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# S5 WITH THE **CBF**

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In [1] semantics are provided for quantificational modal logic through S4 which validate the **CBF** (converse Barcan formula) '$L(\forall X)A \supset (\forall X)LA$' while allowing a counter-model to the **BF** (Barcan formula) '$(\forall X)LA \supset L(\forall X)A$'. With a slight change, reminiscent of the semantics of Kripke, *cf.* [2], the Hughes and Cresswell models will serve in quantificational S5 to both validate the **CBF** and exclude the **BF**.

With our version of QS5 understood to contain two runs of variables—the first called *individual variables* and occurring bound only, and the second called *individual constants*, which take the place of free variables—together with the usual signs '$\forall$', '$\sim$', '$\supset$', '(', ')', and ',', and the *modal operator* '$L$', take an *atomic wff* to be any formula of the sort $F^m(C_1, C_2, \ldots, C_m)$, where $F^m$ is an $m$-place predicate letter and $C_1, C_2, \ldots, C_m$ are individual constants. Then, understand by a *truth-value assignment* any function from the atomic wffs to $\{\mathbf{T}, \mathbf{F}\}$. Next, let '$\Phi$' be a set of truth-value assignments such that each element $\varphi_i$ of $\Phi$ is associated with a (possibly empty) set of individual constants $E_i$. Finally, call any pair of the sort $\langle \Phi, \varphi_i \rangle$ a *truth-pair*, and we have our key semantic concept.

The definitions of truth, falsity, and so forth on a truth-pair are as follows:

A. A wff $A$ shall be said to be *unvalued* on a truth-pair $\langle \Phi, \varphi_i \rangle$ if:

(1) $A$ is of the sort $F^m(C_1, C_2, \ldots, C_m)$ and one or more of $C_1$, $C_2$, etc. are not members of $E_i$ (the associated set of individual constants for $\varphi_i$),
(2) $A$ is of the sort $\sim B$ and $B$ is unvalued (on $\langle \Phi, \varphi_i \rangle$),
(3) $A$ is of the sort $B \supset C$ and either $B$ or $C$ is unvalued,
(4) $A$ is of the sort $(\forall X)B$ and $B(C/X)$ is unvalued for at least one $C$ in $E_i$, and[1]

(5) $A$ is of the sort $LB$ and $B$ is unvalued on some $\langle \Phi, \varphi_j \rangle$.

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1. We do not permit overlapping identical quantification or unbound occurrences of individual variables in our wffs. Thus if $(\forall X)A$ is a wff, $A(C/X)$ is sure to be one too, and not to contain any further instances of '$X$'. For full details see [3].

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B. A wff $A$ that is not unvalued on $\langle\Phi, \varphi_i\rangle$ is *true* if:

(1) $A$ is of the sort $F^m(C_1, C_2, \ldots, C_m)$ and is assigned **T** by $\varphi_i$,
(2) $A$ is of the sort $\sim B$ and $B$ is not true (on $\langle\Phi, \varphi_i\rangle$),
(3) $A$ is of the sort $B \supset C$ and either $B$ is not true or $C$ is,
(4) $A$ is of the sort $(\forall X)B$ and $B(C/X)$ is true for every $C$ in $E_i$,
(5) $A$ is of the sort $LB$ and $B$ is true on every $\langle\Phi, \varphi_j\rangle$.

C. A wff $A$ is *false* on a truth-pair $\langle\Phi, \varphi_i\rangle$ otherwise.[2]

As the reader may wish to note, these truth conditions are similar to those of Bochvar (see [4] for details) for three-valued classical logic. To insure the validity of all classical tautologies (but not necessarily all of their modal substitution instances) we shall call a wff $A$ *valid* in QS5 if $A$ *is not false* on any truth-pair.

Given this semantics, the same counter-examples brought forward by Kripke in [2] and Hughes and Cresswell in [1] will falsify the **BF** here. For example, suppose $LA(C/X)$ is true on $\langle\Phi, \varphi_i\rangle$ for every $C$ in $E_i$, but that $A(C'/X)$ is false on some $\langle\Phi, \varphi_j\rangle$ where $C'$ is a member of $E_j$ but not of $E_i$. Then $(\forall X)LA$ will be true on $\langle\Phi, \varphi_i\rangle$ whereas $L(\forall X)A$ will be false, thus falsifying $(\forall X)LA \supset L(\forall X)A$.

As for the validity of the **CBF**, suppose $L(\forall X)A$ is true on $\langle\Phi, \varphi_i\rangle$. Then for every $\varphi_j$ in $\Phi$, $(\forall X)A$ is true on $\langle\Phi, \varphi_j\rangle$. Hence, $A(C/X)$ is true on each $\langle\Phi, \varphi_j\rangle$. In the case where every $E_j$ is identical to $E_i$ or a superset of $E_i$, clearly $LA(C/X)$ is true on $\langle\Phi, \varphi_i\rangle$ for every $C$ in $E_i$, and, thus, so is $(\forall X)LA$. Two cases therefore remain:

(1) Suppose one or more $E_j$ is such that $E_i \cap E_j = \emptyset$. Then for each $C$ in $E_i$, $A(C/X)$ would be unvalued on $\langle\Phi, \varphi_j\rangle$. Hence $LA(C/X)$ would be unvalued on $\langle\Phi, \varphi_j\rangle$ as would $(\forall X)LA$.

(2) Suppose one or more $E_j$ is such that $E_j \subset E_i$. Then for each $C$ in $E_i$, $A(C/X)$ would either be true on $\langle\Phi, \varphi_j\rangle$ (if $C$ is a member of $E_j$) or unvalued on $\langle\Phi, \varphi_j\rangle$ (if $C$ is not a member of $E_j$). Hence, $LA(C/X)$ is unvalued on $\langle\Phi, \varphi_i\rangle$ for at least one member $C$ of $E_i$, thus $(\forall X)LA$ is unvalued on $\langle\Phi, \varphi_i\rangle$.

Thus, if $L(\forall X)A$ is true on a truth-pair, then $(\forall X)LA$ is either true or unvalued on the pair. Hence $L(\forall X)A \supset (\forall X)LA$ (= **CBF**) is valid in QS5.

## REFERENCES

[1] Hughes, G. E., and M. J. Cresswell, *An Introduction to Modal Logic,* Methuen and Co., London (1968).

[2] Kripke, S. A., "Semantical considerations on modal logic," *Acta Philosophica Fennica,* vol. 16 (1963), pp. 83-94.

2. These truth conditions are adapted from [1], p. 172.

[3] Leblanc, H., *Truth-Value Semantics*, North-Holland, Amsterdam, forthcoming.

[4] Rescher, N., *Introduction to Many-Valued Logic*, McGraw-Hill Co., New York (1969).

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