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Categories of Topological Spaces and Scattered Theories

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Abstract We offer a topological treatment of scattered theories intended to help to explain the parallelism between, on the one hand, the theorems provable using Descriptive Set Theory by analysis of the space of countable models and, on the other, those provable by studying a tree of theories in a hierarchy of fragments of infinintary logic. We state some theorems which are, we hope, a step on the road to fully understanding counterexamples to Vaught's Conjecture. This framework is in the early stages of development, and one area for future exploration is the possibility of extending it to a setting in which the spaces of types of a theory are uncountable.

1 Introduction

1.1 Overview One usually expects that if different methods are applied to a problem, then the theorems that they generate will be different in kind, and perhaps complementary. This arguably applies to the model-theoretic, and set-theoretic, approaches to Vaught's Conjecture. The former approach has, inter alia, yielded theorems about circumstances under which Vaught's Conjecture is true, for example, for superstable theories of finite rank as proved by Buechler in [3] or for ω-stable theories as proved by Harrington, Makkai, and Shelah in [9]. The latter approach has, in contrast, yielded a variety of theorems giving properties of any counterexample, such as the existence of many saturated models.

One might therefore be struck by the similarity of content and even of "feel" between the theorems obtained by Sacks, as described in [8], using analysis of a certain tree of theories, the "Vaughtian tree," and those obtained by Steel, Becker, and others using descriptive set theory; see [10], [1], and [2]. Using the Vaughtian trees, one proves the existence of many saturated models and of an elementary chain of

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atomic models; using descriptive set theory, one again proves the existence of saturated models and shows that those too can be arranged in an elementary chain. And there are many other interesting results proved by these two techniques exhibiting similar parallelism.

What accounts for this similarity? After all, the Sacks analysis of a theory T is quite constructive in flavor and can be carried out in $L(T)$; in contrast, the theorems in the descriptive set theory approach often presuppose a determinacy axiom and are obtained by applying topological ideas to a space of countable models of T.

In this paper, we would like to propose an idea linking these two approaches. That idea is a *type category* of a scattered theory T, whose objects are the spaces of n-ary types for each n, with appropriate arrows. From this category both the Vaughtian tree and the space of models of T can be obtained. We begin by defining these type categories and then show how to obtain the spaces of models from them; then we define a set of trees, one of which will be the Vaughtian tree mentioned earlier; then we will use these to prove a theorem about embeddings of atomic and saturated models; and then we prove some structure theorems about models of a counterexample to Vaught's Conjecture.

It is worth emphasizing that the ideas described in this paper were developed with counterexamples to Vaught's Conjecture in mind. We therefore do not consider theories with uncountably many n-ary types, in the appropriate languages; this is a topic for further exploration.

1.2 Terminology and conventions For results and (most of our) terminology about infinitary logic, we refer to [5]. We define a *fragment* of $\mathcal{L}_{\omega_1,\omega}$ to be a subset closed under Boolean operations, first-order quantification, the taking of subformulas, and permutation of the set of variable letters. If A is a set, then $\mathbf{Set}_A$ is the category whose objects are all elements of A and the arrows are all functions between elements of A. If $m \leq n \leq \omega$, we use the notation $\iota_{m,n}$ to refer to the inclusion map from m to n. If $n \in \omega$, and $i < n$, then we use the notation d_i^n to refer to the map from $n-1$ to n given by

$$d_i^n : j \mapsto \begin{cases} j & \text{if } j < i \\ j+1 & \text{otherwise.} \end{cases}$$

If $n \in \omega \setminus \{0, 1\}$, and $i < n-1$, then we use the notation s_i^n to refer to the map from n to $n-1$ given by

$$s_i^n : j \mapsto \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{otherwise.} \end{cases}$$

If $n \in \omega$ and A is a subset of n, then we use the notation e_A to refer to the function enumerating A; that is, e_A is the unique one-to-one increasing function whose domain is $|A|$ and whose range is A.

One can write down a large number of identities connecting these; for instance, $e_{n\setminus\{i\}} = d_i^n$, $\iota_{n,m} = e_n$, and $d_{n-1}^n = \iota_{n,n+1}$. Any topological notions needed in the text may be found (except where otherwise indicated) in standard works such as [4].

2 Categories of Types

2.1 The definition of a type category We now define the central concept of this paper.

Definition 2.1 A *weak type category* is a contravariant functor S from $\mathbf{Set}_\omega$ to the category of topological spaces such that

1. for each n, Sn is a nonempty, zero-dimensional Hausdorff space;
2. whenever $f : n \to m$, $Sf : Sm \to Sn$ is continuous; and
3. we say that $p \in Sm$ *has the weak amalgamation property in* S iff whenever
   (a)
$$\begin{array}{ccc} m & \xrightarrow{f_1} & n_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ n_2 & \xrightarrow{g_2} & l \end{array}$$
   commutes,
   (b) f_1, f_2, g_1, and g_2 are one-to-one,
   (c) $\operatorname{ran} g_1 \cap \operatorname{ran} g_2 = \operatorname{ran} g_1 \circ f_1$,
   (d) U, V are nonempty and open, respectively, in $(Sf_1)^{-1}\{p\}$ and $(Sf_2)^{-1}\{p\}$,

   then there exists $q \in Sl$ such that $Sf_1 \circ Sg_1(q) = p$, $(Sg_1)(q) \in U$, and $(Sg_2)(q) \in V$. Then for every $m \in \omega$, for every $p \in Sm$, p has the weak amalgamation property in Sm.

If in addition all maps Sf are open, then S is a *type category*.

Condition 3 implies directly that if $f : m \to n$ is one-to-one, then $Sf : Sn \to Sm$ is onto. (This is provable indirectly as follows: if $f : m \to n$ is one-to-one, find $g : n \to m$ such that $g \circ f$ is the identity on m. Then since S is a functor, $S\imath_{m,m}$, which is the identity on Sm, is equal to $(Sf) \circ (Sg)$, so Sf is onto.)

The connection with logic is as follows.

Definition 2.2 Let $\mathcal{L}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$, and let T be a countable theory in $\mathcal{L}$. The *type category of* T *over* $\mathcal{L}$ is the type category $S_T^{\mathcal{L}}$ such that, for each n, $S_T^{\mathcal{L}} n$ is the space of complete n-ary $\mathcal{L}$-types from models of T with the usual topology (that is, that generated by sets of the form $\{p \in Sn \mid \varphi(x_0, \ldots, x_{n-1}) \in p\}$, for $n \in \omega$ and $\varphi \in \mathcal{L}$), and such that if $f : m \to n$, then for each $p \in S_T^{\mathcal{L}} n$,

$$(S_T^{\mathcal{L}} f)(p)(x_0, \ldots, x_{m-1}) = \{\varphi(x_0, \ldots, x_{m-1}) \mid \varphi(x_{f(0)}, \ldots, x_{f(m-1)}) \in p\}.$$

We should check that the maps $S_T^{\mathcal{L}} f$ are well-defined.

Proposition 2.3 *Suppose $p(x_0, \ldots, x_{n-1})$ is an n-ary type in $\mathcal{L}$ from a model of T; say $\mathfrak{M} \models T$ and p is the type in $\mathfrak{M}$ of a tuple $\langle a_0, \ldots, a_{n-1}\rangle$. Suppose $f : m \to n$. Then $(S_T^{\mathcal{L}} f)(p)(x_0, \ldots, x_{m-1})$ is the type in $\mathfrak{M}$ of $\langle a_{f(0)}, \ldots, a_{f(m-1)}\rangle$ and is, in particular, an m-ary type.*

Proof We show that $(S_T^{\mathcal{L}} f)(p)(x_0, \ldots, x_{m-1})$ is the set of $\mathcal{L}$-formulas satisfied by $\langle a_{f(0)}, \ldots, a_{f(m-1)}\rangle$. For $\varphi(x_0, \ldots, x_{m-1}) \in (S_T^{\mathcal{L}} f)(p)$ if and only if $\varphi(x_{f(0)}, \ldots, x_{f(m-1)}) \in p$, and, since p is the type of $\langle a_0, \ldots, a_{n-1}\rangle$, this is equivalent to the statement that $\mathfrak{M} \models \varphi(a_{f(0)}, \ldots, a_{f(m-1)})$, as required. □

We do also need to know that the type category of a theory is a type category.

Theorem 2.4 *The type category $S_T^{\mathcal{L}}$ of a countable theory T in a fragment $\mathcal{L}$ of $\mathcal{L}_{\omega_1,\omega}$ is a type category.*

Proof We check the conditions of Definition 2.1.

1 Each $S^{\mathcal{L}}_T n$ is zero-dimensional, because $\mathcal{L}$ is closed under negation. It is T_0 because distinct points of $S^{\mathcal{L}}_T n$ correspond to distinct types, and if p and q are distinct n-ary types, then there must be some formula φ which is contained in one and not the other; so there is an open set containing one of p and q and not the other. $S^{\mathcal{L}}_T n$ is then a zero-dimensional T_0 space and, so, is Hausdorff.

2 If $f : n \to m$, and $\varphi(x_0, \dots, x_{n-1})$ is an n-ary formula, then

$$\begin{aligned}(S^{\mathcal{L}}_T f)^{-1}\big(\{p \in S^{\mathcal{L}}_T n \mid \varphi(x_0, \dots, x_{n-1}) \in p\}\big)\\ = \{q \in S^{\mathcal{L}}_T m \mid \varphi(x_{f(0)}, \dots, x_{f(n-1)}) \in q\},\end{aligned}$$

and so the inverse image under $S^{\mathcal{L}}_T f$ of every basic open set is open, so $S^{\mathcal{L}}_T f$ is continuous.

3 Suppose $m, n_1, n_2, l, f_1, f_2, g_1, g_2, p, U$, and V satisfy the conditions given in clause 3 of Definition 2.1. Then by definition of the topology, there exist formulas $\varphi_1(x_0, \dots, x_{n_1-1})$ and $\varphi_2(x_0, \dots, x_{n_2-1})$ such that

$$(S^{\mathcal{L}}_T f_1)^{-1}\{p\} \cap \{r \in S^{\mathcal{L}}_T n_1 \mid \varphi_1 \in r\} \subseteq U$$

and

$$(S^{\mathcal{L}}_T f_2)^{-1}\{p\} \cap \{r \in S^{\mathcal{L}}_T n_2 \mid \varphi_2 \in r\} \subseteq V,$$

and such that

$$\exists x_m, \dots, x_{n_i-1}\, \varphi_i(x_{h_i(0)}, \dots, x_{h_i(n_i-1)}) \in p,$$

for $i = 1, 2$, where $h_i : n_i \to n_i$ is a permutation such that $h_i \circ f_i$ is the identity on m. But then,

$$\begin{aligned}(\exists x_m, \dots, x_{n_1+n_2-m-1})\, \big(\varphi_1(x_{h_1(0)}, \dots, x_{h_1(n_1-1)})\\ \wedge\, \varphi_2(x_{h_3(0)}, \dots, , x_{h_3(n_1+n_2-m-1)})\big)\\ \in p\end{aligned}$$

also, where $h_3 = e_{m\cup[n_1,n_2-m)} \circ h_2$. This fact implies the existence of a type $q \in S^{\mathcal{L}}_T l$ such that $(S^{\mathcal{L}}_T(f_1 \circ g_1))(q) = p$, and

$$\varphi_i(x_{g_i(0)}, \dots, x_{g_i(n_i-1)}) \in q$$

for $i = 1, 2$ as required.

Hence $S^{\mathcal{L}}_T$ is a weak type category.

We now prove that all maps $S^{\mathcal{L}}_T f$ are open; the essential reason for this being that the language $\mathcal{L}$ is closed under existential quantification.

If $f : n \to m$, and $\psi(x_0, \dots, x_{m-1})$ is an m-ary formula, $k \geq n$, and if $h = e_{m\setminus \operatorname{ran} f}$, and $j = \big|m \setminus \operatorname{ran} f\big|$, then

$$\begin{aligned}(S^{\mathcal{L}}_T f)\{p \in S^{\mathcal{L}}_T m \mid \psi(x_0, \dots, x_{m-1}) \in p\}\\ = (S^{\mathcal{L}}_T f)\{p \in S^{\mathcal{L}}_T m \mid \exists x_{h(0)}, \dots, x_{h(j-1)}\, \psi(x_0, \dots, x_{m-1}) \in p\},\end{aligned}$$

for the inclusion of the left-hand side in the right is obvious; as for the reverse inclusion, if $\exists x_{h(0)}, \ldots, x_{h(j-1)}\ \psi(x_0, \ldots, x_{m-1}) \in p$, then, defining a function $l : m \to m + j$ so that

$$l : j \mapsto \begin{cases} j & \text{if } j \in \operatorname{ran} f \\ m + h^{-1}(j) & \text{otherwise,} \end{cases}$$

we may find $q \in (S\iota_{m,m+j})^{-1}(p)$ such that

$$\psi(x_{l(0)}, \ldots, x_{l(m-1)}) \in q.$$

Now extend l to a permutation l' of $m + j$ such that

$$l' : j \mapsto \begin{cases} j & \text{if } j \in \operatorname{ran} f \\ m + h^{-1}(j) & \text{if } j \in m \setminus \operatorname{ran} f', \\ l^{-1}(j) & \text{otherwise,} \end{cases}$$

then

$$\psi(x_0, \ldots, x_{m-1}) \in (S_T^{\mathcal{L}} l')(q),$$

and so

$$(S_T^{\mathcal{L}} f)(S_T^{\mathcal{L}} \iota_{m,m+j})(S_T^{\mathcal{L}} l')(q) \in (S_T^{\mathcal{L}} f)\bigl\{r \in S_T^{\mathcal{L}} m \bigm| \psi(x_0, \ldots, x_{m-1}) \in r\bigr\}.$$

We wish to show $(S_T^{\mathcal{L}} f)(p) = (S_T^{\mathcal{L}} f)(S_T^{\mathcal{L}} \iota_{m,m+j})(S_T^{\mathcal{L}} l')(q)$. But $l' \circ \iota_{m,m+j} \circ f = \iota_{m,m+j} \circ f$, so $(S_T^{\mathcal{L}} f)(S_T^{\mathcal{L}} \iota_{m,m+j})(S_T^{\mathcal{L}} l')(q) = (S_T^{\mathcal{L}} f)(S_T^{\mathcal{L}} \iota_{m,m+j})(q) = (S_T^{\mathcal{L}} f)(p)$. Now

$$\begin{aligned} &(S_T^{\mathcal{L}} f)\{p \in S_T^{\mathcal{L}} m \mid \exists x_{h(0)}, \ldots, x_{h(j-1)}\ \psi(x_0, \ldots, x_{m-1}) \in p\} \\ &\qquad = \{p \in S_T^{\mathcal{L}} n \mid \exists x_n, \ldots, x_{k-1}\ \psi(x_{g(0)}, \ldots, x_{g(m-1)}) \in p\}, \end{aligned}$$

where $g : m \to k$ satisfies $f \circ g(i) = i$ for all $i \in \operatorname{ran} f$ and maps $m \setminus \operatorname{ran} f$ one-to-one onto $[n, k)$, and so the image of any basic open set is open, and so $S_T^{\mathcal{L}} f$ is an open map. Hence $S_T^{\mathcal{L}}$ is a type category. □

2.2 Properties of type categories It will be useful to have a criterion for when a weak type category is a type category.

Lemma 2.5 *Suppose S is a weak type category, and for every $n \in \omega$ and $i < n$, Sd_i^n and (if $n > 0$ and $i < n - 1$) Ss_i^n is an open map. Then S is a type category.*

Proof Obvious, since if $\sigma : n \leftrightarrow n$, then $S\sigma$ and $S\sigma^{-1}$ are both continuous and therefore open, and any function from one integer to another is a composition of permutations and functions d_i^n and s_i^n. □

A type category may possess some extra properties.

Definition 2.6 We say a weak type category S is *compact* if for every n, Sn is zero-dimensional, compact and metrizable, and we say it is *Polish* if Sn is zero-dimensional and Polish for every n. The category is *countable* if Sn is countable for every n. The category is *amalgamative* if whenever

1.

$$\begin{array}{ccc} m & \xrightarrow{f_1} & n_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ n_2 & \xrightarrow{g_2} & l \end{array}$$

commutes,
2. f_1, f_2, g_1, and g_2 are one-to-one,
3. $\operatorname{ran} g_1 \cap \operatorname{ran} g_2 = \operatorname{ran} g_1 \circ f_1$,
4. $p \in Sm$, and
5. $q \in Sn_1$, and $r \in Sn_2$, and $(Sf_1)(q) = (Sf_2)(r) = p$,

then there exists $s \in Sl$ such that $(Sg_1)(s) = q$ and $(Sg_2)(s) = r$.

These properties are related to properties of the language using standard, well-known theorems about spaces of types, as follows.

Theorem 2.7 *Let S be the type category of a theory T in a language $\mathcal{L}$.*

1. *If $\mathcal{L}$ is a countable first-order language, then S is compact and amalgamative; and*
2. *if $\mathcal{L}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and T is either*
   (a) *a countable theory in $\mathcal{L}$ or else*
   (b) *a sentence of $\mathcal{L}$,*
   then S is Polish.

Proof If $\mathcal{L}$ is a first-order language, then each Sn is compact by the Compactness Theorem, and if $\mathcal{L}$ is countable, then Sn is metrizable by Urysohn's Metrization Theorem, since it has a countable basis, and is Hausdorff and zero-dimensional, and hence T_3. Amalgamativity of S is again a consequence of the Compactness Theorem.

If $\mathcal{L}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$, then S is Polish because each Sn is G_δ in a compact metrizable space, the countably many open sets in question being given by the requirement that the set

$$\{\neg\theta_i : i \in \omega\} \cup \left\{ \bigwedge_{i\in\omega} \theta_i \right\}$$

should not be realizable, for each of the countably many infinite conjunctions $\bigwedge_{i\in\omega} \theta_i$ occurring as elements of $\mathcal{L}$. □

We also have a partial converse.

Theorem 2.8 *Any compact, amalgamative type category S is the type category of some theory T in a countable language of first-order logic. Moreover, T can be chosen to be Π_2.*

Proof First observe that because each Sn is compact and metrizable, it has only countably many clopen subsets, and these form a basis for the topology on Sn since Sn is zero-dimensional. So, to each $n \in \omega$ and each clopen subset U of Sn, assign an n-ary predicate letter $P^n_U(x_0, \dots, x_{n-1})$, and let T be the theory which includes formulas chosen as follows:

1. $\forall x_0, \dots, x_{n-1} \big(P^n_U(x_0, \dots, x_{n-1}) \leftrightarrow \neg P^n_{Sn\setminus U}(x_0, \dots, x_{n-1}) \big)$, whenever $n \in \omega$ and U is a clopen subset of Sn;

2. $\forall x_0, \ldots, x_{n-1} \neg P^n_{\varnothing}(x_0, \ldots, x_{n-1})$ for all $n \in \omega$;
3. $\forall x_0, \ldots, x_{n-1}\Big(P^n_{U\cap V}(x_0, \ldots, x_{n-1}) \leftrightarrow \big(P^n_U(x_0, \ldots, x_{n-1}) \wedge P^n_V(x_0, \ldots, x_{n-1})\big)\Big)$ whenever U and V are clopen subsets of Sn;
4. $\forall x_0, \ldots, x_{n-1}\Big(P^n_{U\cup V}(x_0, \ldots, x_{n-1}) \leftrightarrow \big(P^n_U(x_0, \ldots, x_{n-1}) \vee P^n_V(x_0, \ldots, x_{n-1})\big)\Big)$ whenever U and V are clopen subsets of Sn;
5. noting that, because each Sn is compact and Hausdorff, all the maps Sf are closed and thus send clopen sets to clopen sets, we also add to T a formula
$$\forall x_0, \ldots, x_{n-1}\, \big(P^n_{(Sf)(U)}(x_0, \ldots, x_{n-1}) \leftrightarrow \exists y_{g(0)}, \ldots, y_{g(m-n-1)}\, P^m_U(y_0, \ldots, y_{m-1})\big),$$
where $y_{f(i)} = x_i$ for all i, whenever U is a clopen subset of Sm, $f : n \xrightarrow{1\text{–}1} m$, and $g : m - n \to m$ is equal to $e_{m\setminus \operatorname{ran} f}$,
6. if $f : n \to m$ is onto, U is a clopen subset of Sm, then
$$\forall x_0, \ldots, x_{n-1}\left(P^n_{(Sf)(U)}(x_0, \ldots, x_{n-1}) \leftrightarrow \left(\Big(\bigwedge_{f(i)=f(j)} x_i = x_j\Big) \wedge P^m_U(x_{g(0)}, \ldots, x_{g(m-1)})\right)\right),$$
whenever $g : m \to n$ is any one-to-one function such that $f \circ g = \iota_{m,m}$.

Having defined this theory, it is straightforward to check that for each n, Sn is, in a natural way, homeomorphic to the space of quantifier-free n-ary types over this theory, and, by axiom scheme 5, we can eliminate quantifiers. T is clearly Π_2. □

2.3 Models of type categories In this section, we show how to construct the space of models of a theory from its type category. It is natural to think of this as being an inverse limit construction, and given this intuition, there is nothing surprising in it.

First we say what we mean by a model of a type category. This definition is designed to correspond to the notion of a model of a logical theory.

Definition 2.9 Suppose S is a type category. A *simplicial object* M for S is a function whose domain is all finite subsets of ω such that for all $A \in \operatorname{dom} M$, $M(A) \in S\big(|A|\big)$, and such that if $A \subseteq B$ and $e_A = e_B \circ g$, then $M(A) = (Sg)(M(B))$. The simplicial object is *existentially closed* if for all $A \in \operatorname{dom} M$, if $|A| = n$, and $n < m$, and U is open in $(S\iota_{n,m})^{-1}\{M(A)\}$, then there exist $g : m \leftrightarrow m$ and $B \supseteq A$ of cardinality m such that $e_A = e_B \circ g \circ \iota_{n,m}$, and $(Sg)M(B) \in U$. We refer to an existentially closed simplicial object as *a model of S*.

Let us check that this concept makes sense.

Proposition 2.10 *Suppose $S^{\mathcal{L}}_T$ is the type category of a theory T in a countable fragment $\mathcal{L}$ of $\mathcal{L}_{\omega_1,\omega}$. Suppose M is a model of $S^{\mathcal{L}}_T$. Then there is a model $\mathfrak{M}$ of T with universe ω such that for all finite subsets A of ω, $M(A)$ is the type satisfied by $\langle e_A(0), \ldots, e_A(|A|-1)\rangle$ in $\mathfrak{M}$. Moreover, all models of T with universe ω arise in this way.*

Proof Define $\mathfrak{M}$ as a structure by saying, if $P(x_0, \ldots, x_{n-1})$ is an atomic predicate, that

$$\mathfrak{M} \models P(i_0, \ldots, i_{n-1})$$

if and only if $\varphi(x_{\sigma(0)}, \ldots, x_{\sigma(n-1)}) \in M(A)$, where for all j, $e_A(j) = i_{\sigma(j)}$. We then need to verify, by induction on formulas $\varphi(x_0, \ldots, x_{n-1})$, that

$$\mathfrak{M} \models \varphi(i_0, \ldots, i_{n-1})$$

if and only if $\varphi(x_{\sigma(0)}, \ldots, x_{\sigma(n-1)}) \in M(A)$, where for all j, $e_A(j) = i_{\sigma(j)}$.

The only case that is nontrivial is the case where φ is an existential formula $\exists x_n\, \psi(x_0, \ldots, x_n)$. But if M is existentially closed, we may find i_n such that, appealing to the inductive hypothesis for ψ,

$$\mathfrak{M} \models \psi(i_0, \ldots, i_n),$$

and so

$$\mathfrak{M} \models \exists x_n\, \psi(i_0, \ldots, i_{n-1}, x_n)$$

as required.

Conversely, we may construct an existentially closed simplicial object M from a model $\mathfrak{M}$ of T with universe ω by defining $M(A)$ to be the type satisfied by $\langle e_A(0), \ldots, e_A(|A|-1)\rangle$ in $\mathfrak{M}$. $\square$

2.4 Model theory of type categories We define satisfaction and elementary embedding in the obvious way.

Definition 2.11 Suppose M is a model of a type category S and $\langle a_0, \ldots, a_{n-1}\rangle$ is a tuple on ω. Suppose $\left|\{a_0, \ldots, a_{n-1}\}\right| = j$, and $g : n \to j$ is the function having the property that for each i, $a_i = e_{\{a_0,\ldots,a_{n-1}\}} \circ g(i)$. Then if $p \in Sn$, we say $M \models p(a_0, \ldots, a_{n-1})$ if and only if

$$p = (Sg)\big(M(\{a_0, \ldots, a_{n-1}\})\big).$$

Suppose M and N are models of a type category S. Then $f : \omega \xrightarrow{1\text{–}1} \omega$ is an *elementary embedding* from M to N if and only if, for all tuples $\langle a_0, \ldots, a_{n-1}\rangle$ on ω, and for all $p \in Sn$, $M \models p(a_0, \ldots, a_{n-1})$ if and only if $N \models p(a_0, \ldots, a_{n-1})$. We say $f : \omega \leftrightarrow \omega$ is an *elementary isomorphism* between M and N if and only if f is an elementary embedding from M to N and f^{-1} is an elementary embedding from N to M.

Some of the standard theory carries over; in particular, we have an Omitting Types Theorem.

Proposition 2.12 *Suppose S is a Polish type category, $n < m$, $p \in Sn$, $f : n \xrightarrow{1\text{–}1} m$, and q is a limit point of $(Sf)^{-1}\{p\}$. Then there is a model M of S such that there exist $a_0, \ldots, a_{n-1}$ such that $M \models p(a_0, \ldots, a_{n-1})$, and there do not exist $a_n, \ldots, a_{m-1}$ such that $M \models q(a_0, \ldots, a_{m-1})$.*

Proof Each $(S\iota_{n,m})^{-1}\{p\}$, for $m \geq n$, is a closed subspace of a Polish space so is Polish and therefore Čech-complete; so let $\{\mathcal{U}^m_k \mid k \in \omega\}$ be a family of open covers of $(S\iota_{n,m})^{-1}\{p\}$ such that whenever, for all k, $V_k \in \mathcal{U}^m_k$, $C_k \subseteq V_k$, C_k is closed, and $C_{k+1} \subseteq C_k$, $\bigcap_{k\in\omega} C_k \neq \varnothing$; since Sn is metrizable we can assume in addition that this intersection always has a unique point.

Recursively construct an increasing sequence of integers $n_k \geq n$ and an open subset U_k of $(S\iota_{n,n_k})^{-1}\{p\}$ with the following properties:

1. for each k and $k' \geq k$, there exists $V_{k'} \in \mathcal{U}_{k'}^{n_k}$ such that $\overline{(S\iota_{n_k,n_{k'}}(U_{k'}))} \subseteq V_{k'}$,
2. for each k, $\overline{(S\iota_{n_k,n_{k+1}})(U_{k+1})} \subseteq U_k$,
3. for each k, for all one-to-one $g : m \to n_k$ extending ι_{n,n_k}, $q \notin (Sf)(U_k)$,
4. for each k, for each $l > n_k$, for each open subset U of $(S\iota_{n_k,l})^{-1}(U_k)$, there exists $k' > k$ such that either $(S\iota_{n_k,l})^{-1}(S\iota_{n_k,n_{k'}})(U_{k'}) \cap U = \varnothing$, or there exists $g : l \xrightarrow{1\text{–}1} n_{k'}$ which is the identity on n_k such that $(Sg)(U_{k'}) \subseteq U$.

We can achieve condition 3 because in the situation described, $(Sf)(U)$ is not nowhere dense in $(S\iota_{n,m})^{-1}\{p\}$, since Sf is an open map. Now define M so that for each k, $M(n_k)$ is the unique element of $\bigcap_{k' \geq k}(S\iota_{n_k,n_{k'}})(U_{k'})$. Conditions 1–3 ensure that $M(n_k)$ exists for each k, and condition 4 ensures existential closure of M. □

2.5 The Polish space of models Now a model of S looks like a direct limit of types, and so it is natural to arrange models of S in a space which is an inverse limit of the Sn.

Definition 2.13 Suppose S is a type category. Then we define the *model space* of S to be a topological space X_S equipped with a group action of S_∞ as follows. X_S is the set of all models of S with a topology generated by all sets of the form

$$\{M \mid M(A) \in U\},$$

for all finite subsets A of ω and all open subsets U of $S(|A|)$. If $\sigma \in S_\infty$, then for all $n \in \omega$ and all elements $a_0, \ldots, a_{n-1}$ of ω, and all elements p of Sn,

$$\sigma(M) \models p(a_0, \ldots, a_{n-1}) \text{ iff } M \models p(a_{\sigma(0)}, \ldots, a_{\sigma(n-1)}).$$

We thus derive the usual Polish S_∞-space of models.

Proposition 2.14 *If S is a Polish type category, then X_S is Polish and the action of S_∞ on it is continuous.*

Proof We can prove that X_S is Polish by embedding it as a G_δ subspace of the inverse limit of the spaces Sn along the maps $S\iota_{n,m}$ for $n \leq m$. The action of S_∞ on it is obviously well-defined and continuous. (In fact, the action of the dense subgroup of all finite permutations can be extracted from the inverse system.) □

It is obvious that if $\mathcal{L}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and T is a countable theory or sentence in $\mathcal{L}$, then $X_{S_T^{\mathcal{L}}}$ is the usual space of countable models.

3 The Scott Hierarchy

We now explore what happens when we proceed to larger fragments of $\mathcal{L}_{\omega_1,\omega}$.

3.1 The Scott hierarchy for languages There are many different ways of defining a hierarchy of fragments of $\mathcal{L}_{\omega_1,\omega}$, as was done by Morley in [7], in order to define and reason about Scott height and Scott rank. These notions tend to give similar values for Scott rank (i.e., the same or different by a finite quantity) on a closed unbounded set. We will choose a notion which is convenient as regards the topological techniques we are using in this paper.

Definition 3.1 Suppose $\mathcal{L}$ is a countable subset of $\mathcal{L}_{\omega_1,\omega}$, T is a countable theory in $\mathcal{L}$, with only countably many types, and $S_T^{\mathcal{L}}$ is the type category of T in $\mathcal{L}$. We define $\Sigma_T\mathcal{L}$ to be the smallest subset of $\mathcal{L}_{\omega_1,\omega}$ which includes

1. $\mathcal{L}$,
2. $\bigwedge p$, $\neg \bigwedge p$, $\exists x \ \bigwedge p$, and $\forall x \neg \bigwedge p$ for p a complete type over $\mathcal{L}$ realized in a model of T.

We define $\Sigma_T^n\mathcal{L}$ by recursion on n in the obvious way: namely, $\Sigma_T^0\mathcal{L} = \mathcal{L}$, and for all n, $\Sigma_T^{n+1}\mathcal{L} = \Sigma_T(\Sigma_T^n\mathcal{L})$. Let $\mathfrak{H}_T\mathcal{L}$ be the smallest fragment of $\mathcal{L}_{\omega_1,\omega}$ including $\bigcup_{n\in\omega} \Sigma_T^n\mathcal{L}$.

It was hardly necessary to modify $\bigcup_{n\in\omega} \Sigma_T^n\mathcal{L}$ to obtain a fragment, in the following sense.

Proposition 3.2 *Suppose $\mathcal{L}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and T is a countable theory over $\mathcal{L}$. Each complete type over $\bigcup_{n\in\omega} \Sigma_T^n\mathcal{L}$ realizable in a model of T can be extended to a unique complete type over $\mathfrak{H}_T\mathcal{L}$.*

Proof We can characterize types in both languages by using the following game. There are two players, O and P. The game begins with two tuples, $\mathbf{a}_1^0$ over a model $\mathfrak{M}_1$ of T and $\mathbf{a}_2^0$ over a model $\mathfrak{M}_2$ of T, of equal arity. O begins by selecting an integer n. The game then consists of n rounds.

In the ith round ($i = 1, 2, \ldots, n$), O selects an element ϵ_i of the set $\{1, 2\}$ and an element $a_{\epsilon_i}^i$ of $\mathfrak{M}_{\epsilon_i}$. P then selects an element $a_{3-\epsilon_i}^i$ of $\mathfrak{M}_{3-\epsilon_i}$. Let $\mathbf{a}_j^i = \mathbf{a}_j^{i-1\frown} a_j^i$ for $j \in \{1, 2\}$.

P wins if, at the end of the game, $\mathbf{a}_1^n$ and $\mathbf{a}_2^n$ satisfy the same $\mathcal{L}$-type; otherwise O wins. The two tuples $\mathbf{a}_1^0$ and $\mathbf{a}_2^0$ are *equivalent* if and only if P has a winning strategy. Two tuples are easily seen to be equivalent if and only if they have the same $\bigcup_{n\in\omega} \Sigma_T^n\mathcal{L}$-type and if and only if they have the same $\mathfrak{H}_T\mathcal{L}$-type. $\square$

Having defined this notion of elevation in the Scott hierarchy, we can, of course, on the assumption that T is scattered, iterate it. (For any T, it may be possible to iterate the elevation operation to some countable ordinal λ, even if not all the way to ω_1. However, we do not, in this paper, address the topic of how λ is to be determined for general nonscattered T, or what happens there.)

Definition 3.3 If T is a theory in a language $\mathcal{L}$, define languages $\mathfrak{H}_T^\alpha\mathcal{L}$, for $\alpha \leq \omega_1$, as follows. $\mathfrak{H}_T^{\alpha+1}\mathcal{L} = \mathfrak{H}_T\mathfrak{H}_T^\alpha\mathcal{L}$, if $\mathfrak{H}_T^\alpha\mathcal{L}$ and all the $\Sigma_T^n\mathfrak{H}_T^\alpha\mathcal{L}$ are countable. If λ is a limit, then $\mathfrak{H}_T^\lambda\mathcal{L}$ is the union of the $\mathfrak{H}_T^\alpha\mathcal{L}$ for $\alpha < \lambda$, if they are all defined.

3.2 The Scott Hierarchy for type categories Now we define the topological notion corresponding to our logical notion of elevation. The details of the following definitions may not be particularly of interest; the important thing is that such details exist. It is also worth noting that they correspond exactly to the logical notions in the preceding section.

It will be useful to extend some logical notions to the situation of the model theory of type categories.

Definition 3.4 Suppose S is a type category. We generate a language $\mathcal{L}_S$ as follows.

1. If $f : n \to \omega$, and $p \in Sn$, then $p(x_{f(0)}, \ldots, x_{f(n-1)}) \in \mathcal{L}_S$. [Note here that the letter p is being used simultaneously to refer to an element of the space Sn and to a predicate letter in the language $\mathcal{L}_S$.]
2. If φ belongs to $\mathcal{L}_S$, so does $\neg\varphi$.
3. If Φ is a subset of $\mathcal{L}_S$, all of whose free variables are taken from some finite set, then $\bigwedge \Phi$ belongs to $\mathcal{L}_S$.
4. If $\varphi \in \mathcal{L}_S$, then $\exists x_i\, \varphi$ is in $\mathcal{L}_S$. As usual, we write $\forall x_i$ for $\neg\exists x_i \neg$.

Suppose M is a model of S. Then we define a satisfaction relation as follows.

1. $M \models p(f(0), \ldots, f(n-1))$ under the circumstances described in Definition 2.11.
2. $M \models \neg\varphi$ iff it is not the case that $M \models \varphi$.
3. $M \models \bigwedge \Phi$ iff for all $\varphi \in \Phi$, $M \models \varphi$.
4. $M \models \exists x_n\, \varphi(a_0, \ldots, a_{n-1}, x_n)$ iff for some a_n, $M \models \varphi(a_0, \ldots, a_n)$.

Now we define a notion of type in a higher language.

Definition 3.5 An *n-ary $\Sigma^0 S$-type* over M is simply a singleton set $\{p\}$, where p is an element of Sn such that, for some $\langle a_0, \ldots, a_{n-1}\rangle$, $M \models p(a_0, \ldots, a_{n-1})$. Then $\langle a_0, \ldots, a_{n-1}\rangle$ is a *realization* of p in M. An *n-ary $\Sigma^1 S$-type* over M is, for some $\langle a_0, \ldots, a_{n-1}\rangle$, the union of the singleton set $\{p\}$, where p is the n-ary $\Sigma^0 S$ type realized by $\langle a_0, \ldots, a_{n-1}\rangle$, with the set of all formulas $\exists x\, q(a_0, \ldots, a_{n-1}, x)$ or $\forall x\, \neg q(a_0, \ldots, a_{n-1}, x)$ which are satisfied by $\langle a_0, \ldots, a_{n-1}\rangle$ in M, for q an element of $S(n+1)$. If $k > 0$, then an *n-ary $\Sigma^{k+1} S$-type* over M is, for some $\langle a_0, \ldots, a_{n-1}\rangle$, the union of the n-ary $\Sigma^k S$ type realized by $\langle a_0, \ldots, a_{n-1}\rangle$ with the set of all formulas $\exists x \bigwedge q(a_0, \ldots, a_{n-1}, x)$ or $\forall x \neg \bigwedge q(a_0, \ldots, a_{n-1}, x)$ which are satisfied by $\langle a_0, \ldots, a_{n-1}\rangle$ in M, for $q(x_0, \ldots, x_n)$ an $(n+1)$-ary $\Sigma^k S$-type. An *n-ary $\Sigma^\omega S$-type* over M is, for some n-tuple $\mathbf{a}$, the union of the $\Sigma^k S$-types realized by $\mathbf{a}$ in M.

Now we define our higher-order type category.

Definition 3.6 Suppose S is a Polish type category. We define another type category $\mathfrak{H}S$, and a map $\mathfrak{h}_S : \mathfrak{H}S \to S$, as follows. $\mathfrak{H}Sn$ is the set of all n-ary $\Sigma^\omega S$-types over models of S. We give $\mathfrak{H}Sn$ the topology generated by the sets $B_k(p')$ for $k \in \omega$ and $p' \in \mathfrak{H}Sn$, where $B_k(p')$ is the set of all $q' \in \mathfrak{H}Sn$ such that p' and q' include precisely the same n-ary $\Sigma^j S$-types for $j \leq k$. If $f : n \to m$, $p' \in \mathfrak{H}Sm$, and $\langle a_0, \ldots, a_{m-1}\rangle$ is a realization of p' in M, then we define $(Sf)(p')$ to be that element q' of $\mathfrak{H}Sn$ such that $\langle a_{f(0)}, \ldots, a_{f(n-1)}\rangle$ realizes q'. If $p' \in \mathfrak{H}Sn$ and $\mathbf{a}$ is a realization of p' in M, then we define $\mathfrak{h}_S(p')$ to be the unique element p of Sn such that $M \models p(\mathbf{a})$.

We show that it is a type category.

Proposition 3.7 *Suppose S is a Polish type category. Then $\mathfrak{H}S$ is a type category.*

Proof We check the conditions of Definition 2.1. It is clear that $\mathfrak{H}S\iota_{m,m}$ is, for each $m \in \omega$, the identity on $\mathfrak{H}Sm$, and that if $f : m \to n$ and $g : n \to k$, then $\mathfrak{H}S(f \circ g) = (\mathfrak{H}Sg) \circ (\mathfrak{H}Sf)$. Now we check the numbered conditions.

1 Since S is Polish, there exist models of S, and if M is a model of S, then for each $n \in \omega$, to each n-tuple on ω there corresponds a $\Sigma^\omega S$-type. Thus, for each n, $\mathfrak{H}Sn$

is nonempty. $\mathfrak{H}Sn$ is zero-dimensional because for each $k \in \omega$, the sets $B_k(p')$, as p' vary over $\mathfrak{H}Sn$, form a partition of $\mathfrak{H}Sn$. It is Hausdorff because, if p' and q' are distinct elements of $\mathfrak{H}Sn$, then for some k, p' and q' include different $\Sigma^k S$-types. Thus $B_k(p') \cap B_k(q') = \varnothing$.

2 We must now show that for all $f : m \to n$, $\mathfrak{H}Sf$ is continuous. It is sufficient to show this for f a permutation, or one of the maps $\iota_{n,n+1}$ or s^n_{n-1}, since every function from one natural number to another is a composition of such maps. Of these special cases, the only nontrivial one is the map $\iota_{n,n+1}$. Accordingly, we argue that if $p' \in \mathfrak{H}Sn$ and $k \in \omega$, then $(\mathfrak{H}S\iota_{n,n+1})^{-1}(B_{k+1}(p'))$ is open.

Suppose that $(\mathfrak{H}S\iota_{n,n+1})(q') = p'$. We argue that $B_{k+1}(q') \subseteq (\mathfrak{H}S\iota_{n,n+1})^{-1}$ $(B_{k+1}(p'))$. For suppose $r' \in B_{k+1}(q')$ and that r' is the $\Sigma^\omega S$-type of a tuple $\langle i_0, \ldots, i_n\rangle$ on ω in a model M of S. Then $(\mathfrak{H}S\iota_{n,n+1})(r')$ is the $\Sigma^\omega S$-type of the tuple $\langle i_0, \ldots, i_{n-1}\rangle$. Now $(\mathfrak{H}S\iota_{n,n+1})(r') \in B_{k+1}(p')$ if and only if $(\mathfrak{H}S\iota_{n,n+1})(r')$ and p' contain the same $\Sigma^{k+l} S$-type; that is, if and only if they contain the same $\Sigma^k S$-type, and also exactly the same formulas $\exists x\, t'(x_0, \ldots, x_{n-1}, x)$ and $\forall x\, \neg t'(x_0, \ldots, x_{n-1}, x)$ for t' an $(n+1)$-ary $\Sigma^k S$-type. Now the formulas $\exists x\, t'(x_0, \ldots, x_{n-1}, x)$ (for t' a $\Sigma^k S$-type) contained in $(\mathfrak{H}S\iota_{n,n+1})(r')$ are precisely the $\Sigma^k S$-types of all tuples $\langle i_0, \ldots, i_{n-1}, i\rangle$ as i varies over ω. But the formulas $\exists x\, t'(x_0, \ldots, x_{n-1}, x_n, x)$ (for t' a $\Sigma^k S$-type) contained in r' are precisely the $\Sigma^k S$-types of all tuples $\langle i_0, \ldots, i_{n-1}, i_n, i\rangle$ as i varies over ω. Because $r' \in B_{k+1}(q')$, these are exactly the formulas $\exists x\, t'(x_0, \ldots, x_{n-1}, x_n, x)$ (for t' a $\Sigma^k S$-type) contained in q', and so the formulas $\exists x\, t'(x_0, \ldots, x_{n-1}, x)$ (for t' a $\Sigma^k S$-type) contained in $(\mathfrak{H}S\iota_{n,n+1})(r')$ are precisely those in p', as required.

3 We now check the weak amalgamation property. Suppose that m, n_1, n_2, l, f_1, f_2, g_1, g_2, p, U, and V are as in clause 3 of Definition 2.1. Without loss of generality, let us assume that for a single $k \in \omega$, for some $q_1 \in \mathfrak{H}Sn_1$ and $q_2 \in \mathfrak{H}Sn_2$, $U = B_k(q_1)$ and $V = B_k(q_2)$. Let M be a model of S, and $\langle i_0, \ldots, i_{n-1}\rangle$ a tuple, such that p is the $\Sigma^\omega S$-type of $\langle i_0, \ldots, i_{n-1}\rangle$ in M. Let q_1'' be the $\Sigma^k S$-type contained in q_1, and let q_2'' be the $\Sigma^k S$-type contained in q_2.

For $i = 1, 2$, let j_i be the cardinality of $n_i \setminus \operatorname{ran} f_i$. Then for each i, p entails $\psi_i = \exists y_0, \ldots, y_{j_i-1}\, q_i''(z_0, \ldots, z_{n_i-1})$, where for each $j < m$, $x_j = z_{f_i(j)}$, and the letters $y_0, \ldots, y_{j_i-1}$ are the letters z_j for which $j \notin \operatorname{ran} f_i$. We must say what we mean by this statement. If $j_i = 0$, it simply means that ψ_i is the $\Sigma^k S$-type included in p. If $j_i = 1$, it means that ψ_i is an existentially quantified statement contained in the $\Sigma^{k+1} S$-type included in p. For other values of j_i, it means that there is a $\Sigma^{k+j_i-1} S$-type s contained in the $\Sigma^{k+1} S$-type included in p such that s entails $\exists y_1, \ldots, y_{j_i-1}\, q_i''(z_0, \ldots, z_{n_i-1})$. Thus, it follows that there exist tuples $\langle i_{1,0}, \ldots, i_{1,n_1-1}\rangle$ and $\langle i_{2,0}, \ldots, i_{2,n_2-1}\rangle$ on ω such that if q_1' is the $\Sigma^\omega S$-type of $\langle i_{1,0}, \ldots, i_{1,n_1-1}\rangle$ and q_2' is the $\Sigma^\omega S$-type of $\langle i_{2,0}, \ldots, i_{2,n_2-1}\rangle$ in M, then q_1'' is contained in q_1' and q_2'' is contained in q_2', and also $i_{1,f_1(j)} = i_j$ and $i_{2,f_2(j)} = i_j$ for all j. Now let $\langle i_{3,0}, \ldots, i_{3,l-1}\rangle$ be a tuple on ω such that for each j, $i_{3,g_1(j)} = i_{1,j}$ and $i_{3,g_2(j)} = i_{2,j}$. Let r be the $\Sigma^\omega S$-type of $\langle i_{3,0}, \ldots, i_{3,l-1}\rangle$. Then $(\mathfrak{H}Sg_1)(r) = q_1' \in U$ and $(\mathfrak{H}Sg_2)(r) = q_2' \in V$. Also $(\mathfrak{H}Sf_1)(q_1') = p$, and $(\mathfrak{H}Sf_2)(q_2') = p$. Thus the weak amalgamation property is established. $\square$

We pause here to make an important observation, which is crucial to understanding the theory we are developing. It is possible that $\mathfrak{H}S$ is not Polish. In that case, the above definition will not "work properly," and theorems we might wish to try to prove will not be true. This was envisaged at the outset, since the theory in this paper was designed with counterexamples to Vaught's Conjecture in view. It is conceivable that one could improve the constructions in this paper to make them more general; the author can at the time of writing see no reason why this should not be possible but has certainly not done it himself.

Proposition 3.8 *If $\mathcal{L}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$, T is a countable theory in $\mathcal{L}$, and S is the type category of $\mathcal{L}$ over T, then $\mathfrak{H}S$ is isomorphic to the type category of $\mathfrak{H}_T\mathcal{L}$ over T. Also, $\mathfrak{h}_S$ is well-defined and onto, and each fiber of it is clopen; hence $\mathfrak{h}_S$ is continuous.*

Proof This is evident from the definitions. □

More compactly, we may say that $\mathfrak{H}S^{\mathcal{L}}_T$ is isomorphic to $S^{\mathfrak{H}_T\mathcal{L}}_T$. We will commit the abuse of regarding these two categories as being *equal*. Now if S is a type category, the definition of $\mathfrak{H}S$ does depend on the topology of S, but only weakly; the topology of S came in in the definition of a model of S; and there, if each Sn is scattered, the only aspect of the topology that matters is the set of triples $\langle p, q, f\rangle$ such that p, q are types, f is a function, and q an isolated member of the suspace $(Sf)^{-1}\{p\}$.

Note that $\mathfrak{h}_S$ depends for its definition on S.

Definition 3.9 If S is a type category, define operators $\mathfrak{H}^\alpha$ and $\mathfrak{h}^\alpha_S$, for $\alpha \leq \omega_1$, as follows. $\mathfrak{H}^{\alpha+1}S = \mathfrak{H}\mathfrak{H}^\alpha S$, and $\mathfrak{h}^{\alpha+1}S = \mathfrak{h}^\alpha_S \circ \mathfrak{h}_{\mathfrak{H}^\alpha S}$. If λ is a limit, then $\mathfrak{H}^\lambda S$ is the inverse limit of the $\mathfrak{H}^\alpha S$ under the maps $\mathfrak{h}^{\beta,\alpha}_S$, and likewise for $\mathfrak{h}^\lambda_S$.

We require that the above definition should make sense.

Proposition 3.10 *Suppose $S^{\mathcal{L}}_T$ is the type category of a theory T over a language $\mathcal{L}$. Then for all $\alpha < \omega_1$, if, for all $\beta \leq \alpha$, $\mathfrak{H}^\beta S^{\mathcal{L}}_T$ is Polish, then there is an isomorphism Π_α from $\mathfrak{H}^\alpha S^{\mathcal{L}}_T$ to $S^{\mathfrak{H}^\alpha_T\mathcal{L}}_T$ such that for all $n \in \omega$, and $\alpha \leq \beta \in \omega_1$, for all $p \in \mathfrak{H}^\alpha S^{\mathcal{L}}_T n$, $\Pi_\alpha(\mathfrak{h}^{\beta,\alpha}_{S^{\mathcal{L}}_T}(p)) = \Pi_\beta(p) \cap \mathfrak{H}^\alpha_T\mathcal{L}$.*

Proof We perform induction over α. We have already noted that the successor case works. One can establish the limit case by an application of the Model Existence Theorem. However, here we follow the alternative route of proving it explicitly. So, suppose that α is a countable limit ordinal. We assume that, for all $\gamma < \alpha$, $\mathfrak{H}^\gamma S^{\mathcal{L}}_T$ is isomorphic to $S^{\mathfrak{H}^\gamma_T\mathcal{L}}_T$ via an isomorphism Π_γ. Moreover, we assume that if $\delta \leq \gamma < \alpha$, then for all $p \in \mathfrak{H}^\gamma S^{\mathcal{L}}_T n$, $\Pi_\delta(\mathfrak{h}^{\gamma,\delta}_{S^{\mathcal{L}}_T}(p)) = \Pi_\gamma(p) \cap \mathfrak{H}^\delta_T\mathcal{L}$. We note first that for every $\beta < \alpha$, if, for every $n \in \omega$, $\mathfrak{H}^{\beta+1}S^{\mathcal{L}}_T n$ is Polish, then, for every $n \in \omega$, $\mathfrak{H}^\beta S^{\mathcal{L}}_T n$ is countable. Thus, $\bigcup_{\beta<\alpha}\bigcup_{n\in\omega}\mathfrak{H}^\beta S^{\mathcal{L}}_T n$ is countable, and thus (using our inductive hypothesis) so is $\mathfrak{H}^\alpha_T\mathcal{L}$.

It is clear how to define Π_α. Suppose that $\langle p_\gamma : \gamma < \alpha\rangle$ is an element of the inverse limit $\mathfrak{H}^\alpha S^{\mathcal{L}}_T n$ for some n; that is, for all γ, $p_\gamma \in \mathfrak{H}^\gamma S^{\mathcal{L}}_T n$, and for all $\delta \leq \gamma$, $p_\delta = \mathfrak{h}^{\gamma,\delta}_{S^{\mathcal{L}}_T}(p_\gamma)$. Then we define $\Pi_\alpha(p_\alpha : \gamma < \alpha)$ to be $\bigcup_{\gamma<\alpha}\Pi_\gamma(p_\gamma)$. If we can prove that Π_α does send the inverse limit of the $S^{\mathfrak{H}^\gamma_T\mathcal{L}}_T n$ to $S^{\mathfrak{H}^\alpha_T\mathcal{L}}_T n$, then it will be clear that Π_α is an isomorphism, and that for all $n \in \omega$ and $\gamma \leq \alpha$, for all $p \in \mathfrak{H}^\alpha S_T^{\mathcal{L}} n$, $\Pi_\gamma(\mathfrak{h}^{\alpha,\gamma}_{S_T^{\mathcal{L}}}(p)) = \Pi_\gamma(p) \cap \mathfrak{H}^\alpha_T \mathcal{L}$.

So, suppose that $\langle p_\gamma : \gamma < \alpha \rangle$ is an element of $\mathfrak{H}^\alpha S_T^{\mathcal{L}} n$. We must show that $\Pi_\alpha(p_\gamma : \gamma < \alpha)$ is an $\mathfrak{H}^\alpha_T \mathcal{L}$-type, and to do that, we must create a countable structure $\mathcal{L}$-structure $\mathfrak{M}$ containing an n-tuple $\mathbf{a}$ such that for each $\gamma < \alpha$, interpreting $\mathfrak{M}$ in the natural way as an $\mathfrak{H}^\gamma \mathcal{L}$-structure, $\Pi_\gamma(p_\gamma)$ is the type of $\mathbf{a}$. We can do this by a Henkin argument, exploiting the fact that $\mathfrak{H}^\alpha_T \mathcal{L}$ is countable.

To see how witnessing of existential quantifiers works in this setting, suppose that $\langle q_\gamma : \gamma < \alpha \rangle$ is an element of $\mathfrak{H}^\alpha S_T^{\mathcal{L}} m$. Suppose that $\beta < \alpha$, and that $\exists x\, \varphi(x_0, x_1, \ldots, x_{m-1}, x)$ belongs to $\Pi_\beta(q_\beta)$.

We must construct an element $\langle r_\gamma : \gamma < \alpha \rangle$ of $(\mathfrak{H}^\alpha S_T^{\mathcal{L}} \iota_{m,m+1})^{-1}(q_\gamma : \gamma < \alpha)$ such that $\varphi(x_0, \ldots, x_m)$ belongs to $\Pi_\beta(r_\beta)$. Because $\mathfrak{H}^\beta S_T^{\mathcal{L}}(m+1)$ is countable and Polish, it is scattered, so we can find an isolated point r_β in $(\mathfrak{H}^\beta S_T^{\mathcal{L}} \iota_{n,n+1})^{-1}(q_\beta)$ such that $\varphi(x_0, \ldots, x_m) \in \Pi(r_\beta)$. Let $\psi(x_0, \ldots, x_m)$ be a formula having the property that r_β is the only element of $(\mathfrak{H}^\beta S_T^{\mathcal{L}} \iota_{n,n+1})^{-1}(q_\beta)$ such that $\Pi_\beta(r_\beta)$ contains $\psi(x_0, \ldots, x_m)$.

Now, let $\langle \alpha_i : i \in \omega \rangle$ be a strictly increasing sequence of ordinals converging to α such that $\alpha_0 = \beta$. Let $r_{\alpha_0} = r_\beta$, and let $\psi_0 = \psi$. Given $r_{\alpha_i} \in \mathfrak{H}^{\alpha_i} S_T^{\mathcal{L}}(m+1)$, and given a formula $\psi_i(x_0, \ldots, x_m)$ of $\mathfrak{H}^{\alpha_i}_T \mathcal{L}$ such that r_{α_i} is the only element of $(S_T^{\mathfrak{H}^{\alpha_i}_T \mathcal{L}} \iota_{n,n+1})^{-1}(q_{\alpha_i})$ such that $\Pi_{\alpha_i}(r_{\alpha_i})$ contains $\psi_i(x_0, \ldots, x_m)$, choose $r_{\alpha_{i+1}}$ as an isolated element of $(\mathfrak{H}^{\alpha_{i+1}} S_T^{\mathcal{L}} \iota_{n,n+1})^{-1}(q_{\alpha_{i+1}})$ with $\Pi_{\alpha_{i+1}}(r_{\alpha_{i+1}})$ containing $\psi_i(x_0, \ldots, x_m)$, and let $\psi_{i+1}(x_0, \ldots, x_m)$ be a formula having the property that $r_{\alpha_{i+1}}$ is the only element of $(S_T^{\mathfrak{H}^{\alpha_{i+1}}_T \mathcal{L}} \iota_{n,n+1})^{-1}(q_{\alpha_{i+1}})$ such that $\Pi_{\alpha_{i+1}}(r_{\alpha_{i+1}})$ contains $\psi_{i+1}(x_0, \ldots, x_m)$. Then $\mathfrak{h}^{\alpha_{i+1},\alpha_i}_{S_T^{\mathcal{L}}}(r_{\alpha_{i+1}}) = r_{\alpha_i}$.

Now define r_γ, for any $\gamma < \alpha$, so that, if $\alpha_i \geq \gamma$, then $r_\gamma = \mathfrak{h}^{\gamma,\alpha_i}_{S_T^{\mathcal{L}}}(r_{\alpha_i})$. (This is clearly well-defined, since the $\mathfrak{h}^{\gamma,\delta}_{S_T^{\mathcal{L}}}$ are a commuting family of maps.) Then $\langle r_\gamma : \gamma < \alpha \rangle$ is an element of the inverse limit $\mathfrak{H}^\alpha S_T^{\mathcal{L}}(m+1)$, and $\psi(x_0, \ldots, x_m) \in \Pi_\beta(r_\beta)$.

We can now carry out the Henkin construction and build a countable structure $\mathcal{L}$-structure $\mathfrak{M}$ containing an n-tuple $\mathbf{a}$ such that $\Pi_\beta(p_\beta)$ is the $\mathfrak{H}^\beta_T \mathcal{L}$-type of $\mathbf{a}$ in $\mathfrak{M}$, and so $\Pi_\alpha(p_\beta : \beta < \alpha)$ is in $S_T^{\mathfrak{H}^\alpha_T \mathcal{L}} n$, as required. □

Proposition 3.11 *If S is a type category, and M is a model of $\mathfrak{H}S$, then $\mathfrak{h}_S \circ M$ is a model of S. Moreover, for all $\alpha \leq \omega_1$, if M is a model of $\mathfrak{H}^\alpha S$, then $\mathfrak{h}^\alpha_S \circ M$ is a model of S.*

Proof We must show that $\mathfrak{h}_S \circ M$ is existentially closed. Suppose U is open in $(S\iota_{n,n+1})^{-1}\{\mathfrak{h}_S(p)\}$. Then for every model M' of S, and for all $\mathbf{a}$ such that $M' \models \mathfrak{h}_S(p)(\mathbf{a})$, there exists $q \in U$ such that for some b, $M' \models q(\mathbf{a}, b)$. Hence, since p is a $\Sigma^\omega S$-type, there exists $q \in U$ such that $\exists x_n q(x_0, \ldots, x_n) \in p$. Hence ${\mathfrak{h}_S}^{-1}(U) \cap (\mathfrak{H}S\iota_{n,n+1})^{-1}(p) \neq \varnothing$; $\mathfrak{h}_S$ is continuous so ${\mathfrak{h}_S}^{-1}(U) \cap (\mathfrak{H}S\iota_{n,n+1})^{-1}(p)$ is open in $(\mathfrak{H}S\iota_{n,n+1})^{-1}(p)$. Hence existential closure in $\mathfrak{H}S$ gives existential closure in S. We have now established the successor stage of the induction in the second paragraph; the limit stage is similar. □

Note that if S is a type category and $\mathfrak{H}^\alpha S$ is Polish, then the models of $\mathfrak{H}^\alpha S$ are in a natural one-to-one correspondence with the models of S; indeed, the map $M \mapsto \mathfrak{h}_S^{\alpha,0} \circ M$ is a continuous bijection from $X_{\mathfrak{H}^\alpha S}$ to X_S.

4 The Trees of Types

4.1 Definition of the trees We have now derived the trees of types.

Definition 4.1 Suppose S is a type category. The category $\mathfrak{T}S$ of *Vaughtian trees* of S is defined as follows. $\mathfrak{T}Sn$ is a tree, whose elements are the elements of $\bigcup_{\alpha<\omega_1} \mathfrak{H}^\alpha Sn$ such that if $p, q \in \mathfrak{T}Sn$, then $p \leq q$ if and only if for some $\alpha \leq \beta$, $p = \mathfrak{h}_S^{\beta,\alpha}(q)$. If $p \in \mathfrak{T}Sn$, then the *level* of p is the unique α such that $p \in \mathfrak{H}^\alpha Sn$. If $f : n \to m$, then $\mathfrak{T}Sf$ is defined so that if $p \in \mathfrak{H}^\alpha Sm$, then $(\mathfrak{T}Sf)(p) = (\mathfrak{H}^\alpha Sf)(p)$.

We are particularly interested in counterexamples to Vaught's Conjecture. Morley proved in [7] that counterexamples can be identified by reference to the structure of their Vaughtian trees, as follows.

Definition 4.2 Suppose S is a type category. We say it is *slender* if

1. for all $\alpha < \omega_1$ and for all $n \in \omega$, $\mathfrak{H}^\alpha Sn$ is countable, and
2. for all $\alpha < \omega_1$, there exist $n \in \omega$, $\beta > \alpha$, and $p \in \mathfrak{H}^\alpha Sn$ such that $(\mathfrak{h}_S^{\beta,\alpha})^{-1}\{p\}$ has more than one element.

Morley's theorem is the following.

Theorem 4.3 *Suppose the Continuum Hypothesis to be false. Suppose $\mathcal{L}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and T is a countable theory in $\mathcal{L}$. Then T is a counterexample to Vaught's Conjecture if and only if $S_T^{\mathcal{L}}$ is slender if and only if T has $\aleph_1$-many countable models.*

The term "slender" for such type categories is due to Becker.

4.2 Branching structure of the trees This section contains joint work with N. Ackerman. The Vaughtian trees have the following properties.

Proposition 4.4 *$\mathfrak{T}Sn$ is upward-closed; that is, every countable ascending chain in $\mathfrak{T}Sn$ has an upper bound.*

Proof Trivially, by definition of $\mathfrak{H}^\lambda S$ for limit λ. □

We give a fine-grained measure of how closely two elements of a Vaughtian tree resemble each other.

Definition 4.5 Suppose p, p' are distinct elements of $\mathfrak{H}^\beta Sn$. We say they are *equal to accuracy $\omega\alpha + k$* (where $\alpha < \beta$) if $\omega\alpha + k$ is (if it exists) the smallest ordinal such that $\mathfrak{h}_S^{\beta,\alpha}(p) = \mathfrak{h}_S^{\beta,\alpha}(p')$, while the $\Sigma^{k+1}S$-types included in $\mathfrak{h}_S^{\beta,\alpha+1}(p)$ and $\mathfrak{h}_S^{\beta,\alpha+1}(p')$ are different. If no such ordinal exists, we say p and p' are equal to accuracy $-\infty$.

When we extend types to types of larger arity, this resemblance deteriorates.

Theorem 4.6 *Suppose p, p' are distinct elements of $\mathfrak{H}^\beta Sn$ which are equal to accuracy $\omega\alpha + k$. Then either there exists $q \in (\mathfrak{H}^\beta S\iota_{n,n+1})^{-1}(p)$ such that there does not exist $q' \in (\mathfrak{H}^\beta S\iota_{n,n+1})^{-1}(p')$ such that q and q' are equal to accuracy*

$\geq \omega\alpha + k$, or there exists $q' \in (\mathfrak{H}^\beta S\iota_{n,n+1})^{-1}(p')$ such that there does not exist $q \in (\mathfrak{H}^\beta S\iota_{n,n+1})^{-1}(p)$ such that q and q' are equal to accuracy $\geq \omega\alpha + k$.

Proof Each of $\mathfrak{h}_S^{\beta,\alpha+1}(p)$ and $\mathfrak{h}_S^{\beta,\alpha+1}(p')$ includes a unique $\Sigma^{k+1}\mathfrak{H}^\alpha S$-type; let $\tilde{p}$ be that included in $\mathfrak{h}_S^{\beta,\alpha+1}(p)$, and $\tilde{p}'$ be that included in $\mathfrak{h}_S^{\beta,\alpha+1}(p')$. Now $\tilde{p}$ and $\tilde{p}'$ are different; however, the $\Sigma^k\mathfrak{H}^\alpha S$-types included in them are the same. Therefore, there exists some $n+1$-ary $\Sigma^k\mathfrak{H}^\alpha S$-type $\tilde{q}$ such that either $\exists x \bigwedge \tilde{q} \in \tilde{p}$ and $\forall x \neg \bigwedge \tilde{q} \in \tilde{p}'$, or $\exists x \bigwedge \tilde{q} \in \tilde{p}'$ and $\forall x \neg \bigwedge \tilde{q} \in \tilde{p}$.

Suppose the former. Now, for every model M of $\mathfrak{H}^\beta S$ and $a_0, \ldots, a_{n-1} \in \omega$ such that $M \models p(a_0, \ldots, a_{n-1})$, there exists $a_n \in \omega$ such that $\mathfrak{h}_S^{\beta,\alpha} \circ M \models \tilde{q}(a_0, \ldots, a_n)$. Select some such M and a_n and let $\overline{q}$ be the $\Sigma^\omega\mathfrak{H}^\alpha S$-type realized by $\langle a_0, \ldots, a_n\rangle$, and let q be the $\Sigma^\omega\mathfrak{H}^\beta S$-type realized by $\langle a_0, \ldots, a_n\rangle$. Then $\mathfrak{h}_S^{\beta,\alpha+1}(q) = \overline{q}$, and $\mathfrak{H}^\beta S\iota_{n,n+1}(q) = p$. However, there does not exist q' such that $\mathfrak{H}^\beta S\iota_{n,n+1}(q') = p'$ and q and q' are equal to accuracy $\omega\alpha + k$, or else q' would entail $\tilde{q}$ and so $\exists x \bigwedge \tilde{q}$ would belong to p', giving a contradiction. □

However, it does not deteriorate very fast.

Theorem 4.7 *Suppose p, p' are distinct elements of $\mathfrak{H}^\beta Sn$ which are equal to accuracy $\omega\alpha + k$. Suppose $\gamma < \omega\alpha + k$. Then for all $q \in \mathfrak{H}^\beta S(n+1)$ with $\mathfrak{H}^\beta S\iota_{n,n+1}(q) = p$, there exists $q' \in \mathfrak{H}^\beta S(n+1)$ with $\mathfrak{H}^\beta S\iota_{n,n+1}(q') = p'$ such that q and q' are equal to accuracy at least γ.*

We may phrase this by saying that information about differences between types is concealed above *slant lines*, that is to say, decreasing functions from ω to ω_1. This is the motivating idea behind the counterexample to Vaught's Conjecture described in [6].

There are some results about slender theories in this paper and many in the literature. However, all of these fall short of a constructive characterization.

Question 4.8 *Find a characterization of slender type categories which is sufficiently explicit that (a) it can easily be used to construct slender type categories, and (b) it is easy to use it to tell whether a type category is slender or not.*

We now address the question of the branching structure of the Vaughtian trees. The next theorem tells us that any given branch either has no side branches or has very many of them.

Theorem 4.9 *Suppose that S is a type category, and $p \in Sn$. Then $(Sf)^{-1}\{p\}$ is topologically discrete for all $m \geq n$ and $f : n \xrightarrow{1\text{–}1} m$ if and only if $(\mathfrak{h}_S)^{-1}\{p\}$ has exactly one element. Moreover, suppose $(Sf)^{-1}\{p\}$ is topologically discrete for all $m \geq n$ and $f : n \xrightarrow{1\text{–}1} m$, and suppose that p' is unique such that $\mathfrak{h}_S(p') = p$. Then $(\mathfrak{H}Sf)^{-1}\{p'\}$ is topologically discrete for all $m \geq n$ and $f : n \xrightarrow{1\text{–}1} m$.*

Proof Suppose first that $(Sf)^{-1}\{p\}$ is topologically discrete for all $m \geq n$ and $f : n \xrightarrow{1\text{–}1} m$. We argue that all realizations of p realize the same $\Sigma^\omega S$-types by showing that for each k they realize the same $\Sigma^k S$-types. Indeed, we prove that for all $k \in \omega$, all $m \geq n$, and all $q \in (S\iota_{n,m})^{-1}\{p\}$, all realizations of q realize the same $\Sigma^k S$-types.

We argue by induction on k. Suppose $\tilde{q}$ is a $\Sigma^{k+1}S$-type such that $q \in \tilde{q}$. Then because $(S\iota_{n,n+1})^{-1}\{q\}$ is a discrete topological space, for every $r \in (S\iota_{n,n+1})^{-1}\{q\}$, there must be a $\Sigma^k S$-type $\tilde{r}$ such that $r \in \tilde{r}$ and $\exists x \bigwedge \tilde{r} \in \tilde{q}$, by the definition of existential closure of models of S. But now for all $m \geq n+1$ and $f : n+1 \xrightarrow{1\text{–}1} m$, $(Sf)^{-1}\{r\}$ is discrete. Hence by the inductive hypothesis for k, as applied to r, $\tilde{r}$ is determined by r. Hence, if $\hat{q}$ is the $\Sigma^k S$-type included in $\tilde{q}$, we may write $\tilde{q}$ as

$$\tilde{q} = \hat{q} \cup \left\{\exists x \ \bigwedge \tilde{r} \;\middle|\; r \in (S\iota_{n,n+1})^{-1}\{q\}\right\}$$
$$\cup \left\{\forall x \neg \bigwedge \tilde{r} \;\middle|\; r \notin (S\iota_{n,n+1})^{-1}\{q\}\right\}.$$

Thus $\tilde{q}$ is determined by q, and the inductive hypothesis is preserved at $k+1$. Hence all realizations of p realize the same $\Sigma^k S$-types for all k, and it follows from this that $\mathfrak{h}_S{}^{-1}\{p\}$ has a unique element.

Now suppose that $m \geq n$, $f : n \xrightarrow{1\text{–}1} m$, $(Sf)(q) = p$, and q is a limit point of $(Sf)^{-1}\{p\}$. Then, by Proposition 2.12, we may find models M and M' of S such that there exist tuples $\langle a_0, \ldots, a_{n-1}\rangle$ and $\langle a'_0, \ldots, a'_{n-1}\rangle$ such that $M \models p(a_0, \ldots, a_{n-1})$ and $M' \models p(a'_0, \ldots, a'_{n-1})$, and there exists a tuple $\langle b_0, \ldots, b_{m-1}\rangle$ such that for all $i < m$, $a_i = b_{f(i)}$ and $M \models q(b_0, \ldots, b_{m-1})$, but there does not exist a tuple $\langle b'_0, \ldots, b'_{m-1}\rangle$ such that for all $i < m$, $a'_i = b'_{f(i)}$ and $M' \models q(b'_0, \ldots, b'_{m-1})$. It follows that $\langle a_0, \ldots, a_{n-1}\rangle$ and $\langle a'_0, \ldots, a'_{n-1}\rangle$ realize different $\Sigma^\omega S$-types, and so $\mathfrak{h}_S{}^{-1}\{p\}$ has more than one element.

If $(Sf)^{-1}\{p\}$ is topologically discrete for all $m \geq n$ and $f : n \xrightarrow{1\text{–}1} m$, then also, whenever $m \geq n$ and $f : n \xrightarrow{1\text{–}1} m$, and $(Sf)(q) = p$, $(Sg)^{-1}\{q\}$ is topologically discrete for all $k \geq m$ and $g : m \xrightarrow{1\text{–}1} k$. Hence, $(\mathfrak{h}_S)^{-1}\{q\}$ has a unique element. In the topological space $\mathfrak{H}Sm$, $(\mathfrak{h}_S)^{-1}\{q\}$ is clopen. So $(\mathfrak{H}Sf)^{-1}\{p'\}$ is topologically discrete for all $m \geq n$ and $f : n \xrightarrow{1\text{–}1} m$, where p' is the unique element of $(\mathfrak{h}_S)^{-1}\{p\}$. □

Corollary 4.10 *Suppose $p \in \mathfrak{H}^\alpha Sn$ is not isolated in $\mathfrak{H}^\alpha Sn$. Then for all $\beta < \alpha$, $(\mathfrak{h}_S^{\beta+1,\beta})^{-1}\{\mathfrak{h}_S^{\alpha,\beta}(p)\}$ has at least two elements.*

We can thus derive a sharper characterization of slenderness.

Corollary 4.11 *A type category S is slender if and only if for every $\alpha \in \omega_1$ and $n \in \omega$, $\mathfrak{H}^\alpha Sn$ is countable, and for every $\alpha \in \omega_1$, for every $n \in \omega$, there exists $p \in \mathfrak{H}^\alpha Sn$ such that $(\mathfrak{h}_S^{\alpha+1,\alpha})^{-1}\{p\}$ has at least two elements.*

Proof Suppose that for some $\alpha \in \omega_1$ and $n \in \omega$, for every $p \in \mathfrak{H}^\alpha Sn$, $(\mathfrak{h}_S^{\alpha+1,\alpha})^{-1}\{p\}$ has just one element. Then by Theorem 4.9, for all $p \in \mathfrak{H}^\alpha Sn$, for all $m \geq n$, and for all $f : n \xrightarrow{1\text{–}1} m$, $(\mathfrak{H}^\alpha Sf)^{-1}\{p\}$ is topologically discrete. Hence, for all $k \geq n$, for all $q \in \mathfrak{H}^\alpha Sk$, for all $m \geq k$, and for all $f : k \xrightarrow{1\text{–}1} m$, $(\mathfrak{H}^\alpha Sf)^{-1}\{q\}$ is topologically discrete.

Then again by Theorem 4.9, for every $k \geq n$, for every $p \in \mathfrak{H}^\alpha Sk$, $(\mathfrak{h}_S^{\alpha+1,\alpha})^{-1}\{p\}$ has just one element. It now follows, from the definition of the topology on $\mathfrak{H}^{\alpha+1}Sk$, that for each $k \geq n$, $\mathfrak{H}^{\alpha+1}Sk$ is topologically discrete. Thus, for each $p \in \mathfrak{H}^{\alpha+1}S(n-1)$, for all $m \geq n-1$ and for all $f : n-1 \xrightarrow{1\text{–}1} m$, $(\mathfrak{H}^{\alpha+1}Sf)^{-1}\{p\}$

is topologically discrete, and so we can prove that for each $k \geq n-1$, $\mathfrak{H}^{\alpha+2}Sk$ is topologically discrete. Continuing in this way, $\mathfrak{H}^{\alpha+n+1}Sk$ is discrete for all k. Applying Theorem 4.9 and induction on β, we can see that, firstly, $\mathfrak{H}^{\beta}Sk$ is discrete for all k and for all $\beta \geq \alpha+n+1$, and secondly, that for all $\beta \geq \alpha+n+1$, $\mathfrak{h}_S^{\beta,\alpha+n+1}$ is one-to-one. It now follows that S is not slender. □

The following is obvious.

Proposition 4.12 *If S is slender, then for all n, $\mathfrak{T}Sn$ is uncountable, all its levels are countable, and it is scattered: that is, a Cantor-Bendixson rank can be assigned to the elements of $\mathfrak{H}^{\omega_1}Sn$ (which are in natural one-to-one correspondence with maximal branches of $\mathfrak{T}Sn$); or equivalently, $\mathfrak{T}Sn$ does not contain a copy of the Cantor tree.*

We abuse terminology by discussing the Cantor-Bendixson rank of maximal branches in $\mathfrak{T}Sn$ instead of the Cantor-Bendixson rank of elements of $\mathfrak{H}^{\omega_1}Sn$, exploiting this natural one-to-one correspondence.

From this proposition the next easily follows.

Proposition 4.13 *If S is slender, then every uncountable, downward closed subset of $\mathfrak{T}Sn$ contains an uncountable maximal branch.*

Proof Suppose that D is an uncountable downward closed subset of $\mathfrak{T}Sn$. Let ν be a sufficiently large cardinal, and let M be a countable elementary substructure of $\langle H_\nu, \in\rangle$ containing S and D as elements. Then, for each $\beta \in \omega_1 \cap M$, $\mathfrak{H}^{\beta}Sn$ is an element of M. Because $\mathfrak{H}^{\beta}Sn$ is countable, it is also a subset of M.

Let $\lambda = M \cap \omega_1$; then λ is a countable limit ordinal. Let $\langle \alpha_m : m \in \omega\rangle$ be a strictly increasing sequence of ordinals converging to λ. We note that because $\mathfrak{H}^{\lambda}Sn$ is the inverse limit of the $\mathfrak{H}^{\alpha}Sn$ for $\alpha < \lambda$ and $\mathfrak{H}^{\lambda}Sn$ is countable, it is impossible to find a subsequence $\langle \alpha_{m_i} : i \in \omega\rangle$ of $\langle \alpha_m : m \in \omega\rangle$ and elements p_s, for $s \in {}^{<\omega}2$ such that

1. if $\operatorname{dom} s = i$, then $p_s \in \mathfrak{H}^{\alpha_{m_i}}Sn$,
2. if $s \subseteq t$, $\operatorname{dom} s = i$, and $\operatorname{dom} t = j$, then $p_s = \mathfrak{h}_S^{\alpha_j,\alpha_i}(p_t)$,
3. for all s, there are uncountably many members of D above p_s,
4. for all s, $p_{s^\frown 0}$ and $p_{s^\frown 1}$ are different.

Hence, there exist $m \in \omega$ and $p \in D \cap \mathfrak{H}^{\alpha_m}Sn$ such that the following statement is true about p:

> For all $\alpha \in \lambda$ there exists $\beta \in \lambda$ such that $\beta \geq \alpha$, and there is only one element q of $\mathfrak{H}^{\beta}Sn$ such that q is above p and there are uncountably many elements of D above q.

(Here, by "q is above p," we mean "$\mathfrak{h}_S^{\beta,\alpha_m}(q) = p$.") That is,

> For all $\alpha \in \omega_1 \cap M$ there exists $\beta \in \omega_1 \cap M$ such that $\beta \geq \alpha$, and there is only one element q of $\mathfrak{H}^{\beta}Sn$ such that q is above p and there are uncountably many elements of D above q.

Now because M is an elementary substructure of $\langle H_\nu, \in\rangle$, the following statement must be true:

> For all $\alpha \in \omega_1$ there exists $\beta \in \omega_1$ such that $\beta \geq \alpha$, and there is only one element q of $\mathfrak{H}^{\beta}Sn$ such that q is above p and there are uncountably many elements of D above q.

Let C be the unbounded set of ordinals $\beta \geq \alpha$ such that there is only one element q of $\mathfrak{H}^{\beta}Sn$ such that $h^{\beta,\alpha_m}(q) = p$ and there are uncountably many elements of D above q, and let q_β be, for $\beta \in C$, the element of $\mathfrak{H}^{\beta}Sn$ so defined. Then, if $\beta, \gamma \in C$, and $\beta < \gamma$, then $\mathfrak{h}_S^{\gamma,\beta}(q_\gamma)$ must be q_β, for all the elements of D that are above q_γ must be above some element of $\mathfrak{H}^{\beta}Sn$, and q_β itself is the only possibility.

There is thus an uncountable maximal branch B containing all the q_β for $\beta \in C$. Since D is downward closed, each q_β belongs to D, and hence B is included in D, as required. □

Here, and elsewhere, a branch is simply a linearly ordered subset of a tree.

4.3 Atomic and saturated models Suppose S is a slender type category. Referring to the Vaughtian trees $\mathfrak{T}Sn$, we define the predicate $\Phi(p, \alpha)$, for $p \in \mathfrak{T}Sn$ and $\alpha < \omega_2$, by recursion on α thus:

$$\Phi(p, \alpha) \text{ iff } \{q \not\perp p \mid \neg\exists\beta < \alpha\ \Phi(q, \beta)\} \text{ contains exactly one maximal branch.}$$

Here the notation $q \not\perp p$ means that q and p have a common *upper* bound, that is, $q \leq p$ or $p \leq q$. Then, of course, any maximal branch contains co-initially many points p such that $\Phi(p, \alpha)$ holds, where α is the Cantor-Bendixson rank of the branch, and if $\Phi(p, \alpha)$, then there exists a unique branch through p of rank $\geq \alpha$, and this branch in fact has rank α.

Of course, not every node necessarily satisfies such a predicate. However, for every $p \in \mathfrak{T}Sn$, it is the case that p belongs to a branch of Cantor-Bendixson rank α for only certain values of α. So, let us define a related predicate $\Psi(p, \alpha)$ as follows:

$$\Psi(p, \alpha) \text{ iff } \{q \not\perp p \mid \exists r \geq q\ \Phi(r, \alpha)\} \text{ contains at least one maximal branch,}$$

equivalently, if and only if $\exists q \geq p\, \Phi(q, \alpha)$. Now let us say that p *generates a branch* if $\Phi(p, \alpha)$ holds for some α and define the *branch generated by* p to be the downward-closure of

$$\{q \geq p : \Phi(q, \alpha)\}$$

(which is an uncountable maximal branch and has rank α). Our next task is to work out how to express the notion that every uncountable branch is generated by some point, without referring to the branches. We do this by an indirect route.

Let us define the *scattered height* of $p \in \mathfrak{T}Sn$ to be the supremum of the set $\{\beta + 1 \mid \exists q \geq p\, \Phi(q, \beta)\}$. (The reason for the terminology is that p defines a clopen subset of the topological space of branches, or equivalently of $\mathfrak{H}^{\omega_1}Sn$, which is scattered, and the definition we have given is that of the scattered height of that space of branches.)

Lemma 4.14 *If $p \leq q$, then the scattered height of p is greater than or equal to that of q. If $\Phi(p, \alpha)$, then the scattered height of p is $\alpha + 1$.*

Proof Obvious. □

Lemma 4.15 *If $p \in \mathfrak{T}Sn$ and the scattered height of p is a limit μ, then there are only countably many $q \geq p$ such that the scattered height of q is μ.*

Proof Otherwise, the set of $q \geq p$ of scattered height μ would include an uncountable branch B by Proposition 4.12. But some element q of B would generate B. Now necessarily, by definition of the scattered height of p, for some $\beta < \mu$, $\Phi(q, \beta)$ holds. But then the scattered height of q is $\beta + 1 < \mu$, giving a contradiction. □

Lemma 4.16 *Suppose that $p \in \mathfrak{T}Sn$, and the scattered height of p is a successor $\beta + 1$. Then p belongs to just countably many uncountable branches of rank β, and so there exists $\gamma \in \omega_1$ such that if for some $\delta > \gamma$, $q \in \mathfrak{H}^\delta Sn$, $q \geq p$, and the scattered height of q is $\beta + 1$, then $\Phi(q, \beta)$ holds.*

Proof If p belonged to uncountably many branches of rank β, then the set of $q \geq p$ such that q belonged to uncountably many branches of rank β would be uncountable, and so would contain an uncountable branch by Proposition 4.12, which would have rank at least $\beta + 1$. But then the scattered height of p would be at least $\beta + 2$, giving a contradiction. □

Note that $\Phi(p, \alpha)$ and $\Psi(p, \alpha)$ can now be defined without any reference to branches; $\Phi(p, \alpha)$ holds if and only if

1. for uncountably many γ, there exists $q \geq p$ such that q is in the γth level of $\mathfrak{T}Sn$ and for cofinally many $\beta < \alpha$, $\Psi(q, \beta)$ is true but $\Phi(q, \beta)$ is not, and
2. there exists γ such that if q and r are above p and also above level γ, and if for cofinally many $\beta < \alpha$, $\Psi(q, \beta)$ and $\Psi(r, \beta)$ are true but $\Phi(q, \beta)$ and $\Phi(r, \beta)$ are not, then $q \leq r$ or $r \leq q$.

If we can find in the set H_{ω_1} of all hereditarily countable sets some surrogate for ordinals $\alpha \in \omega_2$ which we can use to refer to the ranks of branches, then we can carry out the definition of the statements $\Phi(p, \alpha)$ and $\Psi(p, \alpha)$ in the first-order structure $\langle H_{\omega_1}, \in \rangle$. We can do this—once we have observed that the property of belonging to $\mathfrak{T}Sn$ is definable in $\langle H_{\omega_1}, \in \rangle$—by letting the elements p of $\mathfrak{T}Sn$ themselves stand for the ordinals in question; so by recursion on the ordinal α such that $\Phi(p, \alpha)$, we can define surrogate formulas $\Phi'(p, q)$, $\Psi'(p, q)$, and $\mathcal{L}(p, q)$, in which $\mathcal{L}(p, q)$ stands for 'there exist α and $\beta \geq \alpha$ such that $\Phi(p, \alpha)$ and $\Phi(q, \beta)$', $\Phi'(p, q)$ stands for $\mathcal{L}(p, q) \wedge \mathcal{L}(q, p)$, and $\Psi'(p, q)$ stands for 'there exists α such that $\Phi(q, \alpha)$ and $\Psi(p, \alpha)$'. We do this as follows. We declare that $\Psi'(p, q)$ if and only if there exists $r \geq p$ such that $\Phi'(r, q)$. If $\Psi'(p, q)$ is true but $\Phi'(p, q)$ is false, then we declare that $\Psi'(q, p)$ is false.

We now declare that $\Phi'(p, q)$ is true if and only if *either* for all r, $s \geq p$, $r \leq s$, or $s \leq r$, and for all r, $s \geq q$, $r \leq s$, or $s \leq r$, *or else*

1. for uncountably many γ, there exists $r \geq p$ such that r is in the γth level of $\mathfrak{T}Sn$ and for all $s \geq q$, if $\Phi'(s, s)$ is true and $\Phi'(s, q)$ is false, then $\Psi'(r, s)$ is true but $\Phi'(r, s)$ is not, and
2. there exists γ such that if r and s are above p and also above level γ, and if for all $t \geq q$ such that $\Phi'(t, t)$ is true but $\Phi'(t, q)$ is false, $\Psi'(r, t)$ and $\Psi'(s, t)$ are true but $\Phi'(r, t)$ and $\Phi'(s, t)$ are not, then $r \leq s$ or $s \leq r$,

and the same conditions hold with the roles of p and q reversed.

We define $\mathcal{L}(p, q)$ to hold if and only if $\Phi'(p, p)$, $\Phi'(q, q)$, and $\Psi'(p, q)$ all hold. Then $\mathcal{L}$ imposes a well-order on the equivalence classes of Φ' from which the ordinal ranks of the branches can be recovered.

We can also define scattered height in $\langle H_{\omega_1}, \in \rangle$ by expressing the concept 'p has scattered height α' as

1. $\forall \beta < \alpha \, \exists \gamma \geq \beta \, \Psi(p, \gamma)$, and
2. if $\beta \geq \alpha$, then $\neg \Psi(p, \beta)$,

(only, of course, we must use some surrogate for the ordinals α, β, and γ, as described above).

Now although the *proofs* of the above lemmas refer to branches in $\mathfrak{T}Sn$, the *statements* can be rephrased so that they do not, as follows:

1. if the scattered height of p is a limit μ, then there exists γ such that if $q \geq p$ and for some $\delta > \gamma$, $q \in \mathfrak{H}^{\delta}Sn$, then the scattered height of q is less than μ;
2. if the scattered height of p is a successor $\beta + 1$, then then there exists γ such that if $q \geq p$, $q \in \mathfrak{H}^{\delta}Sn$ for some $\delta > \gamma$, and q has scattered height $\beta + 1$, then $\Phi(q, \beta)$.

Thus these statements, which are first-order over $\langle H_{\omega_1}, \in\rangle$, are true in $\langle H_{\omega_1}, \in\rangle$. Thus we can prove the following (recall that any countable elementary substructure $\langle M, \in\rangle$ of $\langle H_{\omega_1}, \in\rangle$ is transitive, so its intersection with ω_1 is an ordinal and is in fact a limit).

Lemma 4.17 *Suppose that $\langle M, \in\rangle$ is a countable elementary substructure of $\langle H_{\omega_1}, \in\rangle$. Let $\lambda = M \cap \omega_1$. Suppose $p \in \mathfrak{H}^{\lambda}Sn$. Then p generates a branch. Moreover, there exists $\alpha < \lambda$ such that $\mathfrak{h}_S^{\lambda,\alpha}(p)$ generates the same branch.*

Proof Suppose λ is the supremum of the increasing sequence $\langle \alpha_n \mid n \in \omega\rangle$, and that for each n, $p_n = \mathfrak{h}_S^{\lambda,\alpha_n}(p)$. Let η_n be the scattered height of p_n. Then the sequence $\langle \eta_n \mid n \in \omega\rangle$ is nonstrictly decreasing, so it is eventually constant. So, without loss of generality (by omitting the first few terms of the sequence if necessary), let us suppose that $\eta_n = \eta$ for all n.

Then η is the scattered height of p_0. Because $\langle M, \in\rangle$ is an elementary substructure of $\langle H_{\omega_1}, \in\rangle$, η is not a limit, for otherwise there exists $\gamma < \lambda$ such that if $q \geq p_0$ and $q \in \mathfrak{H}^{\delta}Sn$ for some $\delta \in (\gamma, \lambda)$, then the scattered height of q is less than η. But this contradicts the fact that the scattered height of p_n is η for all n. So η is a successor $\beta + 1$. By a similar argument, there exists $\gamma < \lambda$ such that if $q \geq p_0$, and $q \in \mathfrak{H}^{\delta}Sn$ for some $\delta \in (\gamma, \lambda)$, and the scattered height of q is $\beta + 1$, then $\Phi(q, \beta)$ holds.

Thus for all but finitely many n, $\Phi(p_n, \beta)$ holds. So each corresponding p_n generates a branch of rank β. Since the p_n are linearly ordered, these branches must all be the same and must include p. Therefore p generates this branch also. $\square$

We draw the obvious corollary.

Corollary 4.18 *There exists a closed unbounded subset C of ω_1 such that if $\lambda \in C$, and $p \in \mathfrak{H}^{\lambda}Sn$, then p generates a branch.*

This result permits us to make the following definition.

Definition 4.19 Suppose $\alpha \in C$ and $\beta \in (\alpha, \omega_1]$. Suppose $p \in \mathfrak{H}^{\alpha}Sn$. Then we define $\mathfrak{h}_S^{\alpha,\beta}(p)$ to be the unique element of $\mathfrak{H}^{\beta}Sn$ belonging to the branch generated by p.

We have an obvious result.

Lemma 4.20 *Suppose $\mathfrak{h}_S^{\alpha,\beta}$ and $\mathfrak{h}_S^{\beta,\gamma}$ are both defined. Then $\mathfrak{h}_S^{\alpha,\gamma}$ is defined, and*

$$\mathfrak{h}_S^{\alpha,\gamma} = \mathfrak{h}_S^{\beta,\gamma} \circ \mathfrak{h}_S^{\alpha,\beta}.$$

Now for one that is rather less obvious.

Lemma 4.21 *Suppose that $\langle M, \in\rangle$ is a countable elementary substructure of $\langle H_{\omega_1}, \in\rangle$, $\alpha = M \cap \omega_1$, $\beta \in \omega_1$, and $f : m \to n$. Then the diagram*

$$\begin{array}{ccc} \mathfrak{H}^\alpha Sn & \xrightarrow{\mathfrak{H}^\alpha Sf} & \mathfrak{H}^\alpha Sm \\ {\scriptstyle \mathfrak{h}_S^{\alpha,\beta}}\downarrow & & \downarrow{\scriptstyle \mathfrak{h}_S^{\alpha,\beta}} \\ \mathfrak{H}^\beta Sn & \xrightarrow{\mathfrak{H}^\beta Sf} & \mathfrak{H}^\beta Sm \end{array}$$

commutes.

Proof This is obvious if $\beta \le \alpha$, so suppose $\beta > \alpha$. Suppose $p \in Sn$, and $q = \mathfrak{H}^\alpha Sf(p)$. By Lemma 4.17, there exists $\zeta < \alpha$ such that if $p' = \mathfrak{h}_S^{\alpha,\zeta}(p)$ and $q' = \mathfrak{h}_S^{\alpha,\zeta}(q)$, then p' generates the branch containing p, and q' generates the branch containing q. Also, $q' = \mathfrak{H}^\zeta Sf(p')$.

Suppose that $\Phi(p', \xi)$ and $\Phi(q', \eta)$. Suppose it were the case that there existed $\gamma \in (\zeta, \alpha)$ and $p'' \in \mathfrak{H}^\gamma Sn$, $q'' \in \mathfrak{H}^\gamma Sm$ such that $p'' > p'$ and $q'' > q$, $\Phi(p'', \xi)$ and $\Phi(q'', \eta)$, and $q'' \ne \mathfrak{H}^\gamma Sf(p'')$, then necessarily we would have $p'' = \mathfrak{h}_S^{\alpha,\gamma}(p)$ and $q'' = \mathfrak{h}_S^{\alpha,\gamma}(q)$, and this would give rise to a contradiction, since $q = \mathfrak{H}^\alpha Sf(p)$.

Since $\langle M, \in\rangle$ is an elementary substructure of $\langle H_{\omega_1}, \in\rangle$, there does not exist $\gamma \in \omega_1$ and $p'' \in \mathfrak{H}^\gamma Sn$, $q'' \in \mathfrak{H}^\gamma Sm$ such that $p'' > p'$ and $q'' > q$, $\Phi(p'', \xi)$ and $\Phi(q'', \eta)$, and $q'' \ne \mathfrak{H}^\gamma Sf(p'')$. Hence the diagram

$$\begin{array}{ccc} \mathfrak{H}^\alpha Sn & \xrightarrow{\mathfrak{H}^\alpha Sf} & \mathfrak{H}^\alpha Sm \\ {\scriptstyle \mathfrak{h}_S^{\alpha,\beta}}\downarrow & & \downarrow{\scriptstyle \mathfrak{h}_S^{\alpha,\beta}} \\ \mathfrak{H}^\beta Sn & \xrightarrow{\mathfrak{H}^\beta Sf} & \mathfrak{H}^\beta Sm \end{array}$$

commutes, as required. □

At elements of the closed unbounded set C, we also have amalgamativity. For we show that $\mathfrak{H}^{\omega_1} S$ is amalgamative and use elementary reflection.

Proposition 4.22 *Suppose that S is a slender type category. Then $\mathfrak{H}^{\omega_1} S$ is amalgamative. Also, if $\alpha \in C$, then $\mathfrak{H}^\alpha S$ is amalgamative.*

Proof Suppose $p \in \mathfrak{H}^{\omega_1} Sm$, $q_1 \in \mathfrak{H}^{\omega_1} Sn_1$, $q_2 \in \mathfrak{H}^{\omega_1} Sn_2$, $f_1 : m \xrightarrow{1\text{–}1} n_1$, $f_2 : m \xrightarrow{1\text{–}1} n_2$, $g_1 : n_1 \xrightarrow{1\text{–}1} l$, $g_2 : n_2 \xrightarrow{1\text{–}1} l$, $\operatorname{ran} g_1 \cap \operatorname{ran} g_2 = \operatorname{ran} g_1 \circ \operatorname{ran} f_1$, and the diagram

$$\begin{array}{ccc} m & \xrightarrow{f_1} & n_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ n_2 & \xrightarrow{g_2} & l \end{array}$$

commutes. Then, for each $\alpha < \omega_1$, if we let $U = (\mathfrak{h}_S^{\omega_1,\alpha})^{-1}\{\mathfrak{h}_S^{\omega_1,\alpha}(q_1)\}$ and $V = (\mathfrak{h}_S^{\omega_1,\alpha})^{-1}\{\mathfrak{h}_S^{\omega_1,\alpha}(q_2)\}$, then by definition of a type category there exists $r^\alpha \in \mathfrak{H}^{\omega_1} Sl$ such that $\mathfrak{H}^{\omega_1} Sg_1(r) \in U$, $\mathfrak{H}^{\omega_1} Sg_2(r) \in V$, and $\mathfrak{H}^{\omega_1} S(g_1 \circ f_1)(r) = p$. Now, $\mathfrak{H}^\alpha Sg_i(\mathfrak{h}_S^{\omega_1,\alpha}(r)) = \mathfrak{h}_S^{\omega_1,\alpha}(q_i)$ for $i = 1, 2$, and obviously also if $\beta \le \alpha$, then $\mathfrak{H}^\beta Sg_i(\mathfrak{h}_S^{\omega_1,\beta}(r)) = \mathfrak{h}_S^{\omega_1,\beta}(q_i)$ for $i = 1, 2$.

Thus, the set

$$\left\{ r \;\middle|\; \exists \alpha \in \omega_1 \left(r \in \mathfrak{H}^\alpha Sl \wedge \mathfrak{H}^\alpha Sg_i(r) = \mathfrak{h}_S^{\omega_1,\alpha}(q_i) \text{ for } i = 1,\ 2 \right) \right\}$$

is uncountable and downward-closed in $\mathfrak{T}Sl$, and therefore contains an uncountable branch B. Suppose r' generates this branch and belongs to $\mathfrak{H}^\gamma Sl$. Then for all $\alpha \in (\gamma, \omega_1)$, $\mathfrak{h}_S^{\gamma,\alpha}(r')$, which, by definition, belongs to B, satisfies the defining property of B, namely, that $\mathfrak{H}^\alpha Sg_i(\mathfrak{h}_S^{\gamma,\alpha}(r)) = \mathfrak{h}_S^{\omega_1,\alpha}(q_i)$ for $i = 1, 2$. Hence also, $\mathfrak{H}^{\omega_1}Sg_i(\mathfrak{h}_S^{\gamma,\omega_1}(r)) = q_i$ for $i = 1, 2$, and so amalgamativity of $\mathfrak{H}^{\omega_1}$ is proved.

Now suppose the branch containing p is generated by p', the branch containing q_i is generated by q_i' for $i = 1, 2$. Then the following statement is true about p', q_1' and q_2':

> There exists $r' \in \mathfrak{T}Sl$ such that r' generates a branch, and for all γ at least as great as the levels of p', q_1', q_2', and r', if r'' is the element of the branch generated by r' on level γ and p'', q_1'', and q_2'' are the elements, respectively, of the branches generated by p', q_1', and q_2' on level γ, then $(Sg_i)(r'') = q_i''$ and $(Sf_i)(q_i'') = p''$ for $i = 1, 2$.

This is all expressible in first order in $\langle H_{\omega_1}, \in\rangle$, and is true in $\langle H_{\omega_1}, \in\rangle$, and is therefore satisfied by $\langle H_{\omega_1}, \in\rangle$ as a first-order structure.. Hence it is also true in any countable elementary substructure $\langle M, \in\rangle$ of $\langle H_{\omega_1}, \in\rangle$ which contains p', q_1', and q_2'. It follows that if $\alpha \in C$, then $\mathfrak{H}^\alpha S$ is amalgamative. □

We define atomic and saturated models of an element of a type category in the way that one would expect.

Definition 4.23 Suppose S is a type category, $p \in Sn$, and M is a model of S. Then M is an *atomic model of* p if and only if $M \models p(0, 1, \ldots, n-1)$, and for all finite subsets A of ω including $\{0, 1, \ldots, n-1\}$, if $q = M(A)$, then q is isolated in $(S\iota_{n,|A|})^{-1}\{p\}$. M is a *saturated model of* p if and only if $M \models p(0, 1, \ldots, n-1)$, and whenever $M \models q(i_0, \ldots, i_{k-1})$ and $r \in (S\iota_{k,k+1})^{-1}\{q\}$, then there exists i_k such that $M \models r(i_0, \ldots, i_k)$.

If $\mathcal{L}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$, T is a theory in $\mathcal{L}$, and $S_T^{\mathcal{L}}$ is scattered (as is the case if $S_T^{\mathcal{L}}$ is slender), then it is well known that for any $p \in S_T^{\mathcal{L}}n$, there exists an atomic model $\mathfrak{A}_p$ of p.

We can now prove the following.

Corollary 4.24 *If $\alpha \in C$ and $p \in \mathfrak{H}^\alpha S0$, then there exists a saturated model $\mathfrak{S}_p$ of p.*

Proof $\mathfrak{H}^\alpha S$ is amalgamative, and all $\mathfrak{H}^\alpha Sn$ are countable. □

There is a correspondence between a type in a theory and the orbit under the automorphism group of a tuple realizing that type in a saturated model. Thus we now have, as a corollary, the following.

Theorem 4.25 *Suppose T is a countable theory in a countable fragment $\mathcal{L}$ of $\mathcal{L}_{\omega_1,\omega}$, and $S_T^{\mathcal{L}}$ is slender. Then there exists a closed unbounded subset C of ω_1 such that if $p \in \mathfrak{H}^\gamma S0$ generates a branch, then for all $\alpha \in C$ such that $\alpha > \gamma$, there exist atomic and saturated models $\mathfrak{A}_\alpha$ and $\mathfrak{S}_\alpha$ of $\mathfrak{h}_S^{\gamma,\alpha}(p)$, and for $\alpha < \beta$ elements of C there exist functions $a_{\alpha,\beta} : \mathfrak{A}_\alpha \to \mathfrak{A}_\beta$, $s_{\alpha,\beta} : \mathfrak{S}_\alpha \to \mathfrak{S}_\beta$, and $i_\alpha : \mathfrak{A}_\alpha \to \mathfrak{S}_\alpha$ such that*

1. *i_α, $a_{\alpha,\beta}$, and $s_{\alpha,\beta}$ are $\mathfrak{H}^\alpha\mathcal{L}$-elementary embeddings (strictly, $a_{\alpha,\beta}$ is an embedding of $\mathfrak{A}_\alpha$ into the reduct of $\mathfrak{A}_\beta$ to $\mathfrak{H}_T^\alpha\mathcal{L}$, and similarly for $s_{\alpha,\beta}$);*

2. *moreover, for $\alpha < \beta$, if $\mathbf{x}$ is a tuple on $\mathfrak{S}_\alpha$ satisfying a type $p \in \mathfrak{H}^\alpha Sn$, then $s_{\alpha,\beta}(\mathbf{x})$ satisfies the type $\mathfrak{h}_S^{\alpha,\beta}(p)$ in $\mathfrak{S}_\beta$;*
3. *if $\alpha < \beta < \gamma$, then $a_{\alpha,\gamma} = a_{\beta,\gamma} \circ a_{\alpha,\beta}$ and $s_{\alpha,\gamma} = s_{\beta,\gamma} \circ s_{\alpha,\beta}$;*
4. *if $\alpha < \beta$ are elements of C, then the diagram*

$$\begin{tikzcd} \mathfrak{A}_\alpha \arrow[r, "a_{\alpha,\beta}"] \arrow[d, "i_\alpha"'] & \mathfrak{A}_\beta \arrow[d, "i_\beta"] \\ \mathfrak{S}_\alpha \arrow[r, "s_{\alpha,\beta}"] & \mathfrak{S}_\beta \end{tikzcd}$$

commutes;

5. *if λ is a limit point of C, then $\mathfrak{A}_\lambda = \bigcup_{\alpha<\lambda} \operatorname{ran} a_{\alpha,\lambda}$, and $\mathfrak{S}_\lambda = \bigcup_{\alpha<\lambda} \operatorname{ran} s_{\alpha,\lambda}$.*

Sacks has proved the following (see [8]).

Theorem 4.26 *Suppose S is a slender type category. Then there is a branch B of rank 1 in $\mathfrak{T}S0$, with associated atomic models $\mathfrak{A}_\alpha$, satisfying the following conditions. If $\alpha < \beta$ are admissible, then there exists a function $a_{\alpha,\beta} : \mathfrak{A}_\alpha \to \mathfrak{A}_\beta$ such that*

1. *$a_{\alpha,\beta}$ is an $\mathfrak{H}^\alpha\mathfrak{L}$-elementary embedding;*
2. *if $\alpha < \beta < \gamma$, then $a_{\alpha,\gamma} = a_{\beta,\gamma} \circ a_{\alpha,\beta}$;*
3. *if λ is a limit of admissibles, then $\mathfrak{A}_\lambda = \bigcup_{\alpha<\lambda} \operatorname{ran} a_{\alpha,\lambda}$.*

The decrease of generality in restricting attention to one nonisolated branch is of little importance, since from a slender type category we can generate one in which $\mathfrak{T}S0$ has only one nonisolated branch. However, the increase of generality resulting in the expansion of the closed unbounded set to the set of all admissibles is very striking.

The following result was stated by Steel in [10] and a proof given by Becker in [2], where again we have the embedding holding on the class of all admissible ordinals.

Theorem 4.27 *Assume Projective Determinacy. Suppose S is a slender type category such that $\mathfrak{T}S0$ has only one nonisolated branch. Let $\mathfrak{S}_\alpha$ be the associated saturated models, for admissible α. Then if $\alpha < \beta$ are admissible, then there exists a function $s_{\alpha,\beta} : \mathfrak{S}_\alpha \to \mathfrak{S}_\beta$ such that*

1. *$s_{\alpha,\beta}$ is an $\mathfrak{H}^\alpha\mathfrak{L}$-elementary embedding;*
2. *if $\alpha < \beta < \gamma$, then $s_{\alpha,\gamma} = s_{\beta,\gamma} \circ s_{\alpha,\beta}$;*
3. *if λ is a limit of admissibles, then $\mathfrak{S}_\lambda = \bigcup_{\alpha<\lambda} \operatorname{ran} s_{\alpha,\lambda}$.*

This gives us a way of defining the functions $\mathfrak{h}_S^{\alpha,\beta}$ when α is admissible, and if we assume Projective Determinacy, then it seems plausible that Theorem 4.25 should be provable for slender type categories with just one nonisolated branch in $\mathfrak{T}S0$, with the closed unbounded set C improved to the set of all countable admissible ordinals.

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