Notre Dame Journal of Formal Logic
Volume XVI, Number 4, October 1975
NDJFAM

THE DEVELOPMENT OF LEWIS' THEORY OF STRICT IMPLICATION

E. M. CURLEY

In an autobiographical article published in 1930, C. I. Lewis described his first contact with Russell and Whitehead's *Principia Mathematica*. After remarking that Josiah Royce had been, of his teachers at Harvard, the one who had exercised the greatest influence on him, Lewis went on to say that

> Royce was also responsible for my interest in logic, or at least for the direction which it took. In 1910-11 I was his assistant in two courses in that subject, and he put into my hands one of the first copies of *Principia Mathematica*, volume i, which came to Cambridge. It is difficult now to appreciate what a novelty this work then was to all of us. Its logistic method was so decidedly an advance upon Schröder and Peano. The principles of mathematics were here deduced from definitions alone, without other assumptions than those of logic. I spent the better part of a year on it.
>
> However, I was troubled from the first by the presence in the logic of *Principia* of the theorems peculiar to material implication . . . [1]

This dissatisfaction with the logical calculus that formed the foundation of *Principia* eventually produced the Lewis systems of strict implication, and with them, modern modal logic.

The theorems "peculiar to material implication" were, as Lewis never tired of pointing out, very numerous indeed. There were the well-known ones:

$$2.02 \qquad q \supset (p \supset q)$$
$$2.21 \qquad -p \supset (p \supset q)$$

which Russell and Whitehead read as "a true proposition is implied by any proposition" and "a false proposition implies any proposition."[2] But there were also many others, not so well-known, which followed equally from the axioms, definitions, and rules of the system, e.g.:

Received January 29, 1973

$$
\begin{array}{ll}
2.51 & -(p \supset q) \supset (p \supset -q) \\
2.521 & -(p \supset q) \supset (q \supset p) \\
5.1 & (p \;\&\; q) \supset (p \equiv q) \\
5.21 & (-p \;\&\; -q) \supset (p \equiv q)
\end{array}
$$

If we read these as Russell and Whitehead would have us read our first two "peculiar theorems," then we shall render them respectively, "Given any two propositions, if the first does not imply that the second is true, then it implies that the second is false," "For any two propositions, if the first does not imply the second, then the second implies the first," "Any two true propositions imply one another," and "Any two false propositions imply one another." These are only a small sample of the theorems Lewis objected to, but they will do, since they all involve essentially the same point.

Lewis' early criticisms of *Principia* indicate an interesting combination of logical and metaphysical concerns. He could not, of course, deny that, in some sense, the peculiar theorems of *Principia Mathematica* were true—even analytically true. As he puts it in one place,

> In themselves, they are neither mysterious sayings, nor great discoveries, nor gross absurdities. They exhibit, in sharp outline, the meaning of 'implies' which has been incorporated into the algebra.[3]

In this mood he was given to comparing the logic of *Principia Mathematica* with non-Euclidean geometry and claiming that both were "true" only in the sense of being consistent pure mathematical systems. But just as with non-Euclidean geometry there arose—or seemed to arise—the further question whether, in addition to being formally consistent, it also described accurately the character of the world in which we live, so also with the logic of *Principia Mathematica* there arose the question whether it applied to "our world." To both questions the early Lewis was inclined to answer "no":

> These theorems are absurd only in the sense that they are utterly inapplicable to our modes of inference and proof. Properly, they are not rules for drawing inferences at all, but only propositions about the nature of any world to which this system of material implication would apply. *In such a world, the all-possible must be real, the true must be necessary, the contingent cannot exist, the false must be absurd and impossible, and the contrary to fact supposition must be quite meaningless.*[4]

The claim that any world to which material implication would apply must have a certain metaphysical character was based on the following logical considerations. In *Principia Mathematica* it was asserted—apparently—that a true proposition was implied by any other, and *a fortiori* by its own negation. There was, however, historical precedent for defining necessary truths as those which are implied by any proposition whatever, or which are implied by their own denial. Assuming some such definition of necessity and assuming that "implies" is being used univocally here (both fairly large assumptions), it would follow that Russell and Whitehead were committing themselves to the claim that every true proposition is necessarily true, i.e.,

(1) $p \supset \Box p$

from which it would follow that every possibly true proposition is true, i.e.,

(2) $\Diamond p \supset p$

and that every false proposition is impossible, i.e.,

(3) $-p \supset \Box -p$

More recent logicians might argue that, if Russell and Whitehead were committed to anything of this sort, it would simply mean that the modal concepts were empty for them. Since the converses of (1)-(3) are all theses of any standard modal logic, to prefix a possibility or necessity operator to a proposition would have no effect. The resulting proposition would be equivalent to the original one.

But the early Lewis seems to have felt that Russell and Whitehead had raised a serious metaphysical issue which was not easy to settle. Just as it was hard to determine whether or not Euclid's parallel postulate was true, since we could only survey small portions of our space at one time, so we could not pronounce dogmatically on the question whether every truth was a necessary truth:

> If we ask now whether the actual world is such a one as material implication may apply to, the answer is not self-evident . . . We do not discover the necessity of all facts, nor the absurdity of every contrary-to-fact hypothesis. Nor are we able to verify that ubiquity of the implication relation demanded by material implication. One may thus maintain that the real is not the all-possible, that reality is, in some part, contingent and not necessary . . . and consequently, that the system of material implication is false as an applied logic. But an obvious reply has it that this is a generalization from our ignorance—that our belief in the contingent and the false but not absurd is due to the smallness of our ken. A decision on metaphysical grounds would thus be doubtful.[5]

Lewis is clearly thinking of Spinoza, and one imagines that it may have been disconcerting to him to find, as he grew older, that the logic which he thought involved a commitment to Spinozism was to become a rallying point for contemporary Humeans. Before we dismiss these metaphysical concerns as quaint but misguided, we should remember that Lewis was right in sensing a legitimate issue here, as the post-war debate over counterfactual conditionals shows. If he did not define the issue in a clear and convincing way, at least he stimulated others to do so.[6]

More pertinent, however, are those criticisms of *Principia* which may be classed—at least in a broad sense—as logical. The point Lewis was chiefly anxious to insist upon was that the definition of implication adopted by Russell and Whitehead was very much at variance with the "ordinary meaning" of implication. On Russell and Whitehead's account a true proposition was supposed to be implied by any proposition and a false proposition to imply any proposition; but on any ordinary understanding of implication this simply is not true. The implications a proposition has are never a function of its truth-value and knowing that a proposition is true, or false, never adds anything to our knowledge of what it implies. Implication is rather a function of the intension or meaning a proposition has—it is its meaning which determines what the proposition does and does not imply, not its truth-value.

This point might be conceded nowadays, I think—with perhaps a few cautionary remarks on the un-wisdom of reading the sign for the material conditional as "implies," or even as "If . . . then . . .," an alternative reading which Russell and Whitehead suggest as being sometimes more convenient. $A \supset B$ simply means what it is defined as meaning, viz. $-A \vee B$. It would be more conducive to clarity to read $A \supset B$ always as "Either not A or B" or "Not both A and not-B" or perhaps "Not A without B."

But there is more at issue here than merely an unfortunate way of reading a technical symbol. It is true that the official position in *Principia Mathematica* is that definitions are "mere typographical conveniences" introduced simply to abbreviate formulae which "would very soon become so lengthy as to be unmanageable," that they are supposed to be neither true nor false, but merely the expressions of the authors' choices in the use of symbolism. Still, the authors of *Principia Mathematica* do want to have it both ways. For after stating their official position on definitions, they go on to say that though definitions are "theoretically superfluous," they nevertheless often convey important information:

> When what is defined is (as often occurs) something already familiar, such as cardinal or ordinal numbers, the definition contains an analysis of a common idea, and may therefore express a notable advance. Cantor's definition of the continuum illustrates this: his definition amounts to the statement that what he is defining is the object which has the properties commonly associated with the word "continuum," though what precisely constitutes these properties had not before been known. In such cases, a definition is a "making definite": it gives definiteness to an idea which had previously been more or less vague.[7]

As a result the authors are in the curious position of maintaining that those portions of their work which are superfluous in theory are nevertheless among its most important parts.

But what is the status of the definition of implication? Is it supposed to be merely a typographical convenience or does it give an analysis of a common idea which had previously been more or less vague? Their discussion of the definition suggests the latter alternative:

> When a proposition follows from a proposition p so that if p is true, q must also be true, we say that p implies q. The idea of implication, in the form in which we require it, can be defined. The meaning to be given to implication in what follows may at first sight appear somewhat artificial; but although there are other legitimate meanings, the one here adopted is very much more convenient for our purposes than any of its rivals. The

> essential property that we require of implication is this: "What is implied by a true proposition is true." It is in virtue of this property that implication yields proofs. But this property by no means determines what is implied by a false proposition. What it does determine is that, if p implies q, then it cannot be the case that p is true and q false, i.e. it must be the case that either p is false or q is true. The most convenient interpretation of implication is to say, conversely, that if either p is false or q is true, then "p implies q" is to be true.[8]

It seems to me not outrageous to suppose that here Russell and Whitehead take themselves to be dealing with a term "already familiar" to us; that they isolate one necessary condition of the correct employment of that term in its ordinary meaning, viz. that it must not be the case that the implying proposition is true and the implied proposition false; and that they are introducing into their definition what, from the point of view of ordinary use, is an artificial simplicity by construing this necessary condition as also sufficient. This makes definite something which the ordinary meaning of "implication" leaves vague—that a false proposition implies any proposition. Lewis had ample excuse for thinking that the authors of *Principia* were making some kind of claim about deducibility, and a false one at that.

But Lewis was concerned about more than the "analysis of a common idea." He also wanted to maintain that the theorems of a logical calculus ought to furnish rules for drawing inferences, and that the theorems of material implication did not do this. By this he seems to have meant, not that the rules which the calculus of *Principia Mathematica* furnishes are invalid in the sense that they might lead one from true premises to a false conclusion, but that they are not useful rules of inference. Thus, while Lewis was cautiously sceptical on the question "Whether the actual world is such a one as material implication may apply to" he thought it quite plain that "pragmatically . . . material implication is obviously a false logic."

What is at issue here may best be brought out by considering the rule of *modus ponens* for material implication, which we may write

$$A \supset B, A, \therefore B$$

and read "From premises of the form $A \supset B$ and A it is permissible to infer B." Lewis thought it relevant to inquire how the first premise might be verified and it seemed to him that there were three possible ways:

(i) The antecedent, A, was known to be false.
(ii) The consequent, B, was known to be true.
(iii) There was known to be a necessary connection between A and B.

But if $A \supset B$ were asserted on the first ground, it would be impossible to proceed to B by *modus ponens* because we would not be able to assert the second premise. On the other hand, if $A \supset B$ were asserted on the second ground, there would be no point in arguing to B by *modus ponens* since B would already be known to be true. This left only the third possibility. But if $A \supset B$ were asserted on the grounds of a necessary connection between A and B, the rule actually being used would be *modus ponens* for strict implication, i.e.,

$$A \prec B, A, \therefore B$$

and not *modus ponens* for material implication. It was presumably for this reason that Lewis, in setting up his various systems, always took *modus ponens* for strict implication as his rule of detachment, in spite of the fact that *modus ponens* for material implication can always be derived (since $((p \supset q) \& p) \prec q$ is a thesis in all the Lewis systems).

Russell and Whitehead would have conceded much of this, while denying its relevance.[9] But Lewis was led by these considerations to distinguish sharply between the implication of *Principia Mathematica* and his own strict implication. The way in which he initially did this is of some interest in the light of subsequent developments.[10] He does not deny that it is appropriate to define implication in terms of negation and disjunction in the usual way, i.e., $A \supset B = -A \vee B$. But, he argues, the "Either . . . or . . ." which is used to define implication is ambiguous: sometimes it means that one or the other, but not both, of the disjuncts is true (extensional exclusive disjunction); sometimes it means that one or the other of the disjuncts is true, without excluding their joint truth (extensional inclusive disjunction: $-A \vee B$); and sometimes it means that one or the other of the disjuncts, possibly both, *must be* true (intensional inclusive disjunction: $-A \boxed{\vee} B$). Exclusive disjunction does not interest Lewis. Extensional inclusive disjunction is the connective in terms of which implication is defined in *Principia*. Lewis customarily exemplifies it by some such proposition as

(1) Either Caesar is dead or the moon is made of green cheese,

which is said to be true simply because one of the disjuncts is true. Intensional inclusive disjunction is used by Lewis—at least in his early logical papers—to define strict implication ($A \prec B = -A \boxed{\vee} B$) and Lewis illustrates it with the proposition

(2) Either Mathilda does not love me or I am beloved,

which Lewis says is such that if you reject either of the disjuncts you are bound to accept the other.

The principal features of intensional disjunction which interest Lewis are that, whereas extensional disjunction *does not*, intensional disjunction *does* support counterfactual inference (i.e., though (1) above is true, it does not follow that if "Caesar died" were false, the moon would be made of green cheese; whereas it does follow from (2) that if "Mathilda does not love me" were false, I would be beloved); and that, whereas the truth of an extensional disjunction *cannot*, the truth of an intensional disjunction *can* be known while the truth of the disjuncts is still problematic. And it is primarily for this reason that he feels that *useful* rules of inference must be formulated in intensional terms.

As Lewis originally conceived of intensional disjunction, the intensional disjunction of two propositions was not equivalent to the absolute logical necessity of their extensional disjunction, i.e., (to put it a bit anachronistically as we shall see)

$$A \boxed{\vee} B \neq \Box(A \vee B)$$

For though Lewis seems to have been content to take $\Box(A \vee B)$ as a sufficient condition for $A \boxed{\vee} B$, he did not regard it as a necessary one:

> Intensional disjunction is not restricted to the purely formal or *a priori* type . . . Suppose a wholly reliable weather forecast for the 16th of the month to be 'warm'. This implies that . . . either today is not the 16th or the weather is warm. On the supposition made, this is an intensional disjunction. One might know its truth even if one could not find a calendar and were suffering from chills and fever. But strike out the initial assumption and this disjunction becomes, if still true, extensional. Knowledge of its truth now depends upon verification of one or both of its members. We may say that extensional disjunction concerns actualities; intensional disjunction, possibilities. But one or more facts being given, the possibilities are thereby narrowed, and an intensional disjunction which is not *a priori* may be implied.[11]

So in his earliest discussion of intensional disjunction, Lewis was prepared to count logically contingent disjunctions as intensional provided that they were implied by something "given."

We have here the germ of the important distinction between absolute and relative modalities, a distinction which Lewis later developed in his and Langford's *Symbolic Logic*.[12] There Lewis remarks that the modal terms "possible," "necessary," and "impossible" are highly ambiguous in ordinary discourse and that the meanings given to them in modal logic are not the "colloquially more frequent" ones. For the modal logician a formula like

$$\Diamond p$$

is to be read as asserting that p is logically conceivable or that it does not involve a contradiction. Analogously,

$$-\Diamond p$$

is interpreted as saying that p does involve a contradiction and is logically inconceivable. And

$$\Box p$$

says that the denial of p involves a contradiction and is impossible. These modal concepts Lewis characterizes as absolute, since "they concern only the relation which the . . . proposition has to itself or its negative—what can be analyzed out of it by sheer logic."

But the colloquially more frequent use of modal terms, according to Lewis, signifies that the proposition in question bears some logical relation to some other propositions which are known to be true or taken as given. In this usage, to say that p is possible is to say that it is consistent with what is given, i.e. (using "Q" to represent what is given):

$$p \circ Q$$

or, equivalently

$$-(Q \prec -p)$$

To say that p is impossible is to say that it is inconsistent with what is given or known:

$$-(p \circ Q) \text{ or } Q \prec -p$$

And to say that something is necessary is to say that its negation is inconsistent with the given, or more simply, that it follows from the given:

$$-(-p \circ Q) \text{ or } Q \prec p$$

These modal concepts Lewis characterizes as relative.

In terms of this later distinction, then, we may say that the early Lewis recognizes either of two conditions as sufficient for an intensional disjunction.

$$A \boxed{\vee} B \text{ if } \Box(A \vee B)$$

and

$$A \boxed{\vee} B \text{ if } Q \prec (A \vee B)$$

If satisfaction of at least one of these conditions is necessary for an intensional disjunction, then we have

$$A \boxed{\vee} B \text{ if and only if either } \Box(A \vee B) \text{ or } Q \prec (A \vee B)$$

as our definition of intensional disjunction.

Initially, Lewis seems to have contemplated two possible ways of developing his project for a system of strict implication.[13] According to one, the Russell-Whitehead symbol for disjunction would be interpreted intensionally and all those axioms of *Principia* which remain true on that interpretation would be retained—this would be everything except the principle of addition $(p \supset (q \vee p))$. Conjunction would have to be introduced as a new primitive, since the De Morgan equivalence $((p \,\&\, q) = -(-p \vee -q))$ by which it is defined in *Principia* would not hold for intensional disjunction. The principle of addition would be replaced by the principle of simplification $((p \,\&\, q) \supset p)$. He seems to have thought of this procedure as producing a fragment of the classical propositional calculus, containing intensional analogues of many of its theorems, but incapable of expressing many others which require a concept of extensional disjunction for their expression. But, of course, with negation and conjunction available, extensional disjunction could easily be introduced by definition.

The other method for developing a calculus of strict implication—and roughly, the one ultimately adopted—would be to retain both extensional and intensional disjunction, symbolise them differently and define implication in terms of intensional disjunction. The principle of addition would be retained (for extensional disjunction), as would the De Morgan equivalence. The result would be a system containing the classical propositional calculus as a proper part but containing as well many theorems which the classical calculus cannot express.

In neither of these early sketches of a system was there any provision for expressing modal concepts other than strict implication and strict disjunction—Lewis seems at first not even to have contemplated introducing symbols for necessity, possibility, and consistency, and formulating logical principles dealing with these notions. Thus the features of the Lewis systems which today are most interesting were not part of the original project.

Again—the analogy between alternative logics and alternative geometries is very prominent in Lewis' early writings on logic.[14] But neither of the ways in which he considered developing his alternative to the classical calculus would have produced a system related to it as Euclidean and non-Euclidean geometries are related. For none of the distinctive theorems of his systems are inconsistent with any theorem of *Principia.* Indeed, they could not be, on pain of absurdity.

Lewis' project for a system of strict implication developed fairly rapidly in the years after the first papers of 1912-13. By the end of 1914[15] he was taking impossibility as primitive and using it to define both strict implication and intensional disjunction. He was also beginning to explore the logical properties of such notions as consistency. Substantially the same procedure was followed in the *Survey of Symbolic Logic*, which appeared in 1918 and worked out the logical principles of Lewis' modal concepts in considerably more detail. But this system very quickly suffered catastrophe when Post showed that one of its axioms $(p \prec q) = (-\Diamond q \prec -\Diamond p)$ led to the consequence that

$$-p = -\Diamond p$$

which collapsed the system into material implication.[16] The system did not reach definitive form until 1932, with the publication of Lewis and Langford's *Symbolic Logic.* By that time the notion of intensional disjunction had dropped out altogether, possibility rather than impossibility was taken as primitive, and strict implication was defined in the now familiar way:

$$p \prec q = -\Diamond(p \,\&\, -q)$$

This definition, of course, led to its own paradoxes,' with which subsequent discussion had made us familiar—e.g., that a necessary proposition is implied by any proposition whatever, that an impossible proposition implies any proposition whatever, that all necessary propositions imply one another, etc.

Lewis himself was uncomfortable with the paradoxes of strict implication. In "Logic and Pragmatism" he remarks that while his early work on logic had convinced him that valid inference was a matter of intension, he had initially had doubts as to whether his relation of strict implication corresponded to "the implication relation of ordinary inference." Strict implication had turned out to have properties he had not anticipated. But he soon satisfied himself that implication had those properties as well:

> There was no way to avoid the principles stated by these unexpected theorems without giving up so many generally accepted laws as to leave it dubious that we could have any formal logic at all. (p. 38)

This last possibility was one which he was not prepared to contemplate. No doubt the situation is not so drastic as Lewis imagined. The development—by Anderson and Belnap and others—of modal logics which are both adequately formal and free of his paradoxes shows that we do not have to give up formal logic if we reject strict implication. But Lewis was clearly right to say that any alternative would have to give up a number of 'generally accepted laws' of logic. And it remains an open question whether the omission from such alternative systems of principles like disjunctive syllogism and antilogism does not render them more paradoxical than the system they are designed to replace.

NOTES

1. "Logic and pragmatism" in *Contemporary American Philosophy*, G. P. Adams and W. P. Montague (eds.), Allen and Unwin, London (1930), p. 32.

2. *Principia Mathematica*, paperback edition to 56, Cambridge University Press, Cambridge (1962), p. 99.

3. "Implication and the algebra of logic," *Mind*, vol. 21 (1912), p. 522.

4. "The calculus of strict implication," *Mind*, vol. 23 (1914), Lewis' italics, p. 244. s].

5. *Op. cit.*, p. 246.

6. See Roderick Chisholm, "The contrary-to-fact conditional," *Mind*, vol. 55 (1946), pp. 289-307.

7. *Principia Mathematica*, p. 12. Note that Russell and Whitehead are not consistent about what they take the thing defined to be. Here it is "the object which has the properties commonly associated with the word." In the passage next cited it is "the idea." But when they first introduce the notion of definition (p. 11), they speak of the *definiendum* as a symbol or combination of symbols, such as might appear on a sheet of paper to the left of the sign "$=_{df}$".

8. *Principia*, p. 94. We might note in passing how freely and unself-consciously Russell and Whitehead use modal language in introducing their definition. There are, of course, two possible ways of reading a statement like "if p implies q, then it cannot be the case that p is true and q is false," viz. "It is impossible that p implies q and that p is true and q false": $-\Diamond((p \supset q) \,\&\, p \,\&\, -q)$ and "If p implies q, then it is impossible that p be true and q false" $(p \supset q) \supset -\Diamond(p \,\&\, -q)$. Since Russell and Whitehead take $-(p \,\&\, -q)$ to be a sufficient as well as necessary condition of $p \supset q$, they presumably intend the first reading. But the failure to distinguish carefully these two readings may have helped to make their definition seem plausible as an account of the converse of deducibility.

9. *Cf.* Russell, *Introduction to Mathematical Philosophy*, Allen & Unwin, London (1948), p. 153:

> In order that it may be valid to infer q from p, it is only necessary that p should be true and that the proposition 'not-p or q' should be true. Whenever this is the case it is clear that q must be true. But inference will in fact only take place when the proposition 'not-p or q' is known otherwise than through knowledge of not-p or knowledge of q . . . the circumstances under which this occurs are those in which certain relations of form exist between p and q . . . But this formal relation is only required in order that we may be able to *know* that either the premise is false or the conclusion is true. It is the truth of 'not-p or q' that is required for the validity of the inference; what is required further is only required for the practical feasibility of the inference.

It is sometimes said that Lewis' criticism of *Principia* rests on a confusion of

(i) $A,\ A \supset B,\ \therefore\ B$ is always valid

with

(ii) $A,\ \therefore\ B$ is valid whenever $A \supset B$

(i) is the ordinary rule of detachment, is assumed in *Principia*, and is perfectly correct. (ii), it is said, is false and is what Lewis is attacking, but is not assumed in *Principia*. This may be, but if there is a confusion here, it is a confusion which Russell himself seems on occasion to be guilty of, as this passage illustrates. By 1932, at any rate, Lewis was reasonably clear about the matter. *Cf. Symbolic Logic*, pp. 235-247.

10. *Cf.* the espousal of intensional disjunction by Anderson and Belnap. See their "Tautological entailments," *Philosophical Studies*, vol. 13 (1962), pp. 9-24, and my "Lewis and entailment," *Philosophical Studies*, vol. 23 (1972), pp. 113-119.

11. "Implication and the algebra of logic," p. 526.

12. See C. I. Lewis and C. H. Langford, *Symbolic Logic*, Dover, New York (1959), pp. 160-162. The distinction is a very old one, but also one which has interested more recent logicians. *Cf.* Wm. & M. Kneale, *The Development of Logic*, Clarendon Press, Oxford (1962), *passim*; and G. H. von Wright, "A new system of modal logic," in *Logical Studies*, The Humanities Press, New York (1957).

13. "Implication and the algebra of logic," pp. 530-531.

14. See particularly "The calculus of strict implication," *Mind*, vol. 23 (1914), pp. 240-247.

15. See "The matrix algebra for implication," *Journal of Philosophy*, vol. 11 (1914), pp. 589-600.

16. See C. I. Lewis, "Strict implication—an emendation," *Journal of Philosophy*, vol. 17 (1920), pp. 300-302.

Australian National University
Canberra, Australia