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LATTICE–THEORETICAL AND MEREOLOGICAL FORMS OF HAUBER'S LAW

BOLESŁAW SOBOCIŃSKI

The shortest form of an important logical law of Hauber[1]:

(i) $[\alpha\beta\gamma\delta] .\therefore \alpha \cap \beta = \Lambda . \gamma \cap \delta = \Lambda . \alpha \cup \beta = \gamma \cup \delta . \supset : \alpha \subset \gamma . \beta \subset \delta . \equiv . \gamma \subset \alpha . \delta \subset \beta$[2]

is provable easily in any system of logic which contains the so-called algebra of classes (or sets), as a subsystem. It is obvious that the most general form of Hauber's theorem, viz.:

(ii) $[\alpha_1\alpha_2 \ldots \alpha_n\beta_1\beta_2 \ldots \beta_n] .\therefore \alpha_1 \cap \alpha_2 = \Lambda . \alpha_1 \cap \alpha_3 = \Lambda \ldots \alpha_1 \cap \alpha_n = \Lambda . \alpha_2 \cap \alpha_3 = \Lambda \ldots \alpha_2 \cap \alpha_n = \Lambda \ldots \alpha_{n-1} \cap \alpha_n = \Lambda . \beta_1 \cap \beta_2 = \Lambda . \beta_1 \cap \beta_3 = \Lambda \ldots \beta_1 \cap \beta_n = \Lambda . \beta_2 \cap \beta_3 = \Lambda \ldots \beta_2 \cap \beta_n = \Lambda \ldots \beta_{n-1} \cap \beta_n = \Lambda . \alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_n = \beta_1 \cup \beta_2 \cup \ldots \cup \beta_n . \supset : \alpha_1 \subset \beta_1 . \alpha_2 \subset \beta_2 \ldots \alpha_n \subset \beta_n . \equiv . \beta_1 \subset \alpha_1 . \beta_2 \subset \alpha_2 \ldots \beta_n \subset \alpha_n$; *where n is a natural number*: $1 < n < \infty$.[3]

can be proved at once by an application of the same mode of proof which was used in order to obtain (i). It is well known that the classical propositional calculus contains a theorem, viz.:

$\mathcal{P}$ $[pqrs] .\therefore p \vee q . \equiv . r \vee s : p \supset \sim q . r \supset \sim s : \supset : p \supset r . q \supset s . \equiv . r \supset p . s \supset q$

which is analogous to (i), and that a propositional thesis corresponding to (ii) also is provable without any difficulty.

In this note it will be shown that: 1) the Boolean algebraic formulas corresponding to (i) and (ii) are provable not only in Boolean algebra which has been proved by Alves, *cf.* [1], but also in a weaker system, namely in the field of distributive lattice with Boolean zero element, and that: 2) the

1. Sometimes and especially when Hauber's law is given in its propositional form it also is called the law of closed systems, cf. [4], pp. 176-177. Hauber's original formulation of his law can be found in Hoormann's paper [2].

2. A notation used in this paper is the well-known symbolism of Peano-Russell which for the formulas belonging to lattice theory or mereology is adjusted to the requirements of these two systems respectively.

3. Cf. [3], p. 288, formula μ.

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mereological formulas analogous to (i) and (ii) are the theorems of Leśniewski's mereology. An elementary acquaintance with lattice theory and with mereology is presupposed. In the proof lines the bold letters **L**, **DL** and **M** will indicate that a proof is obtained by an application of the theorems belonging to lattice theory, distributive lattice or mereology respectively.

1 Assume that a system

$$\mathfrak{A} = \langle A, \cap, \cup, 0 \rangle$$

is a distributive lattice with a constant element $0 \epsilon A$ and with an additional axiom:

L1 $[a]: a \epsilon A . \supset . 0 \leqslant a$

Then:

L2 $[a]: a \epsilon A . \supset . a \cap 0 = 0$ [*L1*; **L**]
L3 $[a]: a \epsilon A . \supset . a \cup 0 = a$ [*L1*; **L**]
L4 $[abcd]: a, b, c, d \epsilon A . c \cap d = 0 . a \cup b = c \cup d . a \leqslant c . b \leqslant d . \supset . c \leqslant a . d \leqslant b$
PR $[abcd]:$ Hp (5) $. \supset .$
6. $a \cap c = a .$ [1; 4; **L**]
7. $b \cap d = b .$ [1; 5; **L**]
8. $c = c \cap (c \cup d) = c \cap (a \cup b)$ [1; **L**; 3]
$= (c \cap a) \cup (c \cap b) = a \cup [c \cap (b \cap d)]$ [1; **DL**; 6; **L**; 7]
$= a \cup [b \cap (c \cap d)] = a \cup (b \cap 0) = a \cup 0 = a .$ [1; **L**; 2; *L2*; *L3*]
9. $d = d \cap (c \cup d) = d \cap (a \cup b)$ [1; **L**; 3]
$= (d \cap a) \cup (d \cap b) = [d \cap (a \cap c)] \cup (b \cap d) = [a \cap (c \cap d)] \cup b$ [1; **DL**; 6; **L**; **L**; 7]
$= (a \cap 0) \cup b = 0 \cup b = b .$ [1; 2; *L2*; *L3*; **L**]
$c \leqslant a . d \leqslant b$ [8; 9; **L**]
$\mathcal{L}$ $[abcd]: a, b, c, d \epsilon A . a \cap b = 0 . c \cap d = 0 . a \cup b = c \cup d . \supset :$
$a \leqslant c . b \leqslant d . \equiv . c \leqslant b . d \leqslant b$ [*L4*; **L**]

Thus, in the field of $\mathfrak{A}$ formula $\mathcal{L}$ which, obviously, is analogous to (i) is provable. And, it is self-evident that in $\mathfrak{A}$ a formula corresponding to (ii) for: $1 < n < \infty$ can be proved by an application of the same mode of reasoning which is used above.

2 In order to construct an analogue of Hauber's theorem in the field of mereology we should substitute some logical formulas occuring in (i) by the suitable mereological ones. To this end instead of "$\alpha \subset \beta$", "$x \epsilon \alpha \cup \beta$" and "$\alpha \cap \beta = \Lambda$" we shall use "$A \varepsilon$ **el**(B)", "$C \varepsilon$ **Kl**$(A \cup B)$" and "$A \varepsilon$ **ex**(B)" respectively. The latter three formulas have the following meanings in mereology:

a) $A \varepsilon$ **el**(B): an object A is an (mereological) element of an object B.
b) $C \varepsilon$ **Kl**$(A \cup B)$: an object C is a (mereological) class generated by logical union of A and B.
c) $A \varepsilon$ **ex**(B): an object A is outside of an object B, i.e. no element of A is an element of B.

The symbols ε and $\cup$ occuring in the above formulas mean "is" (in the sense of Leśniewski's ontology) and customary "logical addition" respectively. In order to explain some points of the proof given below we have to notice that in ontology on which mereology is based an identity between two objects is defined as follows

Df1 $[AB]: A\varepsilon B . B\varepsilon A . \equiv . A = B$

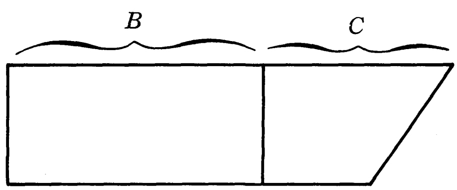
and that ontology contains a theorem, called the characteristic thesis of ontology, viz.:

T1 $[ABa]: A\varepsilon B . B\varepsilon a . \supset . B\varepsilon A$

Now, let us assume an arbitrary but adequate axiom-system of mereology with "**el**" as its single primitive mereological functor. Then, in this theory (in short **M**) the following theorems and definitions which will be used below hold:

M1 $[AB]: A\varepsilon \mathbf{el}(B) . \supset . B\varepsilon B$ [**M**]
M2 $[ABC]: A\varepsilon \mathbf{el}(B) . B\varepsilon \mathbf{el}(C) . \supset . A\varepsilon \mathbf{el}(C)$ [**M**]
M3 $[AB] \therefore A\varepsilon A : [C] : C\varepsilon \mathbf{el}(A) . \supset . [\exists F] . F\varepsilon \mathbf{el}(C) . F\varepsilon \mathbf{el}(B) : \supset . A\varepsilon \mathbf{el}(B)$ [**M**]
D1 $[Aa] \therefore A\varepsilon A : [B] : B\varepsilon a . \supset . B\varepsilon \mathbf{el}(A) : [C] : C\varepsilon \mathbf{el}(A) . \supset .$
$[\exists EF] . E\varepsilon a . F\varepsilon \mathbf{el}(E) . F\varepsilon \mathbf{el}(C) : \equiv . A\varepsilon \mathbf{Kl}(a)$ [**M**]
M4 $[Aa]: A\varepsilon \mathbf{Kl}(a) . \supset . A = \mathbf{Kl}(a)$ [**M**]
M5 $[Aa]: A\varepsilon a . \supset . A\varepsilon \mathbf{el}(\mathbf{Kl}(a))$ [*D1*; **M**]
D2 $[AB] \therefore A\varepsilon A . B\varepsilon B : [C] : C\varepsilon \mathbf{el}(B) . \supset . \sim (C\varepsilon \mathbf{el}(B)) : \equiv . A\varepsilon \mathbf{ex}(B)$ [**M**]
M6 $[ABC]: A\varepsilon \mathbf{el}(B) . B\varepsilon \mathbf{ex}(C) . \supset . \sim (A\varepsilon \mathbf{el}(C))$ [*D2*; **M**]
D3 $[ABC]: A\varepsilon \mathbf{Kl}(B\cup C) . B\varepsilon \mathbf{ex}(C) . \equiv . A\varepsilon B + C$ [**M**]

D3 defines a notion of mereological addition: an object A is the sum of two objects B and C iff A is a class of $B\cup C$ and B is outside of C. This notion can be explained by the following diagram in which we assume that the figures B and C have no common elements:

$A =$ 

$= B + C$

M7 $[ABCD]: A + B = C + D . \equiv . \mathbf{Kl}(A\cup B) = \mathbf{Kl}(C\cup D) . A\varepsilon \mathbf{ex}(B) . C\varepsilon \mathbf{ex}(D)$
[*D3*; *D2*, **M**]
M8 $[ABC]: A\varepsilon B + C . \equiv . A\varepsilon C + B$ [**M**]
M9 $[ABCD]: A\varepsilon B + (C + D) . \equiv . A\varepsilon (B + C) + D$ [**M**]
M10 $[ABCD]: A\varepsilon B + (C + D) . \supset . B\varepsilon \mathbf{ex}(C) . B\varepsilon \mathbf{ex}(D) . C\varepsilon \mathbf{ex}(D)$ [**M**]

Then:

M11 $[A\,B\,C\,D\,V]:\mathbf{Kl}(A\cup B)=C+D\,.\,A\,\varepsilon\,\mathbf{el}(C)\,.\,B\,\varepsilon\,\mathbf{el}(D)\,.\,V\,\varepsilon\,\mathbf{el}(C)\,.\supset.$
$[\exists F]\,.\,F\,\varepsilon\,\mathbf{el}(V)\,.\,F\,\varepsilon\,\mathbf{el}(A)$

PR $[A\,B\,C\,D\,V]::\mathrm{Hp}(4)\,.\supset.\cdot.$

5. $\mathbf{Kl}(A\cup B)\,\varepsilon\,C+D.$ [*Df1*; 1]
6. $\mathbf{Kl}(A\cup B)\,\varepsilon\,\mathbf{Kl}(C\cup D).$ [*D3*; 5]
7. $\mathbf{Kl}(A\cup B)=\mathbf{Kl}(C\cup D).$ [*M4*; 6]
8. $C\,\varepsilon\,\mathbf{ex}(D).$ [*D3*; 5]
9. $C\,\varepsilon\,C\cup D.$ [*D2*; 8]
10. $C\,\varepsilon\,\mathbf{el}(\mathbf{Kl}(C\cup D)).$ [*M5*; 9]
11. $C\,\varepsilon\,\mathbf{el}(\mathbf{Kl}(A\cup B)).$ [11; 7]
12. $V\,\varepsilon\,\mathbf{el}(\mathbf{Kl}(A\cup B))\,.\cdot.$ [*M2*; 4; 11]

$[\exists E\,F]\,.\cdot.$

13. $E\,\varepsilon\,A\cup B.$
14. $F\,\varepsilon\,\mathbf{el}(E).$ [*D1*; 6; 12]
15. $F\,\varepsilon\,\mathbf{el}(V).$
16. $F\,\varepsilon\,\mathbf{el}(C).$ [*M2*; 15; 4]
17. $\sim(F\,\varepsilon\,\mathbf{el}(D)):$ [*M6*; 16; 8]
18. $E=A\,.\mathrm{v}.\,E=B:$ [13; *T1*; *Df1*; 2; 3]
19. $F\,\varepsilon\,\mathbf{el}(A)\,.\mathrm{v}.\,F\,\varepsilon\,\mathbf{el}(B):$ [18; 14]
20. $F\,\varepsilon\,\mathbf{el}(A)\,.\mathrm{v}.\,F\,\varepsilon\,\mathbf{el}(D):$ [19; *M2*; 3]
21. $F\,\varepsilon\,\mathbf{el}(A)\,.\cdot.$ [20; 17]

$[\exists F]\,.\,F\varepsilon\,\mathbf{el}(V)\,.\,F\,\varepsilon\,\mathbf{el}(A)$ [15; 21]

M12 $[A\,B\,C\,D]:\mathbf{Kl}(A\cup B)=C+D\,.\,A\varepsilon\,\mathbf{el}(C)\,.\,B\,\varepsilon\,\mathbf{el}(D)\,.\supset.\,C\varepsilon\,\mathbf{el}(A)$

PR $[A\,B\,C\,D]\,.\cdot.\,\mathrm{Hp}(3)\,.\supset:$

4. $C\,\varepsilon\,C:$ [*M1*; 2]
5. $[V]:\,V\varepsilon\,\mathbf{el}(C)\,.\supset.\,[\exists F]\,.\,F\varepsilon\,\mathbf{el}(V)\,.\,F\varepsilon\,\mathbf{el}(A):$ [*M11*; 1, 2, 3]

$C\,\varepsilon\,\mathbf{el}(A)$ [*M3*; 4; 5]

M13 $[A\,B\,C\,D]:\mathbf{Kl}(A\cup B)=C+D\,.\,A\varepsilon\,\mathbf{el}(C)\,.\,B\varepsilon\,\mathbf{el}(D)\,.\supset.\,D\varepsilon\,\mathbf{el}(B)$ [*M12*; *M8*]

$\mathcal{M}$ $[A\,B\,C\,D]\,.\cdot.\,A+B=C+D\,.\supset:\,A\varepsilon\,\mathbf{el}(C)\,.\,B\varepsilon\,\mathbf{el}(D)\,.\equiv.\,C\varepsilon\,\mathbf{el}(A)\,.\,D\varepsilon\,\mathbf{el}(B)$ [*M12*; *M13*; *M7*]

Clearly, $\mathcal{M}$ is a mereological analogue of (i). Since the theorems *M8*, *M9* and *M10* hold in the field of mereology, it also is clear that a mereological formula corresponding to (ii) is provable in mereology for n: $i<n<\infty$ by an application of the same mode of reasoning which is presented above.

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University of Notre Dame
Notre Dame, Indiana