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On Ideals in H^∞ Whose Closures Are Intersections of Maximal Ideals

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Dedicated to Professor Kôzô Yabuta on his sixtieth birthday

1. Introduction

Let H^∞ be the Banach algebra of bounded analytic functions on the open unit disk D. We denote by $M(H^\infty)$ the maximal ideal space of H^∞, the set of nonzero multiplicative linear functionals of H^∞ endowed with the weak*-topology of the dual space of H^∞. Identifying a point in D with its point evaluation, we think of D as a subset of $M(H^\infty)$. For $\varphi \in M(H^\infty)$, put $\operatorname{Ker}\varphi = \{f \in H^\infty;\ \varphi(f) = 0\}$. Then $\operatorname{Ker}\varphi$ is a maximal ideal in H^∞, and for a maximal ideal I in H^∞ there exists $\psi \in M(H^\infty)$ such that $I = \operatorname{Ker}\psi$. For $f \in H^\infty$, the function $\hat{f}(\varphi) = \varphi(f)$ on $M(H^\infty)$ is called the *Gelfand transform* of f. We can identify f with $\hat{f}$, so that we think of H^∞ as the closed subalgebra of continuous functions on $M(H^\infty)$. Let L^∞ be the Banach algebra of bounded measurable functions on ∂D. The maximal ideal space of L^∞ will be denoted by $M(L^\infty)$. We may think of $M(L^\infty)$ as a subset of $M(H^\infty)$. Then $M(L^\infty)$ is the Shilov boundary of H^∞, that is, the smallest closed subset of $M(H^\infty)$ on which every function in H^∞ attains its maximal modulus. For a subset E of $M(H^\infty)$, we denote the closure of E by $\bar{E}$. A nice reference for this subject is [4].

For $f \in H^\infty$, there exists a radial limit $f(e^{i\theta})$ for almost everywhere. Let h be a bounded measurable function on ∂D such that $\int_0^{2\pi} \log|h|\, d\theta/2\pi > -\infty$. Put

$$f(z) = \exp\left(\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|h(e^{i\theta})| \frac{d\theta}{2\pi}\right), \quad z \in D.$$

A function of this form is called *outer*, and $|f(e^{i\theta})| = |h(e^{i\theta})|$ almost everywhere. A function $u \in H^\infty$ is called *inner* if $|u(e^{i\theta})| = 1$ a.e. on ∂D. For a sequence $\{z_n\}_n$ in D with $\sum_{n=1}^\infty (1-|z_n|) < \infty$, there corresponds a Blaschke product

$$b(z) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|} \frac{z-z_n}{1-\bar{z}_n z}, \quad z \in D.$$

A Blaschke product is called *interpolating* if, for every bounded sequence of complex numbers $\{a_n\}_n$, there exists $h \in H^\infty$ such that $h(z_n) = a_n$ for every n. For a nonnegative bounded singular measure μ ($\mu \neq 0$) on ∂D, let

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$$\psi_\mu(z) = \exp\left(-\int_{\partial D} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu\right), \quad z \in D.$$

Then ψ_μ is inner and is called a *singular* function. It is well known that every function f in H^∞ can be factored in a product $f = qh$, where q is inner and h is outer. It is also well known that every inner function q can be factored in a product $q = BS$, where B is a Blaschke product and S is a singular function [4; 11].

For a subset E of $M(H^\infty)$, let $I(E) = \bigcap\{\operatorname{Ker}\varphi;\ \varphi \in E\}$ be the intersection of maximal ideals associated with points in E. For $f \in H^\infty$, let $Z(f) = \{\varphi \in M(H^\infty);\ \varphi(f) = 0\}$ be the zero set of f. In this paper, we assume that every ideal is nonzero and proper. For an ideal I in H^∞, put $Z(I) = \bigcap\{Z(f);\ f \in I\}$; then $I \subset I(Z(I))$. An ideal I is called *prime* if, for any $f, g \in H^\infty$ with $fg \in I$, we have $f \in I$ or $g \in I$. There are many studies of prime ideals in H^∞; see [5, 15, 16, 17]. Recently, Gorkin and Mortini [7, Thm. 1] have proved that a closed prime ideal I of H^∞ is an intersection of maximal ideals, that is, $I = I(Z(I))$. And they pointed out that if I is a (nonclosed) prime ideal such that $Z(I) \cap M(L^\infty) = \emptyset$, then the closure of I is an intersection of maximal ideals; that is, $\bar{I} = I(Z(I))$, where $\bar{I}$ is the closure of I in H^∞.

Let E be a closed subset of $M(H^\infty) \setminus D$ such that $E \cap M(L^\infty) = \emptyset$. Let $J(E)$ be the set of functions f in H^∞ that vanish on some open subsets (depending on f) of $M(H^\infty) \setminus D$ containing E. Then $J(E)$ is an ideal of H^∞. In [8, Thm. 4.2], Gorkin and Mortini also showed that $\overline{J(E)} = I(Z(J(E)))$.

It is a very interesting problem to determine the class of ideals I satisfying $\bar{I} = I(Z(I))$. But it seems difficult to give a complete characterization of these ideals.

In Section 2, we shall introduce the following condition on ideals I in H^∞ to study the problem when $\bar{I} = I(Z(I))$ holds.

(α) For any $0 < \sigma < 1$ and for any subset A of D such that $Z(I) \cap \bar{A} = \emptyset$, there exists an $h \in I$ such that $\|h\|_\infty \le 1$ and $|h| \ge \sigma$ on A.

We shall prove that if I satisfies condition (α), then $\bar{I} = I(Z(I))$. We shall also give some examples of ideals I satisfying condition (α).

In Section 3, we study an ideal $I(f)$ of H^∞ generated by a noninvertible outer function f in H^∞. We shall show that there exist noninvertible outer functions f and g satisfying $\overline{I(f)} = I(Z(I(f)))$ and $\overline{I(g)} \neq I(Z(I(g)))$. As an application of the theorem given in Section 2, we shall characterize noninvertible outer functions f satisfying $\overline{I(f)} = I(Z(I(f)))$.

2. Closure of Ideals

In order to prove that $I = I(Z(I))$ for a closed prime ideal I of H^∞, Gorkin and Mortini [7] used the following formula given by Guillory and Sarason [10, pp. 177–178]. Let R be an open subset of D such that $\partial R \cap D$ is a system of rectifiable curves. Then

$$\int_{\partial D} \frac{F}{u}\, dz = \int_{\partial R \cap D} \frac{F}{u}\, dz \tag{2.1}$$

holds for $F \in H^\infty$ and an inner function u satisfying $|u(z)| < \beta$ for $z \in R$ and $|u(z)| \geq \alpha$ for $z \in D \setminus R$, where $0 < \alpha < \beta < 1$. Formula (2.1) is used in several papers; see [9; 13; 14]. We note that if u is not inner, equation (2.1) does not hold. In this paper, we need another formula, which is similar to (2.1). The following theorem is interesting in its own right.

THEOREM 2.1. *Let $f \in H^\infty$, $\|f\|_\infty = 1$, and $0 < \varepsilon < 1/2 < \sigma < 1$. Let R be an open subset of D such that $\partial R \cap D$ is a system of rectifiable curves satisfying*

(i) $|f(z)| < \varepsilon$ *for* $z \in R$.

We assign the usual orientation on ∂R. Put $\Gamma = \partial R \cap D$. Let h be a function in H^∞ such that $\|h\|_\infty = 1$,

(ii) $0 < 1/2 \leq |h(z)|$ *for* $z \in D \setminus R$, *and*

(iii) $|h(e^{i\theta})| \geq \sigma$ *for almost every* $e^{i\theta} \in \partial D$ *with* $|f(e^{i\theta})| > \varepsilon$.

Then

$$\left| \int_\Gamma \frac{fF}{h}\, dz - \int_{\partial D} fF\bar{h}\, dz \right| \leq 4(\varepsilon + 1 - \sigma)\|F\|_1$$

for every $F \in H^\infty$, where $\|F\|_1 = \int_0^{2\pi} |F(e^{i\theta})|\, d\theta/2\pi$.

Proof. For $0 < r < 1$, put $D_r = \{z \in D;\ |z| < r\}$ and

$$G_r = D_r \setminus \bar{R}. \tag{2.2}$$

We assign the curves in ∂G_r the usual positive orientation. Let A_r be a subset of ∂D such that

$$rA_r = \partial G_r \cap \partial D_r. \tag{2.3}$$

Let $F \in H^\infty$. Then by Cauchy's theorem,

$$\int_{rA_r} \frac{fF}{h}\, dz + \int_{\partial G_r \cap D_r} \frac{fF}{h}\, dz = 0. \tag{2.4}$$

By (ii) and the dominated convergence theorem, we have

$$\int_{\partial G_r \cap D_r} \frac{fF}{h}\, dz \to -\int_\Gamma \frac{fF}{h}\, dz \ \text{ as } r \to 1. \tag{2.5}$$

Put

$$E = \{e^{i\theta} \in \partial D;\ |f(e^{i\theta})| > \varepsilon\}. \tag{2.6}$$

Then

$$\left| \int_E fF\bar{h}\, dz - \int_{\partial D} fF\bar{h}\, dz \right| \leq \int_{\partial D \setminus E} |fF\bar{h}|\, |dz| \leq \varepsilon \|F\|_1. \tag{2.7}$$

By (iii) and $\sigma > 1/2$, we have

$$\left| \int_E \frac{fF}{h}\, dz - \int_E fF\bar{h}\, dz \right| \leq 4(1-\sigma)\|F\|_1.$$

Therefore, by (2.7),

$$\left| \int_E \frac{fF}{h}\, dz - \int_{\partial D} fF\bar{h}\, dz \right| \leq (\varepsilon + 4(1-\sigma))\|F\|_1. \tag{2.8}$$

By (ii), (2.2), and (2.3), $|h| \geq 1/2$ on rA_r. Hence

$$\left|\int_{A_r\setminus E}\left(\frac{fF}{h}\right)(rz)\,dz\right| \leq 2\int_{\partial D\setminus E}|(fF)(rz)|\,|dz| \to 2\int_{\partial D\setminus E}|(fF)(z)|\,|dz|$$

as $r \to 1$. Then, by (2.6),

$$\limsup_{r\to 1}\left|\int_{A_r\setminus E}\left(\frac{fF}{h}\right)(rz)\,dz\right| \leq 2\varepsilon\|F\|_1.$$

By (2.4),

$$\int_{\partial G_r\cap D_r}\frac{fF}{h}\,dz + r\int_{E\cap A_r}\left(\frac{fF}{h}\right)(rz)\,dz = -r\int_{A_r\setminus E}\left(\frac{fF}{h}\right)(rz)\,dz.$$

Hence

$$\limsup_{r\to 1}\left|\int_{\partial G_r\cap D_r}\frac{fF}{h}\,dz + r\int_{E\cap A_r}\left(\frac{fF}{h}\right)(rz)\,dz\right| \leq 2\varepsilon\|F\|_1. \tag{2.9}$$

For $e^{i\theta} \in E$, by (i) and (2.6) we have $te^{i\theta} \notin \bar{R}$ for t $(0 < t < 1)$ sufficiently close to 1. Then, by (2.2) and (2.3), $\chi_{E\cap A_r}(e^{i\theta}) \to \chi_E(e^{i\theta})$ as $r \to 1$ for almost every point $e^{i\theta}$ in ∂D. Hence by the dominated convergence theorem,

$$r\int_{E\cap A_r}\left(\frac{fF}{h}\right)(rz)\,dz \to \int_E \frac{fF}{h}\,dz \quad \text{as } r \to 1. \tag{2.10}$$

Thus, for $F \in H^\infty$, we obtain

$$\begin{aligned}
&\left|\int_\Gamma \frac{fF}{h}\,dz - \int_{\partial D} fF\bar{h}\,dz\right| \\
&= \lim_{r\to 1}\left|-\int_{\partial G_r\cap D_r}\frac{fF}{h}\,dz - \int_{\partial D} fF\bar{h}\,dz\right| \quad \text{(by (2.5))} \\
&\leq \limsup_{r\to 1}\left|\int_{\partial G_r\cap D_r}\frac{fF}{h}\,dz + \int_E \frac{fF}{h}\,dz\right| + (\varepsilon + 4(1-\sigma))\|F\|_1 \quad \text{(by (2.8))} \\
&= \limsup_{r\to 1}\left|\int_{\partial G_r\cap D_r}\frac{fF}{h}\,dz + r\int_{E\cap A_r}\left(\frac{fF}{h}\right)(rz)\,dz\right| + (\varepsilon + 4(1-\sigma))\|F\|_1 \\
&\leq 4(\varepsilon + 1 - \sigma)\|F\|_1 \quad \text{(by (2.9))},
\end{aligned}$$

where the last equality follows from (2.10). □

Recall condition (α):

(α) For any $0 < \sigma < 1$ and a subset A of D such that $Z(I) \cap \bar{A} = \emptyset$, there exists an $h \in I$ such that $\|h\|_\infty \leq 1$ and $|h| \geq \sigma$ on A.

The main theorem of this paper is the following.

THEOREM 2.2. *Let I be an ideal in H^∞ satisfying condition (α). Then $\bar{I} = I(Z(I))$.*

In order to prove our theorem, we need the following lemma due to Bourgain [2, pp. 165–166]. We denote by H^1 the usual Hardy space on ∂D.

LEMMA 2.3. *Let $f \in H^\infty$ with $\|f\|_\infty \le 1$. Then, for $\varepsilon > 0$, there exists an open subset R of D such that ∂R is a system of rectifiable curves and*

(i) $|f| < \varepsilon$ *on* R,
(ii) $|f| \ge \delta(\varepsilon)$ *on* $D \setminus R$,
(iii) $\int_{\partial R \cap D} |F|\,|dz| \le C\|F\|_1$ *for every* $F \in H^\infty$,

where $\delta(\varepsilon)$ is a fixed positive function of ε (independent of f), $\delta(\varepsilon) < \varepsilon$, and C is a universal constant.

Proof of Theorem 2.2. Let $f \in I(Z(I))$ and $\|f\|_\infty = 1$. We shall prove that $f \in \bar{I}$. Take ε as $0 < \varepsilon < 1/2$. Then, by Lemma 2.3, there exist $\delta(\varepsilon)$ $(0 < \delta(\varepsilon) < \varepsilon)$ and an open subset R of D such that $\partial R \cap D$ is a system of rectifiable curves, say $\Gamma = \partial R \cap D$, satisfying the following conditions:

$$|f(z)| < \varepsilon \quad \text{if } z \in R, \tag{2.11}$$

$$|f(z)| \ge \delta(\varepsilon) \quad \text{if } z \in D \setminus R, \tag{2.12}$$

$$\int_\Gamma |F|\,|dz| \le C\|F\|_1 \quad \text{for } F \in H^\infty, \tag{2.13}$$

where C is a universal constant.

Since $Z(I) \subset Z(f) \subset \{x \in M(H^\infty);\ |f(x)| < \delta(\varepsilon)\}$ and since I satisfies condition (α), there exists a function $h \in I$ such that $\|h\|_\infty = 1$ and $|h| \ge 1 - \varepsilon$ on $\{z \in D;\ |f(z)| \ge \delta(\varepsilon)\}$. Then, by (2.12),

$$|h(z)| \ge 1 - \varepsilon \quad \text{for } z \in D \setminus R. \tag{2.14}$$

Put

$$E = \{e^{i\theta} \in \partial D;\ |f(e^{i\theta})| > \varepsilon\}. \tag{2.15}$$

For $e^{i\theta} \in E$, by (2.11) we have $te^{i\theta} \notin \bar{R}$ for t $(0 < t < 1)$ sufficiently close to 1. Hence, by (2.14),

$$|h| \ge 1 - \varepsilon \ \text{ on } E. \tag{2.16}$$

Applying Theorem 2.1 for $\sigma = 1 - \varepsilon$, we have

$$\left| \int_\Gamma \frac{fF}{h}\,dz - \int_{\partial D} fF\bar{h}\,dz \right| \le 8\varepsilon\|F\|_1 \quad \text{for } F \in H^\infty. \tag{2.17}$$

By (2.11), (2.13), and (2.14), we have

$$\left| \int_\Gamma \frac{fF}{h}\,dz \right| \le \frac{\varepsilon}{1-\varepsilon} \int_\Gamma |F|\,|dz| \le 2C\varepsilon\|F\|_1 \quad \text{for } F \in H^\infty. \tag{2.18}$$

Hence, by (2.17) and (2.18), we obtain

$$\left| \int_{\partial D} fF\bar{h}\,dz \right| \le C_1\varepsilon\|F\|_1 \quad \text{for } F \in H^\infty,$$

where C_1 is another absolute constant. Since L^∞/H^∞ is the dual space of the Banach space zH^1, it follows by the preceding fact and in the same way as in [10, pp. 177–178] that $\|f\bar{h} + H^\infty\| \le C_1\varepsilon$. Hence $\|f|h|^2 + hH^\infty\| \le C_1\varepsilon$. By (2.15) and (2.16), $\|f - f|h|^2\|_\infty \le 2\varepsilon$. Thus we get $\|f + hH^\infty\| \le (2 + C_1)\varepsilon$. Since

$h \in I$, it follows that $hH^\infty \subset I$. Hence we have $f \in \bar{I}$, which completes the proof. □

Generally, the converse assertion of Theorem 2.2 does not hold; a counterexample is $I = zH^\infty$. We shall prove that the converse of Theorem 2.2 is true under some conditions on I; see Corollary 2.7. To show this, we need some notation. For points m_1 and m_2 in $M(H^\infty)$, the pseudohyperbolic distance from m_1 to m_2 is

$$\rho(m_1, m_2) = \sup\{|f(m_2)|;\ \|f\|_\infty \le 1,\ f(m_1) = 0\}.$$

For $\varphi \in M(H^\infty)$, let

$$P(\varphi) = \{m \in M(H^\infty);\ \rho(\varphi, m) < 1\},$$

which is called the *Gleason part* containing φ. Let G be the set of point φ in $M(H^\infty)$ such that $P(\varphi) \neq \{\varphi\}$. By Hoffman's work [12], G is an open subset of $M(H^\infty)$, and for each $\varphi \in G$ there exists an interpolating Blaschke product b such that $\varphi(b) = 0$ as well as a continuous one-to-one map L_φ from D onto $P(\varphi)$ such that $L_\varphi(0) = \varphi$ and $f \circ L_\varphi \in H^\infty$ for every $f \in H^\infty$.

PROPOSITION 2.4. *Let I be an ideal in H^∞.*

(i) *If $I = I(Z(I))$ and $P(\varphi) \subset Z(I)$ for every $\varphi \in Z(I) \cap G$, then I satisfies condition (α).*

(ii) *Let E be a closed subset of $M(H^\infty) \setminus D$ such that $E \cap M(L^\infty) = \emptyset$. Then $J(E)$ satisfies condition (α).*

To prove Proposition 2.4, we need the following lemmas due to Suárez.

LEMMA 2.5 [20, pp. 242–244]. *Let I be an ideal in H^∞. Then for every open subset U of $M(H^\infty)$ such that $Z(I) \subset U$, there exists f in I such that $Z(f) \subset U$.*

For a function f in H^∞, put

$$Z_\infty(f) = (Z(f) \setminus G) \cup \{m \in Z(f) \cap G;\ f \circ L_m \equiv 0 \text{ on } D\}.$$

LEMMA 2.6 [9, Thm. 1.3; 21, Thm. 2.5]. *Let b be a Blaschke product and let E be a closed subset of $M(H^\infty)$ such that $|b| > 0$ on E. Let $0 < \sigma < 1$. Then there is a factorization $b = b_0 b_1 \cdots b_m$ such that b_0 is a product of finitely many interpolating Blaschke products, $|b_j| \ge \sigma$ on E, and $Z_\infty(b_j) = Z_\infty(b)$ for $1 \le j \le m$.*

Proof of Proposition 2.4. (i) Let $0 < \sigma < 1$ and $A \subset D$ such that $Z(I) \cap \bar{A} = \emptyset$. Since $Z(I(Z(I))) = Z(I)$, by Lemma 2.5 there exists $f \in I(Z(I))$ such that $\|f\|_\infty = 1$ and

$$\inf\{|f(z)|;\ z \in A\} > 0. \tag{2.19}$$

By Lemma 2.6, we can write

$$f = bh = b_0 b_1 \cdots b_n h, \tag{2.20}$$

where b is a Blaschke factor of f, $h \in H^\infty$ is zero-free on D, b_0 is a product of finitely many interpolating Blaschke products, and b_j $(1 \le j \le n)$ are Blaschke products such that

$$|b_j| \geq (1+\sigma)/2 \ \text{ on } A \tag{2.21}$$

and $Z_\infty(b_j) = Z_\infty(b)$. Since $P(\varphi) \subset Z(I)$ for $\varphi \in Z(I) \cap G$, we have $f \circ L_\varphi \equiv 0$ on D for every $\varphi \in Z(I) \cap G$. Hence

$$Z(I) \subset Z_\infty(f) = Z_\infty(b) \cup Z_\infty(h) = Z_\infty(b_j h).$$

Thus we get $b_j h \in I(Z(I))$ for every j, $1 \leq j \leq n$. Then $b_j h^{1/k} \in I(Z(I))$ for every positive integer k. By (2.19) and (2.20), $\inf\{|h(z)|;\ z \in A\} > 0$. Therefore, by (2.21), for a sufficiently large k we have

$$|b_j h^{1/k}| \geq \frac{1+3\sigma}{4} \geq \sigma \ \text{ on } A.$$

Hence I satisfies (α).

(ii) By Newman's theorem [18] (see also [11, pp. 179]), for each $x \in E$ there exists a Blaschke product b_x such that $b_x(x) = 0$. Let $\{z_n\}_n$ be the zeros of b_x in D. Then there exists a sequence of positive integers $p_x = (p_1, p_2, \dots)$ such that $p_n \to \infty$ as $n \to \infty$ and $\sum_{n=1}^\infty p_n(1 - |z_n|) < \infty$. Associated with p_x, we have the following Blaschke product:

$$b_x^{p_x}(z) = \prod_{n=1}^\infty \left(\frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z} \right)^{p_n}, \quad z \in D.$$

Then

$$\{\zeta \in M(H^\infty) \setminus D;\ |b_x(\zeta)| < 1\} \subset \{\zeta \in M(H^\infty) \setminus D;\ b_x^{p_x}(\zeta) = 0\}.$$

Hence we may assume that b_x vanishes on a neighborhood of x in $M(H^\infty) \setminus D$. Since E is a compact set, there exist $x_j \in E$ $(j = 1, 2, \dots, n)$ such that $\prod_{j=1}^n b_{x_j}$ vanishes on an open subset of $M(H^\infty) \setminus D$ that contains E. Thus we get $J(E) \neq \{0\}$.

Next, we prove that $J(E)$ satisfies condition (α). The proof is the same as that for (i). Replace $I = I(Z(I))$ by $J(E)$, and follow the proof of (i). In this case, we have $f \in J(E)$. Then there is an open subset U of $M(H^\infty) \setminus D$ such that $E \subset U$ and $f = 0$ on U. Let $f = bh = b_0 b_1 \cdots b_n h$ be the factorization in (2.20). We need to prove $b_j h \in J(E)$. Since b_0 is an interpolating Blaschke product, $b_1 b_2 \cdots b_n h = 0$ on U. For $\zeta \in U \cap G$, $(b_1 b_2 \cdots b_n h) \circ L_\zeta(z)$ vanishes on some open subset of D. Hence $(b_1 b_2 \cdots b_n h) \circ L_\zeta \equiv 0$ on D, so that $Z_\infty(b_1 b_2 \cdots b_n h) \supset U$. Since $Z_\infty(b_j) = Z_\infty(b)$, we have $Z_\infty(b_j h) \supset U$. Thus $b_j h \in J(E)$ for every j, $1 \leq j \leq n$. Hence the proof of (i) works in this case, too. □

COROLLARY 2.7. *Let I be an ideal in H^∞ such that $P(\varphi) \subset Z(I)$ for every $\varphi \in Z(I) \cap G$. Then $\bar{I} = I(Z(I))$ if and only if I satisfies condition (α).*

Proof. Suppose that $\bar{I} = I(Z(I))$. Let $0 < \sigma < 1$ and $A \subset D$ such that $Z(I) \cap \bar{A} = \emptyset$. By Proposition 2.4(i), there exists $h \in I(Z(I))$ such that $\|h\|_\infty = 1$ and $|h| \geq (1+\sigma)/2$ on A. Since $\bar{I} = I(Z(I))$, by the foregoing there exists a $g \in I$ such that $\|g\|_\infty = 1$ and $|g| \geq \sigma$ on A. Hence I satisfies (α). The converse is just Theorem 2.2. □

COROLLARY 2.8. *Let I be an ideal in H^∞ that is algebraically generated by countably many functions. Suppose that $P(\varphi) \subset Z(I)$ for every $\varphi \in Z(I) \cap G$. Then $I(Z(I))$ is the closure of an ideal generated by countably many functions.*

Proof. Suppose that I is an ideal generated by $\{f_n\}_n$ in H^∞. Then $Z(I) = \bigcap_{n=1}^\infty Z(f_n)$. Since $Z(f_n)$ is a G_δ-set, so is $Z(I)$. Let $\{V_k\}_k$ be a sequence of decreasing open subsets of $M(H^\infty)$ such that $Z(I) = \bigcap_{k=1}^\infty V_k$. Since $V_k^c \cap Z(I) = \emptyset$, by the corona theorem [3] there is a subset $A_k \subset D$ such that $\overline{A_k} \supset V_k^c$ and $\overline{A_k} \cap Z(I) = \emptyset$. By Proposition 2.4(i) and our assumption, $I(Z(I))$ satisfies condition (α). Hence there exist $h_k \in I(Z(I))$ such that $\|h_k\|_\infty = 1$ and $|h_k| > 1-1/k$ on V_k^c. Let J be an ideal generated by $\{h_k\}_k$. Then $Z(J) = Z(I)$ and J satisfies condition (α). Thus, by Theorem 2.2, $\bar{J} = I(Z(J)) = I(Z(I))$. □

In Corollary 2.8, the conclusion does not mean that $\bar{I} = I(Z(I))$. For let I be an ideal generated by a single function $\psi = \exp\left(-\frac{1+z}{1-z}\right)$. Then $I = \psi H^\infty$ is a closed ideal of H^∞ and it is not difficult to see that I satisfies the asssumption of Corollary 2.8. Since $\psi^{1/2} \notin I$ and $\psi^{1/2} \in I(Z(I))$, it follows that $\bar{I} = I \neq I(Z(I))$.

By Theorem 2.2 and Proposition 2.4(ii), we have the following.

COROLLARY 2.9 [8, Thm. 4.2]. *Let E be a closed subset of $M(H^\infty) \setminus D$ such that $E \cap M(L^\infty) = \emptyset$. Then $\overline{J(E)} = I(Z(J(E)))$.*

We give other examples of ideals satisfying condition (α).

PROPOSITION 2.10. *The following ideals I in H^∞ satisfy condition (α).*

(i) *I is a prime ideal in H^∞ that does not contain any interpolating Blaschke products.*

(ii) *For a function f in H^∞ not vanishing on D, let I be the ideal in H^∞ algebraically generated by functions $f^{1/n}$, $n = 1, 2, \ldots$.*

(iii) *Let $\mathcal{S}$ be a set of nonnegative bounded singular measures μ $(\mu \neq 0)$ on ∂D. Suppose that $\mathcal{S}$ satisfies the following conditions*:

(a) *for $\mu, \nu \in \mathcal{S}$, there exists a $\lambda \in \mathcal{S}$ such that $\lambda \leq \mu \wedge \nu$, where $\mu \wedge \nu$ is the greatest lower bound of μ and ν*;

(b) *for every $\mu \in \mathcal{S}$ and every positive integer n, there exists a $\lambda \in \mathcal{S}$ such that $n\lambda \leq \mu$.*

Let I be the ideal algebraically generated by singular functions ψ_μ, $\mu \in \mathcal{S}$.

Proof. (i) The proof is given in [7, pp. 187–188] essentially. For the sake of completeness, we run through the proof here. Suppose that I is a prime ideal in H^∞ and does not contain any interpolating Blaschke products. Let $0 < \sigma < 1$ and let A be a subset of D such that $Z(I) \cap \bar{A} = \emptyset$. Then, by Lemma 2.5, there exists an $f \in I$ such that $\|f\|_\infty \leq 1$ and $\inf\{|f(z)|;\ z \in A\} > 0$. Put $f = bF$, where b is a Blaschke product and F is zero-free on D. Since I is prime, $b \in I$ or $F \in I$. Suppose that $F \in I$. Then $F^{1/n} \in I$, $\|F^{1/n}\|_\infty \leq 1$, and $|F^{1/n}| > \sigma$ on A for a sufficiently large n.

Suppose that $b \in I$. Then $\inf\{|b(z)|;\ z \in A\} > 0$. By Lemma 2.6, there is a factorization $b = b_0 b_1 \cdots b_k$ such that b_0 is a product of finitely many interpolating

Blaschke products and $|b_j| \geq \sigma$ on A for every j, $1 \leq j \leq k$. By our assumption, $b_j \in I$ for some j, $1 \leq j \leq k$. Thus I satisfies (α).

It is not difficult to prove that an ideal I with (ii) satisfies (α).

(iii) Let $\mu_1, \mu_2 \in \mathcal{S}$. Then, by (a), there exists a $\mu_3 \in \mathcal{S}$ such that $\mu_3 \leq \mu_1 \wedge \mu_2$; this yields $|\psi_{\mu_3}| \geq |\psi_{\mu_j}|$ for $j = 1, 2$. Thus we get $Z(\psi_{\mu_3}) \subset Z(\psi_{\mu_1}) \cap Z(\psi_{\mu_2})$. Therefore, by the finite intersection property, $\bigcap\{Z(\psi_\mu);\ \mu \in \mathcal{S}\} \neq \emptyset$. Hence I is a proper ideal.

Let $0 < \sigma < 1$ and let $A \subset D$ satisfy $Z(I) \cap \bar{A} = \emptyset$. By Lemma 2.5, there exists an $f \in I$ such that $\inf\{|f(z)|;\ z \in A\} > 0$. We may assume that $f = \psi_\mu$ for some $\mu \in \mathcal{S}$. For each positive integer n, by (b) there exist $\lambda_n \in \mathcal{S}$ such that $n\lambda_n \leq \mu$. Hence $|\psi_\mu^{1/n}| \leq |\psi_{\lambda_n}|$ on D. For a sufficiently large integer n, we have $\sigma \leq |\psi_\mu^{1/n}| \leq |\psi_{\lambda_n}|$ on A. Therefore, condition (α) holds. □

By Theorem 2.2 and Proposition 2.10, we have the following corollary.

COROLLARY 2.11. *Let f be a function in H^∞ that does not vanish on D. Let I be the ideal in H^∞ that is algebraically generated by functions $f^{1/n}$, $n = 1, 2, \dots$. Then $\bar{I} = I(Z(I))$.*

We also have the following.

COROLLARY 2.12. *Let I be a prime ideal in H^∞. Then $\bar{I} = I(Z(I))$.*

Proof. Suppose that I is prime. If I does not contain any interpolating Blaschke product, then our assertion follows from Theorem 2.2 and Proposition 2.10.

Suppose that I contains an interpolating Blaschke product b. Then, by [5, Thm. 4.1; 16, Thm. 3.1], it is known that $\bar{I} = \operatorname{Ker}\varphi$ for some $\varphi \in M(H^\infty)$. Hence $Z(I) = \{\varphi\}$ and $\bar{I} = \operatorname{Ker}\varphi = I(\{\varphi\}) = I(Z(I))$. We can also prove this by using [6, Thm. 2.2]. □

3. Outer Functions

First, we recall Jensen's equality. For a point $\varphi \in M(H^\infty)$, there is a probability measure μ_φ on $M(L^\infty)$ such that $\int_{M(L^\infty)} f\, d\mu_\varphi = \varphi(f)$ for every $f \in H^\infty$. We denote by $\operatorname{supp}\mu_\varphi$ the closed support set of μ_φ. Then

$$\log|\varphi(f)| \leq \int_{M(L^\infty)} \log|f|\, d\mu_\varphi, \quad f \in H^\infty;$$

this is called *Jensen's inequality*. When

$$\log|\varphi(f)| = \int_{M(L^\infty)} \log|f|\, d\mu_\varphi,$$

we say that f satisfies Jensen's equality for a point $\varphi \in M(H^\infty)$; see [11, Chap. 10]. It is well known that every invertible function in H^∞ satisfies Jensen's equality for every point in $M(H^\infty)$. If f is an outer function in H^∞, then f satisfies Jensen's equality for every point $z \in D$.

Let f be a function in H^∞ that is not invertible in H^∞. Then $I = fH^\infty$ is an ideal generated by f. In this section, we study the problem when $\bar{I} = I(Z(I))$

holds for a singly generated ideal I. If f has a nontrivial inner factor then $\bar{I} \neq I(Z(I))$ holds, so we are interested in the case that f is outer.

EXAMPLE 3.1. Let $f(z) = (1-z)/2$. Then f is an outer function and is not invertible in H^∞. Let $I = fH^\infty$ be the ideal generated by f. Then it is not difficult to see that, for $h \in I(Z(I))$,

$$\left\| h - hf\left(\sum_{k=0}^{n-1}\left(\frac{1+z}{2}\right)^k\right)\right\|_\infty = \left\| h - h\left(1 - \left(\frac{1+z}{2}\right)^n\right)\right\|_\infty \to 0 \text{ as } n \to \infty.$$

Thus $h \in \bar{I}$ and hence $\bar{I} = I(Z(I))$.

There is an outer function f that is not invertible in H^∞ such that $I = fH^\infty$ and $\bar{I} \neq I(Z(I))$. We shall give such an example.

EXAMPLE 3.2. Let

$$f(z) = \exp\left(\int_0^1 \frac{e^{i\theta}+z}{e^{i\theta}-z}\log\theta \,\frac{d\theta}{2\pi}\right), \quad z \in D.$$

Then f is an outer function in H^∞ that is not invertible in H^∞, and

$$|f(e^{i\theta})| = \begin{cases} \theta & \text{for } 0 < \theta < 1, \\ 1 & \text{for } 1 < \theta < 2\pi. \end{cases} \tag{3.1}$$

Let $I = fH^\infty$. Since f is outer, by [12, Lemma 2.2] $P(\varphi) \subset Z(f)$ for every $\varphi \in Z(f) \cap G$. Since $Z(I) = Z(f)$, we have $P(\varphi) \subset Z(I)$ for every $\varphi \in Z(I) \cap G$. We shall show that $\bar{I} \neq I(Z(I))$. By Corollary 2.7, it is sufficient to prove that the ideal I does not satisfy condition (α). We have

$$\log|f(z)| = \int_0^1 P_z(e^{i\theta})\log\theta\,\frac{d\theta}{2\pi}, \tag{3.2}$$

where P_z is the Poisson kernel for $z \in D$. By elementary properties of Poisson kernels, there exists a sequence $\{z_n\}_n$ in D such that $z_n \to 1$ and

$$-\frac{1}{2} < \int_0^1 P_{z_n}(e^{i\theta})\log\theta\,\frac{d\theta}{2\pi} < -\frac{1}{3}. \tag{3.3}$$

Put $A = \{z_n\}_n$. Then, by (3.2), $Z(I) \cap \bar{A} = \emptyset$. Let $g \in I$ and $\|g\|_\infty \le 1$. Then $g = fh$ for some $h \in H^\infty$. Since $d\mu_z = P_z\,d\theta/2\pi$, by Jensen's inequality we have

$$\begin{aligned} \log|g(z)| &\le \int_0^{2\pi} P_z(e^{i\theta})\log|g(e^{i\theta})|\,\frac{d\theta}{2\pi} \\ &\le \int_0^1 P_z(e^{i\theta})\log|g(e^{i\theta})|\,\frac{d\theta}{2\pi} \quad (\text{because } \|g\|_\infty \le 1) \\ &\le \int_0^1 P_z(e^{i\theta})\log(\|h\|_\infty\theta)\,\frac{d\theta}{2\pi} \quad (\text{by (3.1)}). \end{aligned}$$

Here we have $\log(\|h\|_\infty\theta)/\log\theta \to 1$ as $\theta \to +0$. Then there exists $K > 1$ such that

$$\log(\|h\|_\infty \theta) < K \log\theta \quad \text{for } 0 < \theta < 1.$$

Hence

$$\log|g(z_n)| \le K \int_0^1 P_{z_n}(e^{i\theta}) \log\theta \,\frac{d\theta}{2\pi}.$$

By (3.3), we have

$$\limsup_{n\to\infty} \log|g(z_n)| \le K \limsup_{n\to\infty} \int_0^1 P_{z_n}(e^{i\theta}) \log\theta \,\frac{d\theta}{2\pi} \le -\frac{K}{3}.$$

It follows that

$$\limsup_{n\to\infty} |g(z_n)| \le e^{-K/3}.$$

Consequently, I does not satisfy condition (α). □

In order to understand our main theorem (Theorem 3.2) in this section, we show that the function f given in Example 3.2 does not satisfy Jensen's equality for a point m such that $m(f) \ne 0$. We use the same notation as in Example 3.2. Let m be a cluster point of $\{z_n\}_n$ in $M(H^\infty)$. Then, by (3.2) and (3.3),

$$-1/2 \le \log|m(f)| \le -1/3. \tag{3.4}$$

We shall prove that

$$\int_{M(L^\infty)} \log|f| \,d\mu_m = 0. \tag{3.5}$$

Since $z_n \to 1$, it follows that $\operatorname{supp}\mu_m \subset \{\varphi \in M(L^\infty);\ \varphi(z) = 1\}$, where z is the identity function on D.

Let $E = \{e^{i\theta}; -1 \le \theta < 0\}$. Then, by (3.1), we have $|f| = \chi_E$ on $\{\varphi \in M(L^\infty); \varphi(z) = 1\}$. Since $\log|m(f)| > -\infty$, by Jensen's inequality $\int_{M(L^\infty)} \log|f| \,d\mu_m > -\infty$. Since $\log|f| = 0$ or $-\infty$ on $\operatorname{supp}\mu_m$, we have

$$\operatorname{supp}\mu_m \subset \{x \in M(L^\infty);\ \log|f(x)| = 0\}.$$

Thus we obtain (3.5). By (3.4) and (3.5), f does not satisfy Jensen's equality for a point m such that $m(f) \ne 0$.

Now our theorem is the following.

THEOREM 3.2. *Let f be an outer function in H^∞ that is not invertible in H^∞. Let $I = fH^\infty$ be the ideal generated by f. Then $\bar{I} = I(Z(I))$ if and only if f satisfies Jensen's equality for every point m in $M(H^\infty)$ such that $m(f) \ne 0$.*

Proof. We may assume that $\|f\|_\infty = 1$. Since f is outer, we have $P(\varphi) \subset Z(I)$ for every $\varphi \in Z(I) \cap G$; see [12, Lemma 2.2]. Hence, by Corollary 2.7, it is sufficient to prove that I satisfies condition (α) if and only if f satisfies Jensen's equality for every point m in $M(H^\infty)$ such that $m(f) \ne 0$.

First, suppose that f does not satisfy Jensen's equality for a point m in $M(H^\infty)$ such that $m(f) \ne 0$. Then Jensen's inequality yields

$$0 < |m(f)| < \exp\left(\int_{M(L^\infty)} \log|f| \,d\mu_m\right).$$

By the corona theorem [3], there exists a net $\{z_\alpha\}_\alpha$ in D such that $z_\alpha \to m$ and $|f(z_\alpha)| > m(f)/2$. Put $A = \{z_\alpha\}_\alpha$. Then $\bar{A} \cap Z(I) = \bar{A} \cap Z(f) = \emptyset$.

Let $g \in I$ and $\|g\|_\infty \le 1$. Then $g = fh$ for some $h \in H^\infty$, and we have

$$\exp\left(\int_{M(L^\infty)} \log|f|\, d\mu_m\right) \exp\left(\int_{M(L^\infty)} \log|h|\, d\mu_m\right) = \exp\left(\int_{M(L^\infty)} \log|g|\, d\mu_m\right) \le 1.$$

Hence, by Jensen's inequality,

$$\begin{aligned} |m(g)| &= |m(f)||m(h)| \\ &\le |m(f)| \exp\left(\int_{M(L^\infty)} \log|h|\, d\mu_m\right) \\ &\le \frac{|m(f)|}{\exp\left(\int_{M(L^\infty)} \log|f|\, d\mu_m\right)} < 1. \end{aligned}$$

Since $m \in \bar{A}$, these inequalities imply that I does not satisfy condition (α).

Next, suppose that

$$\int_{M(L^\infty)} \log|f|\, d\mu_\varphi = \log|\varphi(f)| \quad \text{for every } \varphi \in M(H^\infty),\ \varphi(f) \neq 0. \tag{3.6}$$

Let $0 < \sigma < 1$ and let $A \subset D$ satisfy $Z(I) \cap \bar{A} = \emptyset$. Let $m \in \bar{A}$. Then $m(f) \neq 0$, so that by (3.6) we have $\int_{M(L^\infty)} \log|f|\, d\mu_m > -\infty$. Hence there exists an open and closed subset V_m of $M(L^\infty)$ such that $Z(f) \cap M(L^\infty) \subset V_m$ and

$$\frac{\exp \int_{M(L^\infty)} \log|f|\, d\mu_m}{\exp \int_{M(L^\infty)\setminus V_m} \log|f|\, d\mu_m} > \sigma. \tag{3.7}$$

Let $\tilde{V}_m$ be a measurable subset of ∂D such that $\hat{\chi}_{\tilde{V}_m} = \chi_{V_m}$, where $\hat{\chi}_{\tilde{V}_m}$ is the Gelfand transform of $\chi_{\tilde{V}_m} \in L^\infty$. Let

$$h_m(z) = \exp \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \chi_{\tilde{V}_m} \log|f| \frac{d\theta}{2\pi}, \quad z \in D, \tag{3.8}$$

and

$$g_m(z) = \exp \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \chi_{\partial D\setminus \tilde{V}_m} \log|f| \frac{d\theta}{2\pi}, \quad z \in D. \tag{3.9}$$

Then h_m and g_m are outer functions in H^∞, g_m is invertible in H^∞, and

$$f = h_m g_m. \tag{3.10}$$

Hence $h_m \in I$ and

$$\int_{M(L^\infty)} \log|g_m|\, d\mu_\varphi = \log|\varphi(g_m)| \quad \text{for every } \varphi \in M(H^\infty). \tag{3.11}$$

We have

$$\begin{aligned} |m(h_m)| &= |m(f)| \exp\left(-\int_{M(L^\infty)} \log|g_m|\, d\mu_m\right) \quad \text{(by (3.10) and (3.11))} \\ &= \frac{\exp \int_{M(L^\infty)} \log|f|\, d\mu_m}{\exp \int_{M(L^\infty)\setminus V_m} \log|f|\, d\mu_m} \quad \text{(by (3.6) and (3.9))} \\ &> \sigma \quad \text{(by (3.7))}. \end{aligned}$$

Since $\bar{A}$ is compact, there exist $m_1, m_2, \dots, m_k \in \bar{A}$ such that

$$\max\{|h_{m_1}|, \dots, |h_{m_k}|\} \geq \sigma \quad \text{on } \bar{A}. \tag{3.12}$$

Let $V = \bigcap_{j=1}^k V_{m_j}$, $\tilde{V} = \bigcap_{j=1}^k \tilde{V}_{m_j}$, and

$$h(z) = \exp \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \chi_{\tilde{V}} \log|f| \frac{d\theta}{2\pi}, \quad z \in D. \tag{3.13}$$

Then $\hat{\chi}_{\tilde{V}} = \chi_V$ and $Z(f) \cap M(L^\infty) \subset V$. Moreover, we have $h \in I$ by the same reason as that used for $h_m \in I$. Since $\tilde{V} \subset \tilde{V}_{m_j}$ and $\|f\|_\infty = 1$, by (3.8) and (3.13) we have $|h(z)| \geq |h_{m_j}(z)|$ for $z \in D$. Hence, by (3.12), $|h| \geq \sigma$ on U. Thus I satisfies condition (α), which completes the proof. □

We do not know of any function-theoretic characterization of an outer function f such that f satisfies Jensen's equality for every point in $M(H^\infty)$ with $m(f) \neq 0$. Axler and Shields [1, Prop. 5] showed that a function f in H^∞ with $\operatorname{Re} f > 0$ on D satisfies Jensen's equality for every point in $M(H^\infty)$. For an inner function q, the function $q+1$ satisfies this condition. Put $QA = H^\infty \cap \overline{H^\infty + C}$, where C is the space of continuous functions on ∂D and $\overline{H^\infty + C}$ is the set of complex conjugates of functions in $H^\infty + C$. Wolff [22] proved that, for every $f \in L^\infty$, there exists an outer function $h \in QA$ such that $hf \in H^\infty + C$. If $f \notin H^\infty + C$, then the function h is not invertible in H^∞. Thus there are many outer functions in QA that are not invertible in H^∞. Sarason [19] proved that, if $f \in H^\infty$, then $f \in QA$ if and only if $f|_{\operatorname{supp}\mu_\varphi}$ is constant for every $\varphi \in M(H^\infty) \setminus D$. Hence QA outer functions satisfy Jensen's equality for every $\varphi \in M(H^\infty)$. We have the following corollaries as applications of Theorem 3.2.

Corollary 3.3. *Let $I = fH^\infty$ be an ideal in H^∞ generated by a function f that is not invertible in H^∞, and let* $\operatorname{Re} f > 0$ *on D. Then $\bar{I} = I(Z(I))$.*

Corollary 3.4. *Let $I = fH^\infty$ be an ideal in H^∞ generated by an outer function in QA that is not invertible in H^∞. Then $\bar{I} = I(Z(I))$.*

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