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MATRIX CALCULI SS1M AND SS1I COMPARED WITH AXIOMATIC SYSTEMS

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P. Weingartner developed in [1] a modal matrix calculus which he called SS1M; interpreting it in a certain way he obtained the system SS1I. The purpose of the present note is to state the following facts:

1. Propositional (non-modal) SS1I contains intuitionistic propositional logic (but not conversely) and is contained in classical propositional logic (but not conversely).
2. SS1M contains S0.5.
3. SS1M does not contain S0.9 or S1° (and trivially it does not contain any stronger system).
4. S5, K4, and S9 do not contain SS1M.

The reader is supposed to have [1] at hand. Remember that cv means "characteristic value" which is the highest number assigned to a formula by an assignment of elements of $\{1, 2, 3, 4, 5, 6\}$ to the propositional variables of the formula and the value of the composed formula calculated with the help of the various matrices. 1, 2, 3 are designated values. We write also cv(α) to express "the characteristic value of α." By ϕ we mean an assignment of the kind just mentioned and $\phi(\alpha)$ is the respective value of α. The formulations of the various axiom systems are taken from [4] in the case of intuitionistic logic, from [3] in the case of S1°, and from [2] in the other cases.

1 Intuitionistic Propositional Logic is Contained in Propositional SS1I By "propositional SS1I" we mean the "propositional part" of SS1I, that is the set of theorems of SS1I which have as constant symbols only N', A, C', K', and E', but not L or M. Though these connectives are defined in [1], p. 132 with the help of the corresponding classical ones and LM, we can give matrices for them because the definitions are formulated as LL-equivalences (and two LL-equivalent formulas always take the same value for the same assignment ϕ as seen by the matrix for LLE on p. 103 of [1]).

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p	$N'p$
1	6
2	6
3	6
4	6
5	1
6	1

$C'pq$	1	2	3	4	5	6
1	1	1	1	1	6	6
2	1	1	1	1	6	6
3	1	1	1	1	6	6
4	1	1	1	1	6	6
5	1	1	1	1	1	1
6	1	1	1	1	1	1

$K'pq$	1	2	3	4	5	6
1	1	1	1	1	6	6
2	1	1	1	1	6	6
3	1	1	1	1	6	6
4	1	1	1	1	6	6
5	6	6	6	6	1	1
6	6	6	6	6	1	1

$E'pq$	1	2	3	4	5	6
1	1	1	1	1	6	6
2	1	1	1	1	6	6
3	1	1	1	1	6	6
4	1	1	1	1	6	6
5	6	6	6	6	1	1
6	6	6	6	6	1	1

Now in considering propositional SS1I (abbreviated: SS1Ip) we need not to refer to the matrices for L, M, C, K, and E but only to these just given and to that of A:

Apq	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	1	2
3	1	2	3	1	3	3
4	1	2	1	4	4	4
5	1	1	3	4	5	5
6	1	2	3	4	5	6

We use induction on theorems to prove that intuitionistic propositional logic is contained in SS1Ip. Since SS1Ip is not closed under "intuitionistic" *modus ponens* α, $C'\alpha\beta \vdash \beta$ (take $C'pp$ for α, $ApN'p$ for β, then $\mathrm{cv}(C'pp) = 1$, $\mathrm{cv}(C'C'ppApN'p) = 1$ and $\mathrm{cv}(ApN'p) = 4$), we choose the following system of intuitionistic propositional logic which we call IL (see [4], p. 178):

Axioms of IL are all formulas of the form

$$\text{(1a)} \quad C'pp \qquad \text{(1b)} \quad C'\Lambda\Lambda \qquad \text{(1c)} \quad C'\Lambda p$$

where p is any propositional variable and Λ a certain formula which always takes value 6 (it is unessential if we consider Λ to be a primitive sign or, for example, as a symbol for $K'pN'p$).

Let Γ and Δ be finite ordered sets of formulas, $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. We write $\Gamma\alpha$ to express the formula $C'\gamma_1 C'\gamma_2 \ldots C'\gamma_n\alpha$ (if Γ is empty, then $\Gamma\alpha$ means the formula α). We have the following *rules* of IL:

(2) $\Gamma\alpha \vdash \Delta\alpha$ if Δ results from Γ by changing the order of Γ

(3) $C'\alpha C'\alpha\beta \vdash C'\alpha\beta$

(4) $C'\alpha\gamma,\ C'\beta\gamma \vdash C'A\alpha\beta\gamma$

(5a) $\Gamma\alpha \vdash \Gamma A\alpha\beta$

(5b) $\Gamma\beta \vdash \Gamma A\alpha\beta$
(6) $\Gamma\alpha,\ \Gamma\beta \vdash \Gamma K'\alpha\beta$
(7a) $C'\alpha\gamma \vdash C'K'\alpha\beta\gamma$
(7b) $C'\beta\gamma \vdash C'K'\alpha\beta\gamma$
(8) $\Gamma\alpha,\ C'\beta\gamma \vdash \Gamma C'C'\alpha\beta\gamma$
(9) $\gamma \vdash C'\alpha\gamma$

Lemma 1 *If* $\phi(\Gamma\alpha) > 3$, *then* $\phi(\alpha) > 4$ *and* $\phi(\gamma_i) \leqslant 4$ *for all* $\gamma_i \in \Gamma$.

Proof: If there would be a $\gamma_i \in \Gamma$ with $\mathrm{cv}(\gamma_i) > 4$, then $\phi(C'\gamma_i C'\gamma_{i+1} \ldots C'\gamma_n\alpha) = 1$ and therefore $\phi(\Gamma\alpha) = 1$. If $\phi(\alpha) \leqslant 3$, then it follows $\phi(\Gamma\alpha) \leqslant 3$ directly from the matrix for C'.

Lemma 2 *All formulas derivable in* IL *have* $\mathrm{cv} \leqslant 3$.

Proof: By induction of theorems of IL:

(1) $\mathrm{cv}\ (C'pp) = \mathrm{cv}(C'\Lambda\Lambda) = \mathrm{cv}(C'\Lambda p) = 1 \leqslant 3$.
(2) Let $\Delta\alpha$ be derived from $\Gamma\alpha$ by an application of rule (2) and let $\mathrm{cv}(\Gamma\alpha) \leqslant 3$ (induction hypothesis). If $\mathrm{cv}(\Delta\alpha) > 3$, then there exists an assignment ϕ with $\phi(\Delta\alpha) > 3$ and therefore $\phi\alpha > 4$ and $\phi(\gamma_i) \leqslant 4$ for all $\gamma_i \in \Gamma$. But then $\phi(\Gamma\alpha) > 3$ and hence $\mathrm{cv}(\Gamma\alpha) > 3$.
(3) If $\mathrm{cv}(C'\alpha\beta) > 3$, then there exists an assignment ϕ with $\phi\beta > 4$, $\phi\alpha \leqslant 4$. But then $\phi(C'\alpha C'\alpha\beta) > 3$.
(4) If $\phi(C'A\alpha\beta\gamma) > 3$, then $\phi(\gamma) > 4$ and $\phi(A\alpha\beta) \leqslant 4$. Hence $\phi(\alpha) \leqslant 4$ or $\phi(\beta) \leqslant 4$ and therefore $\phi(C'\alpha\gamma) > 4$ or $\phi(C'\beta\gamma) > 4$.
(5) If $\phi(\Gamma A\alpha\beta) > 3$, then $\phi(\gamma_i) \leqslant 4$ for all $\gamma_i \in \Gamma$ and $\phi(A\alpha\beta) > 4$. But then $\phi(\alpha) > 4$, $\phi(\beta) > 4$ and therefore $\phi(\Gamma\alpha) > 3$ and $\phi(\Gamma\beta) > 3$.
(6) If $\phi(\Gamma K'\alpha\beta) > 3$, then $\phi(\gamma_i) \leqslant 4$ for all $\gamma_i \in \Gamma$ and $\phi(K'\alpha\beta) > 4$. This is possible only for $\phi(\alpha) > 4$ or $\phi(\beta) > 4$; therefore, $\phi(\Gamma\alpha) > 3$ or $\phi(\Gamma\beta) > 3$.
(7) If $\phi(C'K\alpha\beta\gamma) > 3$, then $\phi(K'\alpha\beta) \leqslant 4$ and $\phi(\gamma) > 4$. It follows that $\phi(\alpha) \leqslant 4$, $\phi(\beta) \leqslant 4$ and therefore $\phi(C'\alpha\gamma) > 3$ and $\phi(C'\beta\gamma) > 3$.
(8) If $\phi(\Gamma C'C'\alpha\beta\gamma) > 3$, then $\phi(\gamma_i) \leqslant 4$ for all $\gamma_i \in \Gamma$, $\phi(C'\alpha\beta) \leqslant 4$, $\phi(\gamma) > 4$. If $\phi(\beta) \leqslant 4$, then $\phi(C'\beta\gamma) > 3$. If $\phi(\beta) > 4$, then $\phi(\alpha) > 4$ and therefore $\phi(\Gamma\alpha) > 3$.
(9) If $\phi(C'\alpha\gamma) > 3$, then $\phi(\gamma) > 4$.

Lemma 2 shows that intuitionistic propositional logic is contained in propositional SS1I. The converse does not hold; this follows from the well-known fact that intuitionistic propositional logic has no finite characteristic matrix. Indeed, in SS1I the formula $C'pN'N'p$ (see 4.310 of [1]) or even the formula $K'ApN'pApN'p$ are theorems. So one could try to take C or K or both instead of C' or K' respectively. But C would not be a better interpretation of intuitionistic implication since $CCpN'qCqN'q$ (which is intuitionistically valid) takes value 4 for $p = 4$ and $q = 1$.

To show that propositional SS1I is contained in classical propositional logic we only need to identify the values 1, 2, 3, and 4 with the truth-value **v** and the values 5 and 6 with the truth-value **f**. It follows that all formulas of SS1Ip with $\mathrm{cv}(\alpha) \leqslant 4$ are theorems of the classical propositional calculus. That SS1Ip is weaker than classical propositional logic is shown in [1] (see 4.301). Therefore, SS1Ip is a system properly between intuitionistic and classical propositional logic.

2 SS1M Contains S0.5 Lemmon's S0.5 ([2], p. 256) has L as only primitive modal operator. Axioms are (1) $CLpp$ and (2) $CLCpqCLpLq$, the inference rules are uniform substitution, material detachment and

(3) If α is a theorem of the propositional calculus, then $\vdash L\alpha$.

We prove by induction on theorems of S0.5 that S0.5 is contained in SS1M. The axioms (1) and (2) have cv 2 and 1 (see 2.194 and 2.4541 of [1]) and therefore are theorems of SS1M. SS1M is closed under uniform substitution and material detachment. For example, let $\mathrm{cv}(\alpha) \leqslant 3$, $\mathrm{cv}(C\alpha\beta) \leqslant 3$, $\mathrm{cv}(\beta) > 3$. Then there exists an assignment ϕ of members of $\{1, 2, 3, 4, 5, 6\}$ to the propositional variables of β with $\phi\beta > 3$. We extend ϕ arbitrarily to the propositional variables of α which do not occur in β. Then $\phi\alpha \leqslant \mathrm{cv}(\alpha) \leqslant 3$. Looking at the matrix for C one sees that this is possible only with $\phi C\alpha\beta > 3$. To show that SS1M is closed under (3) we consider the fact that all the theorems of the propositional calculus have cv $\leqslant 2$. We choose the system of Whitehead and Russell. The axioms have cv $\leqslant 2$ (see 3.011-3.014 of [1]), and uniform substitution and material detachment preserve this property (seen by the same argumentation as above but with 2 instead of 3). Hence, if α is a theorem of the propositional calculus, then $\mathrm{cv}(L\alpha) \leqslant 3$.

3 SS1M Does Not Contain S0.9 or S1° The rules

$$KNMK\alpha N\beta NMK\beta N\alpha \vdash KNMKNMN\alpha NNMN\beta NMKNMN\beta NNMN\alpha$$

and

If $KNK\alpha N\beta NK\beta N\alpha$ is a theorem of the propositional calculus, then

$$\vdash KNMK\alpha N\beta NMK\beta N\alpha$$

are permitted in S1° (these are the rules 31.19 and 34.42 of [3], p. 49 and p. 58, written down without use of defined signs because $L(C\alpha\beta)$ and $(LC)\alpha\beta$ are not equivalent in S1°). Choosing $NKpNq$ for α and $KNKpNKpqNKKpqNp$ for β we obtain the derivation of a formula which takes the value 6 for $p = 3$ and $q = 2$; hence there is a formula which is a theorem of S1° but not of SS1M. To show the analogue situation for S0.9 we use the rules

$$E\alpha\beta \vdash (LE)\alpha\beta \text{ and } (LE)\alpha\beta \vdash (LE)L\alpha L\beta$$

(see [2], pp. 226, 247, and 257) to derive in S0.9 the formula

$$(LE)CLpqLEpKpq$$

which is in SS1M equivalent to the formula mentioned in the case of S1° and therefore is not a theorem of SS1M (again it takes value 6 for $p = 3$ and $q = 2$).

4 S5, K4, and S9 Do Not Contain SS1M The formula $ALCpqALCqpApq$ is a theorem of SS1M with cv = 3 but not a theorem of S5 (falsifying Kripke-model: $W = \{a, b, c\}$, $V(a, p) = V(c, p) = V(b, q) = V(c, q) = 0$, $V(a, q) = V(b, p) = 1$; then $V(c, ALCpqALCqpApq) = 0$). It follows that SS1M is not a subsystem of S5.

Since $CpLMp$ is a theorem of SS1M (see [1], p. 195) and not of K4 (otherwise K4 would contain S5 because S4 is a subsystem of K4 and S4 + $CpLMp$ is S5), SS1M cannot be contained in K4.

The formula $LCpLLMp$ is a theorem of SS1M with cv = 3; but it is not a theorem of S9 (S1 + $LCpLLMp$ would yield S5—see [2], p. 258—and so S9 would contain S5 with the effect of inconsistency); this means that SS1M is not contained in S9.

It should be mentioned that SS1M has relatively strong reduction laws; for fully modalized α we have $EL\alpha LL\alpha$ and $EM\alpha MM\alpha$ as theorems of SS1M because fully modalized formulas never take the values 2 and 5 (this is seen by an easy induction proof); further, $EL\beta ML\beta$ and $ELM\beta M\beta$ hold generally in SS1M. But the use of these laws is restricted by the fact that SS1M is not closed under substitution of strict or material equivalents (if so, one could prove that S1 is contained in SS1M); we can apply only the rule $\alpha, LLE\beta\gamma \vdash \delta$ where δ differs from α in having some wff β at one or more places where α has a wff γ.

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