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Upward Stability Transfer for Tame Abstract Elementary Classes

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Abstract Grossberg and VanDieren have started a program to develop a stability theory for tame classes. We name some variants of tameness and prove the following.

Theorem 1 *Let $\mathcal{K}$ be an* AEC *with Löwenheim-Skolem number $\leq \kappa$. Assume that $\mathcal{K}$ satisfies the amalgamation property and is κ-weakly tame and Galois-stable in κ. Then $\mathcal{K}$ is Galois-stable in κ^{+n} for all $n < \omega$.*

With one further hypothesis we get a very strong conclusion in the countable case.

Theorem 2 *Let $\mathcal{K}$ be an* AEC *satisfying the amalgamation property and with Löwenheim-Skolem number $\aleph_0$ that is ω-local and $\aleph_0$-tame. If $\mathcal{K}$ is $\aleph_0$-Galois-stable then $\mathcal{K}$ is Galois-stable in all cardinalities.*

1 Introduction

A tame abstract elementary class is an abstract elementary class (AEC) in which inequality of Galois-types has a local behavior. Tameness is a natural condition, generalizing both homogeneous classes and excellent classes, that has very strong consequences. We examine one of them here.

The work discussed in this paper fits in the program of developing a model theory, in particular a stability theory, for nonelementary classes. Many results to this end were in contexts where manipulations with first-order formulas, or infinitary formulas, were pertinent and consequential. Most often, types in these contexts were identified with satisfiable collections of formulas. The model theory for abstract elementary classes where types are identified roughly with the orbits of an element

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under automorphisms of some large structure moves away from the dependence on ideas from first-order logic.

The main result of this paper is not surprising in light of what is known about first-order model theory, but it does shed light on problems that become more elusive in abstract elementary classes. Grossberg and VanDieren [5] provide a sufficient condition for stability which yields Theorem 1 under GCH. They prove the following. (Their paper assumes μ greater than the Hanf number but this is not needed; see Baldwin [2].)

Fact 1.1 (Corollary 6.4 of [5]) Suppose that $\mathcal{K}$ is a χ-tame AEC for some $\chi \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is Galois-stable in some $\mu \geq \chi$, then $\mathcal{K}$ is stable in every κ with $\kappa^\mu = \kappa$.

The [5] argument generalizes an aspect of Shelah's method for calculating the entire spectrum function. The ZFC-argument here illustrates the relation of tame AEC's to first-order logic. We adapt an argument for Morley's Theorem that ω-stability implies stability in all cardinalities to the context of Galois-types to move Galois-stabililty from a cardinal to its successor. For larger κ some splitting technology is needed and the result is that Galois-stability in κ implies Galois-stability in κ^+ when κ is at least as large as the tameness cardinal.

Combining the result of [5] along with the results of this paper, we gain a better understanding of the stability spectrum for tame AECs.

Corollary 1.2 (Partial stability spectrum) *Suppose that $\mathcal{K}$ is a χ-tame* AEC *for some $\chi \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is Galois-stable in some $\mu \geq \chi$, then $\mathcal{K}$ is stable in every κ with $\kappa^\mu = \kappa$ and in μ^{+n} for all $n < \omega$.*

2 Background

Much of the necessary background for this paper can be found in the exposition Grossberg [4] and the following papers on tame abstract elementary classes, Grossberg and VanDieren [5] and [6]. We will review some of the required definitions and theorems in this section. We will use $\alpha, \beta, \gamma, i, j$ to denote ordinals and $\kappa, \lambda, \mu, \chi$ will be used for cardinals. We will use $(\mathcal{K}, \prec_{\mathcal{K}})$ to denote an abstract elementary class and $\mathcal{K}_\mu$ is the subclass of models in $\mathcal{K}$ of cardinality μ. For an AEC $\mathcal{K}$, $\mathrm{LS}(\mathcal{K})$ represents the Löwenheim-Skolem number of the class. Models are denoted by M, N and may be decorated with superscripts and subscripts. Sequences of elements from M are written as $\bar{a}, \bar{b}, \bar{c}, \bar{d}$. The letters e, f, g, h are reserved for $\mathcal{K}$-mappings and id is the identity mapping.

For the remainder of this paper we will fix $(\mathcal{K}, \prec_{\mathcal{K}})$ to be an abstract elementary class *satisfying the amalgamation property*. It is easy to see that we only make use of the κ-amalgamation property for certain κ and some facts here hold in classes satisfying even weaker amalgamation hypotheses. Since we assume the amalgamation property, we can fix a monster model $\mathfrak{C} \in \mathcal{K}$ and say that the type of a over a model $M \prec_{\mathcal{K}} \mathfrak{C}$ is equal to the type b over M if and only if there is an automorphism of $\mathfrak{C}$ fixing M which takes a to b. (Technically, the existence of a monster model requires the joint embedding property as well as the amalgamation property. However, in the presence of the amalgamation property, joint embedding is an equivalence relation and our fixing of the monster model is the same as restricting to one equivalence class.) In this paper, we work entirely with *Galois-types* (i.e., orbits) and so feel free to write simply *type*. For a model M in $\mathcal{K}$, the set of Galois-types over M is written as ga-S(M). An AEC $\mathcal{K}$ satisfying the amalgamation property is *Galois-stable* in κ provided that for every $M \in \mathcal{K}_\kappa$ the number of types over M is $\leq \kappa$.

Let us recall a few results that follow from Galois-stability in κ.

Definition 2.1 Let $M \in \mathcal{K}_\kappa$. We say that N *is universal over* M provided that for every $M' \in \mathcal{K}_\kappa$ with $M \prec_{\mathcal{K}} M'$, there exists a $\mathcal{K}$-mapping $f : M' \to N$ such that $f \restriction M = \mathrm{id}_M$.

Note that in contrast to most model theoretic literature, in AECs a tradition has grown up of defining 'universal over M' as 'universal over submodels of the same size as M'.

Fact 2.2 (Shelah [7], see [2] or [5] for a proof) If $\mathcal{K}$ is Galois-stable in κ and satisfies the $\leq \kappa$-amalgamation property, then for every $M \in \mathcal{K}_\kappa$ there is some (not necessarily unique) extension N of M of cardinality κ such that N is universal over M.

If $\mathcal{K}$ is Galois-stable in κ, we can construct an increasing and continuous chain of models $\langle M_i \in \mathcal{K}_\kappa \mid i < \sigma \rangle$ for any limit ordinal $\sigma \leq \kappa^+$ such that M_{i+1} is universal over M_i. The limit of such a chain is referred to as a (κ, σ)*-limit model*.

Corollary 2.3 *Suppose $\mathcal{K}$ is κ-Galois-stable and $\mathcal{K}_\kappa$ has the amalgamation property with* $\mathrm{LS}(\mathcal{K}) \leq \kappa$. *Then for any model $M \in \mathcal{K}$ with cardinality κ^+ we can find a κ^+-saturated and (κ, κ^+)-limit model M' such that M can be embedded in M'.*

Proof Write M as an increasing continuous chain M_i of models of cardinality at most κ. We define an increasing chain of models M'_i, each with cardinality κ, and f_i so that f_i is a $\mathcal{K}$-embedding of M_i in M'_i and such that each M'_{i+1} realizes all types over M_i; indeed, M'_{i+1} is universal over M'_i. For this, first choose M^1_i which is universal over M'_i by Fact 2.2. Then amalgamate M_{i+1} and M^1_i over $f_i : M_i \mapsto M'_i$ with $M'_i \prec_{\mathcal{K}} M'_{i+1}$. Now the union of the M'_i is a (κ, κ^+)-limit model which imbeds M. □

Now we turn our attention to two definitions which capture instances in which types are determined by a small set. These two approaches to local character play different roles in this paper.

Definition 2.4 Let $\mathcal{K}$ be an AEC.

1. We say that a class $\mathcal{K}$ is χ*-tame* provided that for every model M in $\mathcal{K}$ with $|M| \geq \chi$ and every p and q, types over M, if $p \neq q$, then there is a model of cardinality χ which distinguishes them. In other words if $p \neq q$, then there exists $N \in \mathcal{K}_\chi$ with $N \prec_{\mathcal{K}} M$ such that $p \restriction N \neq q \restriction N$.
2. A class $\mathcal{K}$ is ω*-local* provided for every increasing chain of types $\{p_i \mid i < \omega\}$ there is a unique p such that $p = \bigcup_{i<\omega} p_i$.

For some of the results in this paper we could replace χ-tameness with the two-parameter version of Baldwin [3], (κ, χ)-tameness, which requires only that distinct types over models of cardinality κ be distinguished by models of cardinality χ. Since we don't actually carry out any inductions to establish tameness, this nicety is not needed here. Note that if $\chi < \kappa$, χ-tame implies κ-tame.

Remark 2.5 If $\mathcal{K}$ is an AEC with the amalgamation property, for every increasing ω-chain of types p_i, there is a type over the union of the domains extending each of the p_i (1.10 of Shelah [8], proved as 3.14 in [3]); however, this extension need not be unique.

Remark 2.6 Clearly, if an AEC is defined by a logic with finitary syntax, has Löwenheim-Skolem number $\aleph_0$, and "Galois-types = syntactic types" then the AEC is both $\aleph_0$-tame and ω-local. Shelah showed, assuming weak GCH, this happens for $L_{\omega_1,\omega}$ classes that are categorical in $\aleph_n$ for every $n < \omega$; it also holds for Zilber's quasi-minimal excellent classes.

A weaker version of tameness requires that only those types over saturated models are determined by small sets. This appears as χ-character in [8] where Shelah proves that, in certain situations, categorical AECs have small character.

Definition 2.7 For an AEC $\mathcal{K}$ and a cardinal χ, we say that $\mathcal{K}$ is χ*-weakly tame* or has χ*-character* if and only if for every saturated model M with $|M| \geq \chi$ and every $p \neq q \in \text{ga-S}(M)$, there exists $N \in \mathcal{K}_\chi$ such that $N \prec_{\mathcal{K}} M$ and $p \upharpoonright N \neq q \upharpoonright N$.

3 $\aleph_0$-tameness

In this section we assume $\mathcal{K}$ has a countable language, has Löwenheim-Skolem number ω, and is $\aleph_0$-tame.

Theorem 3.1 *Suppose* $\text{LS}(\mathcal{K}) = \aleph_0$. *If* $\mathcal{K}$ *is* $\aleph_0$*-tame and* μ*-Galois-stable for all* $\mu < \kappa$ *and* $\text{cf}(\kappa) > \aleph_0$ *then* $\mathcal{K}$ *is* κ*-Galois-stable.*

Proof For purposes of contradiction suppose there are more than κ types over some model M^* in $\mathcal{K}$ of cardinality κ. We may write M^* as the union of a continuous chain $\langle M_i \mid i < \kappa \rangle$ under $\prec_{\mathcal{K}}$ of models in $\mathcal{K}$ which have cardinality $< \kappa$. We say that a type over M_i *has many extensions* to mean that it has $\geq \kappa^+$ distinct extensions to a type over M^*.

Claim 3.2 *For every* i*, there is some type over* M_i *with many extensions.*

Proof of Claim 3.2 Each type over M^* is the extension of some type over M_i and, by our assumption, there are less than κ many types over M_i, so at least one of them must have many extensions. $\square$

Claim 3.3 *For every* i*, if the type* p *over* M_i *has many extensions, then for every* $j > i$*,* p *has an extension to a type* p' *over* M_j *with many extensions.*

Proof of Claim 3.3 Every extension of p to a type over M^* is the extension of some extension of p to a type over M_j. By our assumption there are less than κ many such extensions to a type over M_j, so at least one of them must have many extensions. $\square$

Claim 3.4 *For every* i*, if the type* p *over* M_i *has many extensions, then for all sufficiently large* $j > i$*,* p *can be extended to two types over* M_j *each having many extensions.*

Proof of Claim 3.4 By Claim 3.3 it suffices to establish the result for some $j > i$. So assume that there is no $j > i$ such that p has two extensions to types over M_j each having many extensions. Then, by Claim 3.3 again, for every $j > i$, p has a unique extension to a type p_j over M_j with many extensions. Let S^* be the set of all extensions of p to a type over M^*—so $|S^*| \geq \kappa^+$. Then S^* is the union of S_0 and S_1, where S_0 is the set of all q in S^* such that $p_j \subseteq q$ for all $j > i$, and S_1 is the set of all q in S^* such that q does not extend p_j for some $j > i$. Now if q_1 and q_2 are different types in S^* then, since $\mathcal{K}$ is $\aleph_0$-tame and $\mathrm{cf}(\kappa) > \aleph_0$, their restrictions to some $M_i \prec_{\mathcal{K}} M^*$ with $i < \kappa$ must differ. Hence their restrictions to all sufficiently large M_j must differ. Therefore, S_0 contains at most one type. On the other hand, if q is in S_1 then, for some $j > i$, $q \restriction M_j$ is an extension of p to a type over M_j which is different from p_j, hence has at most κ extensions to a type over M^*. Since there are $< \kappa$ types over each M_j (by our stability assumption) and just κ models M_j there can be at most κ types in S_1. Thus S^* contains at most κ types, a contradiction. $\square$

Claim 3.5 *There is a countable $M \prec_{\mathcal{K}} M^*$ such that there are $2^{\aleph_0}$ types over M.*

Proof of Claim 3.5 Let p be a type over M_0 with many extensions. By Claim 3.4 there is a $j_1 > 0$ such that p has two extensions p_0, p_1 to types over M_{j_1} with many extensions. Iterating this construction we find a sequence of models M_{j_n} and a tree p_s of types for $s \in 2^{<\omega}$ with the 2^n types p_s (where s has length n) all over M_{j_n} and each p_s has many extensions. Invoking $\aleph_0$-tameness, we can replace each M_{j_n} by a countable M'_{j_n} and p_s by p'_s over M'_{j_n} while preserving the tree structure on the p'_s. Let $\hat{M}$ be the union of the M'_{j_n}. Now for each $\sigma \in 2^\omega$, $p_\sigma = \bigcup_{s \subset \sigma} p_s$ is a Galois-type, by Remark 2.5 $\square$

Since Claim 3.5 contradicts the hypothesis of ω-Galois-stability, this establishes Theorem 3.1. $\square$

Now we obtain Theorem 2 from the abstract.

Corollary 3.6 *Suppose* $\mathrm{LS}(\mathcal{K}) = \aleph_0$ *and $\mathcal{K}$ has the amalgamation property. If $\mathcal{K}$ is $\aleph_0$-weakly-tame and ω-Galois-stable then*

1. *$\mathcal{K}$ is Galois-stable in all $\aleph_n$ for $n < \omega$;*
2. *if in addition $\mathcal{K}$ is both ω-local and $\aleph_0$-tame, $\mathcal{K}$ is Galois-stable in all cardinalities.*

Proof of Corollary 3.6 In the proof of Theorem 3.1, if κ is a successor cardinal, then by Corollary 2.3, M^* can be embedded into a saturated model and the proof can be carried through with the weaker assumption of $\aleph_0$-weak-tameness. Thus the first claim follows by induction.

To carry out the induction for all cardinals, we extend the argument in Theorem 3.1 to limit cardinals of cofinality ω. At the stage where we called upon $\aleph_0$-tameness in Claim 3.4, we now use the hypothesis of ω-locality. For limit cardinals of uncountable cofinality, we use the assumption of $\aleph_0$-tameness since we have no guarantee that M^* can be taken to be saturated. $\square$

4 κ-tame: Uncountable κ

Note that the proof of Theorem 3.1 cannot be immediately generalized to deducing stability in κ^+ from stability in κ when the class is tame but not $\aleph_0$-tame. The fact that the countable increasing union of Galois types is a Galois type is very much particular to 'countable' and in general does not hold when we replace countable by uncountable. We solve this with a use of μ-splitting.

Definition 4.1 (Shelah [8]) A type $p \in \text{ga-S}(N)$ *μ-splits* over $M \prec_{\mathcal{K}} N$ if and only if there exist $N_1, N_2 \in \mathcal{K}_{\leq\mu}$ and h, a $\mathcal{K}$-embedding such that $M \prec_{\mathcal{K}} N_l \prec_{\mathcal{K}} N$ for $l = 1, 2$ and $h : N_1 \to N_2$ such that $h \restriction M = \text{id}_M$ and $p \restriction N_2 \neq h(p \restriction N_1)$.

This dependence relation behaves nicely in Galois-stable AECs. The existence of unique nonsplitting extensions from M to M' where M and M' have the same cardinality and M' is universal over M holds for any AEC with amalgamation. There is a full proof as 1.4.13 and 1.4.14 of VanDieren [9]. Existence of nonsplitting extensions to larger cardinalities is more difficult; under the assumption of categoricity, such an extension property is asserted in [8] and a special case is given a short proof in Baldwin [1]. In the more general situation, uniqueness requires tameness; see 6.2 of [8]. Here we state the uniqueness and existence statements upon which we will be calling explicitly.

Lemma 4.2 (Uniqueness [8] and [9]) *Let $N, M, M' \in \mathcal{K}_\mu$ be such that M' is universal over M and M is universal over N. If $p \in \text{ga-S}(M)$ does not μ-split over N, then there is a unique $p' \in \text{ga-S}(M')$ such that p' extends p and p' does not μ-split over N.*

Lemma 4.3 (Existence Fact 3.3 of [8]; see also [5]) *Let $M \in \mathcal{K}_{\geq\kappa}$ be given. Suppose that $\mathcal{K}$ satisfies the $(\leq \|M\|)$-amalgamation property. If $\mathcal{K}$ is Galois-stable in κ, then for every $p \in \text{ga-S}(M)$, there exists $N \in \mathcal{K}_\kappa$ such that $N \preceq_{\mathcal{K}} M$ and p does not κ-split over N.*

Remark 4.4 The arguments in Claim 3.5 and Lemma 4.3 differ. In Claim 3.5, we construct a tree of height ω of Galois types and must find a limit for each branch. In Lemma 4.3, a tree of height κ is constructed by spreading out copies of a given type.

We are able to carry out the following argument under the hypothesis of weakly tame rather than tame so we record the stronger result.

Theorem 4.5 *Let $\mathcal{K}$ be an abstract elementary class with the amalgamation property that has Löwenheim-Skolem number $\leq \kappa$ and is κ-weakly-tame. Then if $\mathcal{K}$ is Galois-stable in κ it is also Galois-stable in κ^+.*

Proof We proceed by contradiction. So we make the following assumption: M^* is a model of cardinality κ^+ with more than κ^+ types over it. By Corollary 2.3, we can extend M^* to a (κ, κ^+)-limit model which is saturated. Since it has at least as many types as the original we just assume that M^* is a saturated, (κ, κ^+)-limit model witnessed by $\langle M_i \mid i < \kappa^+\rangle$.

Let $\{p_\alpha \mid \alpha < \kappa^{++}\}$ be a set of distinct types over M^*. By stability in κ, for every p_α there exists $i_\alpha < \kappa^+$ such that p_α does not κ-split over M_{i_α} (by Lemma 4.3). (Note, we don't need a (κ, κ^+)-limit here but we do below.) By the pigeonhole principle there exists $i^* < \kappa^+$ and $A \subseteq \kappa^{++}$ of cardinality κ^{++} such that for every $\alpha \in A$, $i_\alpha = i^*$.

Now apply the argument of the claims from Section 3 to the p_α for $\alpha \in A$ to conclude there exist $p, q \in S(M^*)$ and $i < i' \in A$ such that neither p nor q κ-splits over M_i or $M_{i'}$ but $p \restriction M_{i'} = q \restriction M_{i'}$. By weak tameness, there exists an ordinal $j > i'$ such that $p \restriction M_j \neq q \restriction M_j$. Notice that neither $p \restriction M_j$ nor $q \restriction M_j$ κ-split over M_i. This contradicts Lemma 4.2 by giving us two distinct extensions of a nonsplitting type to the model M_j which by construction is universal over $M_{i'}$. $\square$

Using Theorem 4.5 with an inductive argument on $n < \omega$, together with the argument for Corollary 3.6 (1), we obtain Theorem 1 from the abstract.

Theorem 4.6 *Let $\mathcal{K}$ be an abstract elementary class that has Löwenheim-Skolem number $\leq \kappa$ and satisfies the amalgamation property and is κ-weakly tame. Then if $\mathcal{K}$ is Galois-stable in κ it is also Galois-stable in κ^{+n} for any $n < \omega$.*

One motivation for working out these arguments was to explore whether or not Galois-superstability (in the sense of few types over models in every large enough cardinality) could be derived from categoricity in the abstract elementary class setting. Following tradition, we write $\mathrm{Hanf}(\mathcal{K})$ for the Hanf number for omitting types in first-order languages with the same size vocabulary as $\mathcal{K}$. Using Ehrenfeucht-Mostowski models as in the first-order case, for an AEC with amalgamation, categoricity in a λ greater than $\mathrm{Hanf}(\mathcal{K})$ implies Galois-stability below λ. In the first-order case, analysis of the stability spectrum function allows one to conclude stability in λ. Although we don't have such a full analysis of the spectrum function, we can immediately conclude the following from Theorem 4.5.

Corollary 4.7 *Suppose λ is a successor cardinal greater than* $\mathrm{Hanf}(\mathcal{K})$. *Let $\mathcal{K}$ be an abstract elementary class with the amalgamation property that has Löwenheim-Skolem number $< \lambda$ and is λ-weakly tame. If $\mathcal{K}$ is λ-categorical, then it is Galois-stable in λ.*

This result is also a consequence of Theorem 4.1 in [6] in which the hypotheses of Corollary 4.7 allow one to construct, for every $M \in \mathcal{K}_\lambda$, a model M' also of cardinality λ so that M' realizes every type over M.

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