *can be shrunk to a point on an* ε*-subset of* $E^3 - X$.

Proof. We suppose with no loss of generality that X lies in a horizontal plane P.

Since each component of P - X is 0-ULC and P - X contains only a finite number of such components with diameters greater than $\varepsilon/3$, there exists a $\delta > 0$ such that each pair of points of the same component of P - X whose distance apart is less than δ can be joined by an arc in P - X of diameter less than $\varepsilon/3$. We show that this is the δ promised by the theorem. Note that $\delta \leq \varepsilon/3$.

Suppose J_1 is a simple closed curve in $E^3 - S$ of diameter less than δ. Without loss of generality we suppose that J_1 is polygonal and contains no vertex on P. We show that if N is the convex hull of the $\varepsilon/3$-neighborhood of J_1, then J_1 can be shrunk to a point in $N - N \cdot X$. In the easy case that J_1 misses P, we can shrink J_1 to a point in $N - N \cdot P$ and hence in $N - N \cdot X$. We show in any case that J_1 can be shrunk by reducing the number of points in $J_1 \cdot P$.

Let $D_1, D_2, \cdots, D_m$ be the closures of the components of P - X intersecting J_1. It follows from Theorem 7 of [2] that there exists a 2-sphere S' in $E^3 - J_1$ that contains Bd D_1 + Bd $D_2 + \cdots +$ Bd D_m and is locally polyhedral

$$\text{mod Bd } D_1 + \text{Bd } D_2 + \cdots + \text{Bd } D_m.$$

It follows from [9] that S' is tame. Hence, J_1 can be shrunk to a point in $E^3 - S'$. In particular, J_1 can be shrunk to a point in $E^3 -$ (Bd D_1 + Bd $D_2 + \cdots +$ Bd D_m).

For convenience we suppose each D_i is a disk. (If some D_i were unbounded, we would replace it with a disk D_i' such that Bd D_i' = Bd D_i and $D_i' \subset D_i$ + Bd C, where C is a large oriented cube whose interior contains $X + J_1$.) Hence we suppose the D_i's are mutually exclusive polyhedral disks.

It follows from Theorem 7.4 that there exists an arc $p_1 x_1 q_1$ of J_1 that intersects P only in its end points, and p_1, q_1 belong to the same component of P - X. Let $p_1 y_1 q_1$ be an arc in P - X of diameter less than $\varepsilon/3$ such that

$$J_2 = (J_1 - p_1 x_1 q_1) + p_1 y_1 q_1$$

is a simple closed curve. Shrink J_1 to J_2 in N - X by leaving the points of $J_1 - p_1 x_1 q_1$ fixed and moving points of $p_1 x_1 q_1$ along rectilinear segments to points of $p_1 y_1 q_1$. In a neighborhood of $p_1 y_1 q_1$ we shove J_2 slightly to one side of P and onto a polygonal simple closed curve J_2' that intersects P in two fewer points than J_1 does. It follows again from Theorem 7.4 that, unless J_2' misses P, some arc $p_2 x_2 q_2$ of J_2' intersects P only at its end points and has its end points on the same component of P - X. Let $p_2 y_2 q_2$ be a polygonal arc in P - X of diameter less than $\varepsilon/3$ such that $J_3 = (J_2' - p_2 x_2 q_2) + p_2 y_2 q_2$ is a polygonal simple closed curve. Push J_2' straight to J_3 in N, and adjust J_3 to J_3' by pushing J_3 in a neighborhood of $p_2 y_2 q_2$ slightly to one side of P.

We continue this procedure and get a sequence of polygonal simple closed curves $J_1, J_2, J_2', J_3, J_3', \cdots, J_n, J_n'$ such that each can be pushed to the next in $N - N \cdot X$ and J_n' misses P. Since J_n' can be shrunk to a point in $N - N \cdot P$, J_1 can be shrunk to a point in an ε-subset of $E^3 - X$.

Although Theorem 8.1 was stated only for simple closed curves, each map of a simple closed curve into E^3 can be approximated by a homeomorphism. Hence, we have shown that each map of Bd I^2 into an ε-subset of $E^3 - S$ can be extended to map I^2 into an ε-subset of $E^3 - X$. It is in this form that we use Theorem 8.1 in the proof of the following result.

THEOREM 8.2. *A 2-sphere* S *in* E^3 *is tame if it is locally tame modulo a tame Sierpiński curve* X *in* S.

Proof. We show that S is tame by showing that $E^3 - S$ is 1-ULC. See Theorem 2 of [4].

It follows from [9] that the closure of each component of S - X is a tame disk. Let δ_1 be a positive number so small that each δ_1-subset of one of these disks lies in a subdisk of diameter less than $\varepsilon/3$. Let δ_2 be a number promised by the previous theorem such that each simple closed curve of diameter less than δ_2 in $E^3 - S$ can be shrunk to a point on a δ_1-subset of $E^3 - X$.

Let f be a map of Bd I^2 into a δ_2-subset of $E^3 - S$. Then f can be extended to map I^2 into a δ_1-subset of $E^3 - X$. Let Y be the component of $I^2 - f^{-1}(S \cdot f(I^2))$ containing Bd I^2. Let g be a map of I^2 into f(Y) + (S - X) such that each component of $I^2 - Y$ is sent into a disk in S - X of diameter less than $\varepsilon/3$. Since each such disk in S - X is tame, we can adjust f slightly in a neighborhood of $I^2 - Y$ by shoving to one side of these disks so that the adjusted $g(I^2)$ misses S.

It may be convenient to use Theorem 8.2 in the following more general form.

THEOREM 8.3. *Suppose* S *is a* 2-*sphere in* E^3, X *is a time Sierpiński curve in* S, *and* U *is an open subset of* S *such that* S *is locally tame at each point of* U - X. *Then* S *is locally tame on* U.

Proof. To show that S is locally tame at a point p of U, use Theorem 8 of [2] to adjust S to a 2-sphere S' such that S' contains X, S' is locally tame mod X, and S' agrees with S in a neighborhood of p. Since S' is tame by Theorem 8.2, S is locally tame at p.

We can extend Theorem 8.2 as follows.

THEOREM 8.4. *A 2-sphere* S *in* E^3 *is tame if it is locally tame modulo the sum of a finite number of tame Sierpiński curves* $X_1, X_2, \cdots, X_n$.

Proof. It follows from repeated applications of Theorem 8.3 that S is locally tame mod $X_2 + X_3 + \cdots + X_n$, locally tame mod $X_3 + X_4 + \cdots + X_n, \cdots$, locally tame mod X_n, and finally locally tame everywhere. Hence it is tame.

Our arguments apply to arcs as well as to Sierpiński curves, so we have the following generalization.

THEOREM 8.5. *Suppose* S *is a 2-sphere in* E^3; $X_1, X_2, \cdots, X_n$ *is a finite collection of subsets of* S *such that each is either a tame finite graph or a tame Sierpiński curve;* U *is an open subset of* S *such that* S *is locally tame at each point of* $U - U \cdot (X_1 + X_2 + \cdots + X_n)$. *Then* S *is locally tame on* U.

9. BURYING SETS IN SIERPINSKI CURVES

THEOREM 9.1. *Suppose* S *is a 2-sphere in* E^3 *and* K *is a tame finite graph (or tame Sierpiński curve) in* S. *Then for each* $\varepsilon > 0$ *there exists a tame Sierpiński curve* X *on* S *such that each point of* K *is an inaccessible point of* X *and each component of* S - X *is of diameter less than* ε.

Proof. We deal only with the case where K is connected, since the general case can be treated in an analogous manner. We suppose diameter $K > \varepsilon$.

It follows from Theorem 1 of [5] that there exists a tame Sierpiński curve X_1 on S such that each component of $S - X_1$ has diameter less than ε. For each integer $i > 1$, let X_i be a tame Sierpiński curve on S such that $X_i \subset V(K, 1/i)$, and such that each component of $S - X_i$ of diameter more than ε/i is at a positive distance from K. The existence of such X_i's follows from the facts [5] that we can get tame Sierpiński curves in S with arbitrarily small holes and that we can cut out parts of these curves that are far from K. The required Sierpiński curve X will lie in $K + \Sigma X_i$.

Let G_i be the decomposition of S such that an element of G_i is either an inaccessible point of X_i or the closure of a component of $S - X_i$. Let G be the decomposition of S such that the element of G containing p is also a component of the intersection of the various elements of the G_i that contain p. No element of G separates S. Also, G has only a null sequence of nondegenerate elements, and none of these intersects K. The sum of these nondegenerate elements is dense in S. Using the fact that the decomposition space of G is a 2-sphere [17], we find that there exists a null sequence of mutually exclusive disks $D_1, D_2, \cdots$ in S such that each of these disks has diameter less than ε, none of them intersects K, and each nondegenerate element of G lies on the interior of one of the disks. Then $S - \Sigma \operatorname{Int} D_i = X$ is the required Sierpiński curve.

We show that X is tame by showing that $K + \Sigma X_i$ lies on a tame 2-sphere. Adjust S to a 2-sphere S' such that S' contains $K + \Sigma X_i$ and S' is locally tame mod $K + \Sigma X_i$. It follows from Theorem 8.5 that S' is locally tame mod K, and from another application of Theorem 8.5 that S' is locally tame everywhere—and hence tame.

The following theorem is an application of Theorem 9.1.

THEOREM 9.2. *Suppose* S *is a 2-sphere in* E^3 *and* T_1 *is a triangulation of* S *such that the* 1-*skeleton of* T_1 *is tame. Then for each* $\varepsilon > 0$, *there exists a triangulation* T_2 *of mesh less than* ε *such that* T_2 *refines* T_1 *and the* 1-*skeleton of* T_2 *is tame.*

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