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Notre Dame Journal of Formal Logic
Volume XX, Number 3, July 1979
NDJFAM

MODAL INTERPRETATIONS OF THREE-VALUED LOGIC. II

MICHAEL J. DUFFY

*Introduction** The present paper continues the investigations reported in [1] and [7]. The reader unfamiliar with [7] is advised to consult it before proceeding in the present paper.[1] Our purpose here is to find a modal interpretation of Słupecki's functionally complete version of Ł[2] by constructing mappings from the wffs of Ł into those of a modified version of S5.[3] The preference for mappings that preserve semantic equivalence will guide our efforts.

1 *The functionally complete version of* Ł *and a modification of* S5 To obtain the functionally complete version of Ł we add to the conditions on well-formedness given in [7] the condition that Tp is a member of the set of wffs whenever p is. We also stipulate that $I(Tp) = \frac{1}{2}$ for every interpretation I and every wff p of Ł. To S5 we add a sentential constant, which we refer to as w, and which behaves syntactically like an atom. An interpretation I for S5 in an arbitrary set $\mathcal{W}$ now satisfies all the requirements given in [7] plus the following: $\mathcal{W}$ must have at least two members; one member, w, is distinguished; and $I(w) = \{\mathsf{w}\}$. (For convenience, we refer to $\{\mathsf{w}\}$ as $\mathbf{W}$.) As we pointed out in a note above, some modification of S5 is necessary so that we become enabled to translate the Słupecki 'T' operator. The reader familiar with [1] may recall that Woodruff's solution to this problem was to add the 'T' operator to S5 also, requiring of every interpretation I in a set $\mathcal{W}$ for S5 that both $\mathcal{W}$ have at least two members and $\emptyset \neq I(Tp) \neq \mathcal{W}$ for every p. An adequate translation[4] @ of Ł into this modified version of S5 would now be given as follows: $@Tp = T@p$. But notice that while $Tp \equiv TTp$ in Ł, $T@p \not\equiv TT@p$ in S5, according to the way this new operator of S5 is interpreted. And since we are here concerned to discover mappings that preserve semantic equivalence, this way of modifying S5 does not meet our needs. Thus we modify S5 by adding the sentential constant, which, we noted, was inspired by remarks in a footnote of [1].

*The first part of this paper appears in *Notre Dame Journal of Formal Logic*, vol. XX (1979), pp. 647-657.

Received October 30, 1976

Henceforth the reader should take 'Ł' and 'S5' to denote, respectively, the functionally complete version of Ł, and S5 with the sentential constant w added as per above. In addition to the definitions of [7] for Ł and S5, we adopt the following for both systems:

$$\Diamond p =_{df} KMpMNp.$$ [5]

2 *3042 mappings from* Ł *to* S5 In this section we develop 3042 mappings from Ł to S5 and show of each that it yields a semantic interpretation of Ł in modal terms. We later show, after the style of [7], that in a strong sense we have exhausted the possible mappings; and we finally show that two (and two only) of the mappings preserve semantic equivalence. In what follows, we use 'f' as a variable over the 3042 mappings. For reasons which will become clear, it will be convenient for us to first present 1,024 of the mappings. We do so by means of the following "gappy" formulas, the numbered wffs in brackets above each blank indicating the range of choices for filling in that blank (the p's and q's are of course variables for wffs of Ł):

Where p is an atom, $fp = p$.

$$fNp = ANMfpK\Diamond fp\overbrace{\underline{\qquad\qquad}}^{\begin{bmatrix}(1)\ fp\\(2)\ Nfp\\(3)\ w\\(4)\ Nw\end{bmatrix}}$$

$$fCpq = AAAANMfpLfqK\Diamond fp\Diamond fqK\Diamond fp\overbrace{\underline{\qquad\qquad}}^{\begin{bmatrix}(1)\ fp\\(2)\ Nfp\\(3)\ w\\(4)\ Nw\end{bmatrix}}K\Diamond fq\overbrace{\ .\ .\ .\ .\ .\ .\ }^{\begin{bmatrix}(1)\ fq\\(2)\ Nfq\\(3)\ w\\(4)\ Nw\end{bmatrix}}$$

$$fTp = AAKLfp\overbrace{\underline{\qquad\qquad}}^{\begin{bmatrix}(1)\ w\\(2)\ Nw\end{bmatrix}}KNMfp\overbrace{\ .\ .\ .\ .\ .\ }^{\begin{bmatrix}(1)\ w\\(2)\ Nw\end{bmatrix}}.K\Diamond fp\overbrace{--------}^{\begin{bmatrix}(1)\ fp\\(2)\ Nfp\\(3)\ w\\(4)\ Nw\end{bmatrix}}$$

If we let h and i be variables over the integers 1 and 2, and j, k, m, n be variables over the integers 1 through 4, then we may think of each mapping as an ordered triple $\langle j, \langle k, m\rangle, \langle h, i, n\rangle\rangle$, the jth choice having filled the blank in translating negations, the kth choice having filled the first blank in translating conditionals, the mth choice having filled the second, etc. Note that in instantiating the variables we may obtain 1,024 triples, and so we have presented that many mappings. (Matrices presented below will show that no two of these mappings are equivalent.) We are thus using 'f' as short for '$\langle j, \langle k,m\rangle, \langle h,i,n\rangle\rangle$', and we will indicate partial instances of this schematic triple by suffixing 'f' with instances or partial instances of the first, second or third member. So, for example, we may use '$f\langle 2,m\rangle$' as short for '$\langle j, \langle 2,m\rangle, \langle h,i,n\rangle\rangle$'.

As in [7] we provide schematic S5 matrices for our translations, $\mathscr{J}$ and $\mathscr{N}$ again being arbitrary proper non-empty subsets of $\mathscr{K}$ and p and q being arbitrary wffs of Ł: (See pages 661-662).

We now justify the above matrices, but the reader uninterested in these details may omit the material within the asterisks without loss of continuity.

For the sake of economy, we treat the gappy formulas given above, filling the blanks only as the exposition demands. As will become clear, this will enable us to treat all of the translations of a given connective at a stroke. In general,

$$\begin{aligned} I(fNp) &= I(ANMfpK\Diamond fp\_\_\_\_) \\ &= I(NMfp) \cup I(K\Diamond fp\_\_\_\_) \\ &= \overline{I(Mfp)} \cup [I(\Diamond fp) \cap I(\_\_\_\_)] \\ &= \overline{*I(fp)} \cup [I(\Diamond fp) \cap I(\_\_\_\_)] \end{aligned}$$

Suppose first that $I(fp) = \mathscr{K}$. Then $\overline{*I(fp)} = I(\Diamond fp) = \emptyset$, and hence $I(fNp) = \emptyset$ in this case. Suppose next that $I(fp) = \emptyset$. Then $\overline{*I(fp)} = \mathscr{K}$, so $I(fNp) = \mathscr{K}$ as well. Suppose finally that $I(fp) = \mathscr{J}$. Then $\overline{*I(fp)} = \emptyset$, so $I(fNp) = I(\Diamond fp) \cap I(\_\_\_\_)$. But $I(\Diamond fp) = \mathscr{K}$ in this case, so $I(fNp) = I(\_\_\_\_)$. Hence $I(f1Np) = I(fp) = \mathscr{J}$, $I(f2Np)$ $I(Nfp) = \overline{\mathscr{J}}$, $I(f3Np) = I(w) = \mathbf{W}$, and $I(f4Np) = I(Nw) = \overline{\mathbf{W}}$. Thus the matrices for $I(fNp)$. Again in general,

$$\begin{aligned} I(fCpq) &= I(AAAANMfpLfqK\Diamond fp\Diamond fqK\Diamond fp\_\_\_\_K\Diamond fq\ \ldots) \\ &= I(AAANMfpLfqK\Diamond fp\Diamond fqK\Diamond fp\_\_\_\_) \cup I(K\Diamond fq\ \ldots) \\ &= I(AANMfpLfqK\Diamond fp\Diamond fq) \cup (I(K\Diamond fp\_\_\_\_) \cup I(K\Diamond fq\ \ldots)) \\ &= (I(ANMfpLfq) \cup I(K\Diamond fp\Diamond fq)) \cup (I(K\Diamond fp\_\_\_\_) \cup I(K\Diamond fq\ \ldots)) \\ &= (I(NMfp) \cup I(Lfq) \cup [I(\Diamond fp) \cap I(\Diamond fq)]) \cup ([I(\Diamond fp) \cap \\ &\qquad I(\_\_\_\_)] \cup [I(\Diamond fq) \cap I(\ \ldots)]) \\ &= (\overline{*I(fp)} \cup \overline{*\overline{I(fq)}} \cup [I(\Diamond fp \cap I(\Diamond fq)]) \cup ([I(\Diamond fp) \cap \\ &\qquad I(\_\_\_\_)] \cup [I(\Diamond fq) \cap I(\ldots)]) \end{aligned}$$

As the reader may verify, the left side of this union equals $\mathscr{K}$ if either $I(fp) = \emptyset$, or $I(fq) = \mathscr{K}$, or both $I(fp) = \mathscr{J}$ and $I(fq) = \mathscr{N}$, so $I(fCpq) = \mathscr{K}$ in these cases; note also that the left side of the union is empty otherwise, and so $I(fCpq) = [I(\Diamond fp) \cap I(\_\_\_\_)] \cup [I(\Diamond fq) \cap I(\ldots)]$ in the remaining cases. It is fairly obvious, then, that if $I(fp) = \mathscr{K}$ and $I(fq) = \emptyset$, then $I(\Diamond fp) = I(\Diamond fq) = \emptyset = I(fCpq)$. Two cases remain. For the first, suppose $I(fp) = \mathscr{K}$ and $I(fq) = \mathscr{N}$. Then $I(\Diamond fp) = \emptyset$, so $I(fCpq) = I(\Diamond fq) \cap I(\ldots)$. But $I(\Diamond fq) = \mathscr{K}$ in this case, so $I(fCpq) = I(\ldots)$. Hence $I(f\langle k, 1\rangle Cpq) = I(fq) = \mathscr{N}$, $I(f\langle k, 2\rangle Cpq) = I(Nfq) = \overline{\mathscr{N}}$, $I(f\langle k, 3\rangle Cpq) = I(w) = \mathbf{W}$, $I(f\langle k, 4\rangle Cpq) = I(Nw) = \overline{\mathbf{W}}$. Finally, suppose $I(fp) = \mathscr{J}$ and $I(fq) = \emptyset$. By reasoning similar to that employed in the last case, $I(fCpq) = I(\_\_\_\_)$ in this case. So now $I(f\langle 1,m\rangle Cpq) = I(fp) = \mathscr{J}$,

$f p$	$f1Np$	$f2Np$	$f3Np$	$f4Np$
$\mathscr{X}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathscr{J}$	$\mathscr{J}$	$\overline{\mathscr{J}}$	$\mathbf{W}$	$\overline{\mathbf{W}}$
$\emptyset$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$

$f\langle 1,1\rangle Cpq\, f\langle 1,2\rangle Cpq\, f\langle 1,3\rangle Cpq\, f\langle 1,4\rangle Cpq\, f\langle 2,1\rangle Cpq\, f\langle 2,2\rangle Cpq\, f\langle 2,3\rangle Cpq\, f\langle 2,4\rangle Cpq$

$f p \backslash f q$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$
$\mathscr{X}$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\overline{\not\mathscr{X}}$	$\emptyset$	$\mathscr{X}$	$\mathbf{W}$	$\emptyset$	$\mathscr{X}$	$\overline{\mathbf{W}}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\overline{\not\mathscr{X}}$	$\emptyset$	$\mathscr{X}$	$\mathbf{W}$	$\emptyset$	$\mathscr{X}$	$\overline{\mathbf{W}}$	$\emptyset$
$\mathscr{J}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{J}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{J}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{J}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{J}$	$\mathscr{X}$	$\mathscr{X}$	$\overline{\mathscr{J}}$	$\mathscr{X}$	$\mathscr{X}$	$\overline{\mathscr{J}}$	$\mathscr{X}$	$\mathscr{X}$	$\overline{\mathscr{J}}$	$\mathscr{X}$	$\mathscr{X}$	$\overline{\mathscr{J}}$
$\emptyset$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$

$f\langle 3,1\rangle Cpq\, f\langle 3,2\rangle Cpq\, f\langle 3,3\rangle Cpq\, f\langle 3,4\rangle Cpq\, f\langle 4,1\rangle Cpq\, f\langle 4,2\rangle Cpq\, f\langle 4,3\rangle Cpq\, f\langle 4,4\rangle Cpq$

$f p \backslash f q$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$
$\mathscr{X}$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\overline{\not\mathscr{X}}$	$\emptyset$	$\mathscr{X}$	$\mathbf{W}$	$\emptyset$	$\mathscr{X}$	$\overline{\mathbf{W}}$	$\emptyset$	$\mathscr{X}$	$\not\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\overline{\not\mathscr{X}}$	$\emptyset$	$\mathscr{X}$	$\mathbf{W}$	$\emptyset$	$\mathscr{X}$	$\overline{\mathbf{W}}$	$\emptyset$
$\mathscr{J}$	$\mathscr{X}$	$\mathscr{X}$	$\mathbf{W}$	$\mathscr{X}$	$\mathscr{X}$	$\mathbf{W}$	$\mathscr{X}$	$\mathscr{X}$	$\mathbf{W}$	$\mathscr{X}$	$\mathscr{X}$	$\mathbf{W}$	$\mathscr{X}$	$\mathscr{X}$	$\overline{\mathbf{W}}$	$\mathscr{X}$	$\mathscr{X}$	$\overline{\mathbf{W}}$	$\mathscr{X}$	$\mathscr{X}$	$\overline{\mathbf{W}}$	$\mathscr{X}$	$\mathscr{X}$	$\overline{\mathbf{W}}$
$\emptyset$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$

$\mathscr{f}p$	$\mathscr{f}\langle 1,1,1\rangle Tp$	$\mathscr{f}\langle 1,1,2\rangle Tp$	$\mathscr{f}\langle 1,1,3\rangle Tp$	$\mathscr{f}\langle 1,1,4\rangle Tp$	$\mathscr{f}\langle 1,2,1\rangle Tp$	$\mathscr{f}\langle 1,2,2\rangle Tp$	$\mathscr{f}\langle 1,2,3\rangle Tp$	$\mathscr{f}\langle 1,2,4\rangle Tp$
$\mathscr{K}$	$\mathbf{W}$	$\mathbf{W}$	$\mathbf{W}$	$\mathbf{W}$	$\mathbf{W}$	$\mathbf{W}$	$\mathbf{W}$	$\mathbf{W}$
$\mathscr{J}$	$\mathscr{J}$	$\overline{\mathscr{J}}$	$\mathbf{W}$	$\overline{\mathbf{W}}$	$\mathscr{J}$	$\overline{\mathscr{J}}$	$\mathbf{W}$	$\overline{\mathbf{W}}$
$\mathscr{O}$	$\mathbf{W}$	$\mathbf{W}$	$\mathbf{W}$	$\mathbf{W}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$

$\mathscr{f}p$	$\mathscr{f}\langle 2,1,1\rangle Tp$	$\mathscr{f}\langle 2,1,2\rangle Tp$	$\mathscr{f}\langle 2,1,3\rangle Tp$	$\mathscr{f}\langle 2,1,4\rangle Tp$	$\mathscr{f}\langle 2,2,1\rangle Tp$	$\mathscr{f}\langle 2,2,2\rangle Tp$	$\mathscr{f}\langle 2,2,3\rangle Tp$	$\mathscr{f}\langle 2,2,4\rangle Tp$
$\mathscr{K}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$
$\mathscr{J}$	$\mathscr{J}$	$\overline{\mathscr{J}}$	$\mathbf{W}$	$\overline{\mathbf{W}}$	$\mathscr{J}$	$\overline{\mathscr{J}}$	$\mathbf{W}$	$\overline{\mathbf{W}}$
$\mathscr{O}$	$\mathbf{W}$	$\mathbf{W}$	$\mathbf{W}$	$\mathbf{W}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$

$I(f\langle 2,m\rangle Cpq) = I(Nfp) = \overline{\mathscr{J}}$, $I(f\langle 3,m\rangle Cpq) = I(w) = \mathbf{W}$ and $I(f\langle 4,m\rangle Cpq) = I(Nw) = \overline{\mathbf{W}}$. Thus the matrices for $fCpq$. For the last,

$$\begin{aligned} I(fTp) &= I(AAKLfp\_\_\_\_KNMfp\ .\ .\ .\ K\Diamond fp{-}{-}{-}) \\ &= I(AKLfp\_\_\_\_KNMfp\ .\ .\ .) \cup I(K\Diamond fp{-}{-}{-}) \\ &= I(KLfp\_\_\_\_) \cup I(KNMfp\ .\ .\ .) \cup I(K\Diamond fp{-}{-}{-}) \\ &= [I(Lfp) \cap I(\_\_\_\_)] \cup [I(NMfp) \cap I(.\ .\ .)] \cup [I(\Diamond fp) \cap I({-}{-}{-})] \\ &= [\overline{*\overline{I(fp)}} \cap I(\_\_\_\_)] \cup [\overline{*I(fp)} \cap I(.\ .\ .)] \cup [I(\Diamond fp) \cap I({-}{-}{-})] \end{aligned}$$

Suppose first that $I(fp) = \mathscr{K}$. In this case $\overline{*\overline{\overline{I(fp)}}} = \mathscr{K}$, but $\overline{*I(\overline{fp})} = \emptyset = I(\Diamond fp)$; hence $I(fTp) = I(\_)$. Thus, $I(f\langle 1,i,n\rangle Tp) = \mathbf{W}$ and $I(f\langle 2,i,n\rangle Tp) = \overline{\mathbf{W}}$. Suppose next that $I(fp) = \emptyset$. Then $\overline{*I(fp)} = \mathscr{K}$, but $*\overline{\overline{I(fp)}} = \emptyset = I(\Diamond fp)$; so $I(fTp) = I(.\ .\ .)$. Thus, $I(f\langle h,1,n\rangle Tp) = \mathbf{W}$ and $I(f\langle h,2,n\rangle Tp) = \overline{\mathbf{W}}$. Finally, suppose $I(fp) = \mathscr{J}$. By reasoning similar to that in the last two cases, we now have $I(fTp) = I({-}{-}{-})$, so $I(f\langle h,i,1\rangle Tp) = \mathscr{J}$, $I(f\langle h,i,2\rangle Tp) = \overline{\mathscr{J}}$, $I(f\langle h,i,3\rangle Tp) = \mathbf{W}$, and $I(f\langle h,i,4\rangle Tp) = \overline{\mathbf{W}}$. Thus the matrices for fTp.

The presence of the sentential constant w in S5 has allowed us to construct more translations for the connectives 'N' and 'C' of Ł than we were able to produce in [7]. We will now show that as a result of the presence of this constant there are also more ways of translating the atoms of Ł into S5. Consider the following three ways of translating the atoms:

(1) p
(2) w
(3) Nw

Where p is atomic, $fp = ALpK\Diamond p\_\_\_\_\_\_\_\_$.

The first translation is of course semantically equivalent in S5 to p. The second and third, however, are wffs of S5 that take the values $\mathscr{K}$ and $\emptyset$ when and only when p takes those values, and that respectively take the values of $\mathbf{W}$, $\overline{\mathbf{W}}$ when and only when $\mathscr{K} \neq I(p) \neq \emptyset$. These features will make the latter two translations suitable candidates for our purposes. Moreover, mappings that translate the atoms in these different ways will not differ trivially.[6] Henceforth our mappings will employ all three of these translations of the atoms (and no others), the translation any particular mapping employs being indicated by a numerical superscript 'a' where $1 \leqslant a \leqslant 3$. (As the occasion demands, we may also superscript 'f' to indicate that we are speaking only of the mappings that translate the atoms in a certain way.)

It would now appear that we have $3 \times 1024 = 3072$ mappings from Ł to S5, but this is only apparent since 16 of the f^2 mappings are equivalent to one another, and so are 16 of the f^3 mappings. This may be seen from the following considerations. The matrices we presented earlier characterize all of the mappings we have discussed. (In particular, the f^2 and f^3 translations of 'N', 'T', and 'C' have the same form as the f^1 translations of these connectives, so the matrices characterize these new mappings.) But where the f^2 and f^3 mappings are concerned, the $\mathscr{J}$'s and $\mathscr{K}$'s of the matrices can only be placeholders for **W** or $\overline{\mathbf{W}}$.[7] For those f^2 (f^3) mappings in whose matrices none of $\overline{\mathscr{J}}$, $\overline{\mathscr{K}}$, and $\overline{\mathbf{W}}(\mathbf{W})$ occur, the only possible value for $\mathscr{J}$ and $\mathscr{K}$ is $\mathbf{W}(\overline{\mathbf{W}})$.[8] Thus, in each of the following two lists, every mapping in the list is equivalent to every other in that list:

LIST I	LIST II
$\langle 3,\langle 3,3\rangle,\langle 1,1,3\rangle\rangle^2$	$\langle 4,\langle 4,4\rangle,\langle 2,2,4\rangle\rangle^3$
$\langle 3,\langle 3,3\rangle,\langle 1,1,1\rangle\rangle^2$	$\langle 4,\langle 4,4\rangle,\langle 2,2,1\rangle\rangle^3$
$\langle 3,\langle 3,1\rangle,\langle 1,1,3\rangle\rangle^2$	$\langle 4,\langle 4,1\rangle,\langle 2,2,4\rangle\rangle^3$
$\langle 3,\langle 3,1\rangle,\langle 1,1,1\rangle\rangle^2$	$\langle 4,\langle 4,1\rangle,\langle 2,2,1\rangle\rangle^3$
$\langle 3,\langle 1,3\rangle,\langle 1,1,3\rangle\rangle^2$	$\langle 4,\langle 1,4\rangle,\langle 2,2,4\rangle\rangle^3$
$\langle 3,\langle 1,3\rangle,\langle 1,1,1\rangle\rangle^2$	$\langle 4,\langle 1,4\rangle,\langle 2,2,1\rangle\rangle^3$
$\langle 3,\langle 1,1\rangle,\langle 1,1,3\rangle\rangle^2$	$\langle 4,\langle 1,1\rangle,\langle 2,2,4\rangle\rangle^3$
$\langle 3,\langle 1,1\rangle,\langle 1,1,1\rangle\rangle^2$	$\langle 4,\langle 1,1\rangle,\langle 2,2,1\rangle\rangle^3$
$\langle 1,\langle 3,3\rangle,\langle 1,1,3\rangle\rangle^2$	$\langle 1,\langle 4,4\rangle,\langle 2,2,4\rangle\rangle^3$
$\langle 1,\langle 3,3\rangle,\langle 1,1,1\rangle\rangle^2$	$\langle 1,\langle 4,4\rangle,\langle 2,2,1\rangle\rangle^3$
$\langle 1,\langle 3,1\rangle,\langle 1,1,3\rangle\rangle^2$	$\langle 1,\langle 4,1\rangle,\langle 2,2,4\rangle\rangle^3$
$\langle 1,\langle 3,1\rangle,\langle 1,1,1\rangle\rangle^2$	$\langle 1,\langle 4,1\rangle,\langle 2,2,1\rangle\rangle^3$
$\langle 1,\langle 1,3\rangle,\langle 1,1,3\rangle\rangle^2$	$\langle 1,\langle 1,4\rangle,\langle 2,2,4\rangle\rangle^3$
$\langle 1,\langle 1,3\rangle,\langle 1,1,1\rangle\rangle^2$	$\langle 1,\langle 1,4\rangle,\langle 2,2,1\rangle\rangle^3$
$\langle 1,\langle 1,1\rangle,\langle 1,1,3\rangle\rangle^2$	$\langle 1,\langle 1,1\rangle,\langle 2,2,4\rangle\rangle^3$
$\langle 1,\langle 1,1\rangle,\langle 1,1,1\rangle\rangle^2$	$\langle 1,\langle 1,1\rangle,\langle 2,2,1\rangle\rangle^3$

Henceforth we consider the first entry in each of these lists to number among our mappings, the other fifteen in each list being discarded (which means we have eliminated thirty mappings from consideration). Thus we have 3042 official mappings. Note also that in the matrices for the f^2 and f^3 mappings we retain, any $\mathscr{J}$ or $\mathscr{K}$ may have either **W** or $\overline{\mathbf{W}}$ as its value, and no two of our 3042 mappings are equivalent.

By means of the following definitions and theorems, we now show that each mapping yields an interpretation of Ł in modal terms.

For any interpretation I of S5 (in a given set $\mathscr{K}$), let If be the function from wffs of Ł to $\{1, \frac{1}{2}, 0\}$ defined as follows:

$$If(p) = \begin{cases} 1, \text{ if } I(fp) = \mathscr{K} \\ 0, \text{ if } I(fp) = \emptyset \\ \frac{1}{2} \text{ otherwise.} \end{cases}$$

Theorem 1 *If is an interpretation of Ł.*

Proof: It suffices to show:

(a) $If(Np) = 1 - If(p)$;
(b) $If(Cpq) = \min(1, (1 - (If(p) - If(q))))$;
(c) $If(Tp) = \frac{1}{2}$.

The proofs of (a) and (b) are like the proofs of Theorem 1 (a) and (b) in [7] with 'f' in place of '$f^m_{\mathfrak{m}}$'. For proof of (c): $If(Tp) = \frac{1}{2}$ iff $\mathscr{K} \neq I(fTp) \neq \emptyset$; but the matrices show $\mathscr{K} \neq I(fTp) \neq \emptyset$, hence $If(Tp) = \frac{1}{2}$.

Theorem 2 *For any interpretation I of Ł, there is an interpretation $\mathcal{J}$ of* S5 *such that $I = \mathcal{J}f$.*

Proof: The proof is like that of Theorem 2 of [7] with 'f' replacing 'f^{n}_{m}' and the proviso that w = 0.

Theorem 3 *For every wff p of Ł:*

(a) *p is valid (in* Ł*) iff fp is valid (in* S5*);*
(b) *p is contravalid (in* Ł*) iff fp is contravalid (in* S5*);*
(c) *p is indeterminate (in* Ł*) iff fp is indeterminate (in* S5*).*

Proof: Like that for Theorem 3 of [7] with 'f' in place of 'f^{n}_{m}'.

3 *A sense in which the mappings are exhaustive* As in [7], we consider at this point whether there are other mappings for which the results of the preceding section may be obtained. Our vocabulary is that of [7], except that where '$\mathcal{A}(p, \ldots, q)$', '$\mathcal{B}(p, \ldots, q)$', etc. denote wffs of S5, they denote wffs compounded from only $p, \ldots, q$, and w.

Now let @ be a mapping from wffs of Ł to those of S5, with @ presumed to satisfy both of the following conditions:

Condition 1: @ has a definition of the following form: For all wffs p, q of Ł:

(i) if p is atomic, $@p = ALpK\Diamond p$____. (Where the blank may be filled by 'p', 'w', or 'Nw'.)
(ii) $@Np = \mathcal{A}(@p)$
(iii) $@Tp = \mathcal{B}(@p)$
(iv) $@Cpq = \mathcal{C}(@p, @q)$.

Condition 2: @ is such that the following definition, in which p is a variable for wffs of Ł and I is an arbitrary interpretation for S5 in an arbitrary $\mathcal{K}$, guarantees that the appropriate analogues of Theorems 1-3 hold true of @[9]:

$$I@(p) = \begin{cases} 1, \text{ if } I(@p) = \mathcal{K} \\ 0, \text{ if } I(@p) = \emptyset \\ \frac{1}{2} \text{ otherwise} \end{cases}$$

We now show that @ is equivalent to some f. In the lemmas and theorem that follow, I and $\mathcal{K}$ are arbitrary, and $\mathcal{P}$ and $\mathcal{W}$ are, as before, arbitrary non-empty proper subsets of $\mathcal{K}$.

Lemma 1 *Let $I(p) = \mathcal{P}$. Then one of the following is sure to hold for $\mathcal{A}(p)$:*

(i) $I(\mathcal{A}(p)) = \mathcal{K}$	(viii) $I(\mathcal{A}(p)) = \overline{\mathcal{P}} \cup \mathbf{W}$
(ii) $I(\mathcal{A}(p)) = \emptyset$	(ix) $I(\mathcal{A}(p)) = \mathcal{P} \cup \overline{\mathbf{W}}$
(iii) $I(\mathcal{A}(p)) = \mathcal{P}$	(x) $I(\mathcal{A}(p)) = \overline{\mathcal{P}} \cup \overline{\mathbf{W}}$
(iv) $I(\mathcal{A}(p)) = \overline{\mathcal{P}}$	(xi) $I(\mathcal{A}(p)) = \mathcal{P} \cap \mathbf{W}$
(v) $I(\mathcal{A}(p)) = \mathbf{W}$	(xii) $I(\mathcal{A}(p)) = \overline{\mathcal{P}} \cap \mathbf{W}$
(vi) $I(\mathcal{A}(p)) = \overline{\mathbf{W}}$	(xiii) $I(\mathcal{A}(p)) = \mathcal{P} \cap \overline{\mathbf{W}}$
(vii) $I(\mathcal{A}(p)) = \mathcal{P} \cup \mathbf{W}$	(xiv) $I(\mathcal{A}(p)) = \overline{\mathcal{P}} \cap \overline{\mathbf{W}}$

Proof: By strong induction on the length of $\mathcal{A}(p)$. Details left to the reader.

In the basis, $\mathcal{A}(p) = p$ or $\mathcal{A}(p) = w$. In the inductive step there are three cases: $\mathcal{A}(p) = N\mathcal{B}(p)$; $\mathcal{A}(p) = M\mathcal{B}(p)$; $\mathcal{A}(p) = C\mathcal{B}(p)\mathcal{C}(p)$. (In effect, the inductive step establishes that the list of values for $\mathcal{A}(p)$ in the lemma is closed under the operations ____, $*$, and $\cup$.)

Lemma 2 *If* $p \underset{T}{\leftrightarrow} q$, *then* $\mathcal{A}(\ldots p \ldots) \underset{T}{\leftrightarrow} \mathcal{A}(\ldots q \ldots)$.

Proof left to the reader.

Lemma 3 *Let* $I(p) = \mathcal{K}$ *and* $I(q) = H$. *Then one of the following is sure to hold for* $\mathcal{A}(p, q)$:

(i) $I(\mathcal{A}(p, q)) = \mathcal{K}$
(ii) $I(\mathcal{A}(p, q)) = \emptyset$
(iii) $I(\mathcal{A}(p, q)) = \mathcal{H}$
(iv) $I(\mathcal{A}(p, q)) = \overline{\mathcal{H}}$
(v) $I(\mathcal{A}(p, q)) = \mathbf{W}$
(vi) $I(\mathcal{A}(p, q)) = \overline{\mathbf{W}}$
(vii) $I(\mathcal{A}(p, q)) = \mathcal{H} \cup \mathbf{W}$
(viii) $I(\mathcal{A}(p, q)) = \overline{\mathcal{H}} \cup \mathbf{W}$
(ix) $I(\mathcal{A}(p, q)) = \mathcal{H} \cup \overline{\mathbf{W}}$
(x) $I(\mathcal{A}(p, q)) = \overline{\mathcal{H}} \cup \overline{\mathbf{W}}$
(xi) $I(\mathcal{A}(p, q)) = \mathcal{H} \cap \mathbf{W}$
(xii) $I(\mathcal{A}(p, q)) = \overline{\mathcal{H}} \cap \mathbf{W}$
(xiii) $I(\mathcal{A}(p, q)) = \mathcal{H} \cap \overline{\mathbf{W}}$
(xiv) $I(\mathcal{A}(p, q)) = \overline{\mathcal{H}} \cap \overline{\mathbf{W}}$.

Proof: Since $I(p) = \mathcal{K}$ and $I(q) = \mathcal{H}$, $I(Mq) = *\mathcal{H} = \mathcal{K} = I(p)$. Hence $p \underset{T}{\leftrightarrow} Mq$, so $\mathcal{A}(p, q) \underset{T}{\leftrightarrow} \mathcal{A}(Mq, q)$ by Lemma 2. But $\mathcal{A}(Mq, q)$ qualifies as a compound $\mathcal{B}(q)$ of q, and hence Lemma 3 by Lemma 1.

Lemma 4 *Let* $I(p) = \mathcal{J}$ *and* $I(q) = \emptyset$. *Then one of the following is sure to hold for* $\mathcal{A}(p, q)$:

(i) $I(\mathcal{A}(p, q)) = \mathcal{K}$
(ii) $I(\mathcal{A}(p, q)) = \emptyset$
(iii) $I(\mathcal{A}(p, q)) = \mathcal{J}$
(iv) $I(\mathcal{A}(p, q)) = \overline{\mathcal{J}}$
(v) $I(\mathcal{A}(p, q)) = \mathbf{W}$
(vi) $I(\mathcal{A}(p, q)) = \overline{\mathbf{W}}$
(vii) $I(\mathcal{A}(p, q)) = \mathcal{J} \cup \mathbf{W}$
(viii) $I(\mathcal{A}(p, q)) = \overline{\mathcal{J}} \cup \mathbf{W}$
(ix) $I(\mathcal{A}(p, q)) = \mathcal{J} \cup \overline{\mathbf{W}}$
(x) $I(\mathcal{A}(p, q)) = \overline{\mathcal{J}} \cup \overline{\mathbf{W}}$
(xi) $I(\mathcal{A}(p, q)) = \mathcal{J} \cap \mathbf{W}$
(xii) $I(\mathcal{A}(p, q)) = \overline{\mathcal{J}} \cap \mathbf{W}$
(xiii) $I(\mathcal{A}(p, q)) = \mathcal{J} \cap \overline{\mathbf{W}}$
(xiv) $I(\mathcal{A}(p, q)) = \overline{\mathcal{J}} \cap \overline{\mathbf{W}}$.

Proof like that for Lemma 3.

Lemma 5 *Let* $I(p) = \mathcal{K}$. *Then one of the following is sure to hold for* $\mathcal{A}(p)$.

(i) $I(\mathcal{A}(p)) = \mathcal{K}$
(ii) $I(\mathcal{A}(p)) = \emptyset$
(iii) $I(\mathcal{A}(p)) = \mathbf{W}$
(iv) $I(\mathcal{A}(p)) = \overline{\mathbf{W}}$.

Proof: By strong induction on the length of $\mathcal{A}(p)$. Details again left to the reader. In the basis $\mathcal{A}(p) = p$ or $\mathcal{A}(p) = w$, and in the inductive step we have the same cases as the proof of Lemma 1. (To establish the inductive step it would suffice to show $\{\mathcal{K}, \emptyset, \mathbf{W}, \overline{\mathbf{W}}\}$ closed under ——, $*$, and $\cup$.)

Lemma 6 *Let* $I(p) = \emptyset$. *Then one of the following is sure to hold of* $\mathcal{A}(p)$:

(i) $I(\mathcal{A}(p)) = \mathcal{K}$
(ii) $I(\mathcal{A}(p)) = \emptyset$

(iii) $I(\mathcal{A}(p)) = \mathbf{W}$
(iv) $I(\mathcal{A}(p)) = \overline{\mathbf{W}}$.

Proof like that of Lemma 5.

Lemma 7

(a) *If* $I(@p) = \emptyset$, *then* $I(@Np) = \mathcal{K}$.
(b) *If* $I(@p) = \mathcal{K}$, *then* $I(@Np) = \emptyset$.
(c) *If* $I(@p) = \mathcal{P}$, *then* $I(@Np) = \mathcal{P}$ *or* $I(@Np) = \overline{\mathcal{P}}$ *or* $I(@Np) = \mathbf{W}$ *or* $\mathrm{I}(@Np) = \overline{\mathbf{W}}$.

Proof: (a) Suppose $I(@p) = \emptyset$. Then $I@(p) = 0$, in which case $I@(Np) = 1$ by Theorem 1. But $I@(Np) = 1$ iff $I(@Np) = \mathcal{K}$. Hence, (a). (b) Proof like that of (a). (c) Suppose $I(@p) = \mathcal{P}$. Then $I@(p) = \frac{1}{2}$, in which case $I@(Np) = \frac{1}{2}$ as well, this last holding iff $\mathcal{K} \neq I(@Np) \neq \emptyset$. Furthermore, given the supposition that $I(@p) = \mathcal{P}$, then one of (i)-(xiv) of Lemma 1 holds of $@Np$. Obviously, (i) and (ii) fail to hold. Moreover, since the present argument is to hold for every I and so every $\mathcal{P}$, (vii)-(xiv) fail as well (because where $\mathcal{P} = \mathbf{W}$ or $\mathcal{P} = \overline{\mathbf{W}}$ the values in (vii)-(xiv) sometimes become $\mathcal{K}$ or $\emptyset$). Hence, one of (iii)-(vi) holds; hence, (c).

Lemma 8

(a) *If* $I(@p) = \mathcal{K}$ *and* $I(@q) = \emptyset$, *then* $I(@Cpq) = \emptyset$.
(b) *If any of the following obtain:*

(i) $I(@p) = \emptyset$
(ii) $I(@q) = \mathcal{K}$
(iii) $I(@p) \neq \mathcal{K}$ and $I(@q) \neq \emptyset$

then $I(@Cpq) = \mathcal{K}$.

(c) *If both* $I(@p) = \mathcal{K}$ *and* $I(@q) = \mathcal{K}$, *then one of the following obtains:*

(i) $I(@Cpq) = \mathcal{K}$
(ii) $I(@Cpq) = \overline{\mathcal{K}}$
(iii) $I(@Cpq) = \mathbf{W}$
(iv) $I(@Cpq) = \overline{\mathbf{W}}$.

(d) *If both* $I(@p) = \mathcal{P}$ *and* $I(@q) = \emptyset$, *then one of the following holds:*

(i) $I(@Cpq) = \mathcal{P}$
(ii) $I(@Cpq) = \overline{\mathcal{P}}$
(iii) $I(@Cpq) = \mathbf{W}$
(iv) $I(@Cpq) = \overline{\mathbf{W}}$.

Proof: Proof of (a) and (b) like proof of Lemma 7 (a) and (b). Proof of (c) like proof of Lemma 7 (c) using Lemma 3 in place of Lemma 1. Proof of (d) like proof of (c) using Lemma 4 in place of Lemma 3.

Lemma 9

(a) *If* $I(@p) = \mathcal{K}$, *then either* $I(@Tp) = \mathbf{W}$ *or* $I(@Tp) = \overline{\mathbf{W}}$.
(b) *If* $I(@p) = \emptyset$, *then either* $I(@Tp) = \mathbf{W}$ *or* $I(@Tp) = \overline{\mathbf{W}}$.
(c) *If* $I(@Tp) = \mathcal{P}$, *then one of the following holds:*

(i) $I(@Tp) = \mathscr{J}$
(ii) $I(@Tp) = \overline{\mathscr{J}}$
(iii) $I(@Tp) = \mathbf{W}$
(iv) $I(@Tp) = \overline{\mathbf{W}}$.

Proof: (a) Suppose $I(@p) = \mathscr{K}$. $I@(Tp) = \frac{1}{2}$, and hence $\mathscr{K} \neq I(@Tp) \neq \emptyset$. Thus, (a) by Lemma 5. (b) Proof like that of (a) using Lemma 6 in place of Lemma 5. (c) Proof of (c) like proof of (b) using Lemma 1 in place of Lemma 6 (and relying on the fact that $\mathscr{J}$ must be arbitrary, thus ruling out (vii)-(xiv) of Lemma 1.)

Theorem 4 *@ is equivalent to some f.*

Proof: By condition 1 on @, @ has the same values for atomic arguments as, say, $f^{\&}$. By Lemma 7, the possible matrices for $@Np$ are the same as for f, so let the actual matrix for $@Np$ be that given for $f r$. Similarly, by Lemma 8, the possible matrices for $@Cpq$ are those given for $fCpq$, so let the matrix for $f\langle s,t\rangle Cpq$ be the actual matrix for $@Cpq$. Finally, in light of Lemma 9, let the actual matrix for $@Tp$ be identical to that given for $f\langle u,v,w\rangle Tp$. A straightforward induction (easily supplied by the reader) then shows that $@p \equiv \langle r,\langle s,t\rangle,\langle u,v,w\rangle\rangle^{\&} p$ for every wff p of Ł, and hence shows the mappings to be equivalent.[10]

4 *The preservation of semantic equivalence* In this section we will prove that $\langle 3,\langle 3,3\rangle,\langle 1,1,3\rangle\rangle^2$ and $\langle 4,\langle 4,4\rangle,\langle 2,2,4\rangle\rangle^3$, which we henceforth refer to as $\mathbf{F}^2$ and $\mathbf{F}^3$ respectively, preserve semantic equivalence; and we further show that no other of the remaining 3040 mappings does so.

Theorem 5

(a) *$p \equiv q$ in Ł iff $\mathbf{F}^2 p \equiv \mathbf{F}^2 q$ in* S5.
(b) *$p \equiv q$ in Ł iff $\mathbf{F}^3 p \equiv \mathbf{F}^3 q$ in* S5.[11]

Proof: (a) Suppose $p \not\equiv q$ in Ł. Assume for reductio that $\mathbf{F}^2 p \neq \mathbf{F}^2 q$ in S5. It is clear in the atomic case, and the matrices make it clear in the remaining cases, that for every I of S5 and every wff p' of Ł, $I(\mathbf{F}^2 p') = \mathscr{K}$ or $I(\mathbf{F}^2 p') = \mathbf{W}$ or $I(\mathbf{F}^2 p') = \emptyset$. Thus, the assumption that $\mathbf{F}^2 p \not\equiv \mathbf{F}^2 q$ is equivalent to the assumption that for some I, one of the following holds:

(i) $I(\mathbf{F}^2 p) = \mathscr{K}$ and $I(\mathbf{F}^2 q) = \mathbf{W}$;
(ii) $I(\mathbf{F}^2 p) = \mathscr{K}$ and $I(\mathbf{F}^2 q) = \emptyset$;
(iii) $I(\mathbf{F}^2 p) = \mathbf{W}$ and $I(\mathbf{F}^2 q) = \mathscr{K}$;
(iv) $I(\mathbf{F}^2 p) = \mathbf{W}$ and $I(\mathbf{F}^2 q) = \emptyset$;
(v) $I(\mathbf{F}^2 p) = \emptyset$ and $I(\mathbf{F}^2 q) = \mathscr{K}$;
(vi) $I(\mathbf{F}^2 p) = \emptyset$ and $I(\mathbf{F}^2 q) = \mathbf{W}$.

But these are respectively equivalent to:

(i′) $I\mathbf{F}^2(p) = 1$ and $I\mathbf{F}^2(q) = \frac{1}{2}$;
(ii′) $I\mathbf{F}^2(p) = 1$ and $I\mathbf{F}^2(q) = 0$;
(iii′) $I\mathbf{F}^2(p) = \frac{1}{2}$ and $I\mathbf{F}^2(q) = 1$;
(iv′) $I\mathbf{F}^2(p) = \frac{1}{2}$ and $I\mathbf{F}^2(q) = 0$;

(v′) $I\mathbf{F}^2(p) = 0$ and $I\mathbf{F}^2(q) = 1$;
(vi′) $I\mathbf{F}^2(p) = 0$ and $I\mathbf{F}^2(q) = \frac{1}{2}$.

And since we know $I\mathbf{F}^2$ to be an interpretation of Ł by Theorem 1, each of (i′)-(vi′) contradicts the supposition that $p \equiv q$ in Ł. Hence, if $p \equiv q$ in Ł, $\mathbf{F}^2 p \equiv \mathbf{F}^2 q$ in S5.

Suppose conversely that $\mathbf{F}^2 p \equiv \mathbf{F}^2 q$ in S5, and assume for reductio that $p \not\equiv q$ in Ł. Let $\mathcal{I}$ be an interpretation of Ł such that $\mathcal{I}(p) \neq \mathcal{I}(q)$, in light of the assumption. By Theorem 2, there is an interpretation I of S5 such that $\mathcal{I} = I\mathbf{F}^2$. Thus the assumption that $p \not\equiv q$ in Ł is equivalent to assuming the disjunction of (i′)-(vi′) above for some I, which is equivalent to assuming (i)-(vi) above, which contradicts the present supposition that $\mathbf{F}^2 p \equiv \mathbf{F}^2 q$ in S5. Hence, if $\mathbf{F}^2 p \equiv \mathbf{F}^2 q$ in S5, $p \equiv q$ in Ł. Hence, (a). (b) Proof of (b) just like proof of (a) with '$\mathbf{F}^3$' in place of '$\mathbf{F}^2$' and '$\overline{\mathbf{W}}$' in place of '$\mathbf{W}$'.

Corollary *Where p and q are non-atomic wffs of* Ł, *then for each* α *from* 1 *through* 3*:*

(a) $p \equiv q$ *in* Ł *iff* $\langle 3,\langle 3,3\rangle,\langle 1,1,3\rangle\rangle^{\alpha} p \equiv \langle 3,\langle 3,3\rangle,\langle 1,1,3\rangle\rangle^{\alpha} q$ *in* S5*;*
(b) $p \equiv q$ *in* Ł *iff* $\langle 4,\langle 4,4\rangle,\langle 2,2,4\rangle\rangle^{\alpha} p \equiv \langle 4,\langle 4,4\rangle,\langle 2,2,4\rangle\rangle^{\alpha} q$ *in* S5.

Proof: Proof of the corollary is identical to the proof of the theorem with the atomic case eliminated.

Theorem 6 *If* f *preserves semantic equivalence, then* $f = \mathbf{F}^2$ *or* $f = \mathbf{F}^3$.

Proof: We suppose that f preserves semantic equivalence, and adduce considerations that reduce the possible values of f to the two given in the theorem. It is well known that $Tp \equiv Tq$ in Ł for every wff p and q. Thus we have $fTp \equiv fTq$ in S5 for every p and q of Ł. The matrices indicate that there are exactly two translations of 'T' for which the equivalence is guaranteed, viz. $f\langle 1,1,3\rangle$ and $f\langle 2,2,4\rangle$.[12] Moreover $Tp \equiv NTp$ in Ł for every p, so we can eliminate translations of 'N' that would obtain the complement of the value of fTp for $fNTp$. This leaves us with the following possible values for f: $\langle 1,\langle k,m\rangle,\langle 1,1,3\rangle\rangle^{\alpha}, \langle 3,\langle k,m\rangle,\langle 1,1,3\rangle\rangle^{\alpha}, \langle 1,\langle k,m\rangle,\langle 2,2,4\rangle\rangle^{\alpha}$, and $\langle 4,\langle k,m\rangle,\langle 2,2,4\rangle\rangle^{\alpha}$.[13]

We recall at this point the argument of [7] designed to show that certain translations of 'C' do not preserve the equivalence between Apq and Aqp holding in Ł.[14] Since the argument applies to the translations in virtue of the matrices that characterize them, the argument eliminates the following translations of 'C' presented here (since these are characterized by the same matrices as their counterparts in [7]): $f\langle 1,1\rangle, f\langle 1,2\rangle, f\langle 2,1\rangle$, and $f\langle 2,2\rangle$. We now run the same test case on the remaining translations of 'C'. We first provide matrices for the translations of Cqp, and then for each mapping's translations of Apq ($=_{df}$ $CCpqq$) and Aqp ($=_{df}$ $CCqpp$). (The method for arriving at these matrices is given in [7].) The matrices show of all but two of the translations of 'C' that they fail to preserve semantic equivalence.

	$f\langle 1,3\rangle Cqp$			$f\langle 1,4\rangle Cqp$			$f\langle 2,3\rangle Cqp$			$f\langle 2,4\rangle Cqp$			$f\langle 3,1\rangle Cqp$			$f\langle 3,2\rangle Cqp$		
fp \ fq	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$
$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$
$\mathscr{J}$	$\mathbf{W}$	$\varkappa$	$\varkappa$	$\overline{\mathbf{W}}$	$\varkappa$	$\varkappa$	$\mathbf{W}$	$\varkappa$	$\varkappa$	$\overline{\mathbf{W}}$	$\varkappa$	$\varkappa$	$\mathscr{J}$	$\varkappa$	$\varkappa$	$\overline{\mathscr{J}}$	$\varkappa$	$\varkappa$
$\emptyset$	$\emptyset$	$\not\varkappa$	$\varkappa$	$\emptyset$	$\not\varkappa$	$\varkappa$	$\emptyset$	$\overline{\not\varkappa}$	$\varkappa$	$\emptyset$	$\overline{\not\varkappa}$	$\varkappa$	$\emptyset$	$\mathbf{W}$	$\varkappa$	$\emptyset$	$\mathbf{W}$	$\varkappa$

	$f\langle 3,3\rangle Cqp$			$f\langle 3,4\rangle Cqp$			$f\langle 4,1\rangle Cqp$			$f\langle 4,2\rangle Cqp$			$f\langle 4,3\rangle Cqp$			$f\langle 4,4\rangle Cqp$		
fp \ fq	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$
$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$
$\mathscr{J}$	$\mathbf{W}$	$\varkappa$	$\varkappa$	$\overline{\mathbf{W}}$	$\varkappa$	$\varkappa$	$\mathscr{J}$	$\varkappa$	$\varkappa$	$\overline{\mathscr{J}}$	$\varkappa$	$\varkappa$	$\mathbf{W}$	$\varkappa$	$\varkappa$	$\overline{\mathbf{W}}$	$\varkappa$	$\varkappa$
$\emptyset$	$\emptyset$	$\mathbf{W}$	$\varkappa$	$\emptyset$	$\mathbf{W}$	$\varkappa$	$\emptyset$	$\overline{\mathbf{W}}$	$\varkappa$	$\emptyset$	$\overline{\mathbf{W}}$	$\varkappa$	$\emptyset$	$\overline{\mathbf{W}}$	$\varkappa$	$\emptyset$	$\overline{\mathbf{W}}$	$\varkappa$

	$f\langle 1,3\rangle Apq$			$f\langle 1,3\rangle Aqp$			$f\langle 1,4\rangle Apq$			$f\langle 1,4\rangle Aqp$			$f\langle 2,3\rangle Apq$			$f\langle 2,3\rangle Aqp$		
fp \ fq	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$
$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$
$\mathscr{J}$	$\varkappa$	$\mathbf{W}$	$\mathscr{J}$	$\varkappa$	$\mathbf{W}$	$\mathbf{W}$	$\varkappa$	$\overline{\mathbf{W}}$	$\mathscr{J}$	$\varkappa$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$	$\varkappa$	$\mathbf{W}$	$\mathscr{J}$	$\varkappa$	$\mathbf{W}$	$\mathbf{W}$
$\emptyset$	$\varkappa$	$\mathbf{W}$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\overline{\mathbf{W}}$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\mathbf{W}$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$

	$f\langle 2,4\rangle Apq$			$f\langle 2,4\rangle Aqp$			$f\langle 3,1\rangle Apq$			$f\langle 3,1\rangle Aqp$			$f\langle 3,2\rangle Apq$			$f\langle 3,2\rangle Aqp$		
fp \ fq	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$
$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$
$\mathscr{J}$	$\varkappa$	$\overline{\mathbf{W}}$	$\mathscr{J}$	$\varkappa$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$	$\varkappa$	$\not\varkappa$	$\mathbf{W}$	$\varkappa$	$\mathscr{J}$	$\mathscr{J}$	$\varkappa$	$\overline{\not\varkappa}$	$\mathbf{W}$	$\varkappa$	$\overline{\mathscr{J}}$	$\overline{\mathscr{J}}$
$\emptyset$	$\varkappa$	$\overline{\mathbf{W}}$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\mathbf{W}$	$\emptyset$	$\varkappa$	$\overline{\not\varkappa}$	$\emptyset$	$\varkappa$	$\mathbf{W}$	$\emptyset$

	$f\langle 3,3\rangle Apq$			$f\langle 3,3\rangle Aqp$			$f\langle 3,4\rangle Aqp$			$f\langle 3,4\rangle Aqp$			$f\langle 4,1\rangle Apq$			$f\langle 4,1\rangle Aqp$		
fp \ fq	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$
$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$
$\mathscr{J}$	$\varkappa$	$\mathbf{W}$	$\mathbf{W}$	$\varkappa$	$\mathbf{W}$	$\mathbf{W}$	$\varkappa$	$\overline{\mathbf{W}}$	$\mathbf{W}$	$\varkappa$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$	$\varkappa$	$\not\varkappa$	$\overline{\mathbf{W}}$	$\varkappa$	$\mathscr{J}$	$\mathscr{J}$
$\emptyset$	$\varkappa$	$\mathbf{W}$	$\emptyset$	$\varkappa$	$\mathbf{W}$	$\emptyset$	$\varkappa$	$\overline{\mathbf{W}}$	$\emptyset$	$\varkappa$	$\mathbf{W}$	$\emptyset$	$\varkappa$	$\not\varkappa$	$\emptyset$	$\varkappa$	$\overline{\mathbf{W}}$	$\emptyset$

	$f\langle 4,2\rangle Apq$			$f\langle 4,2\rangle Aqp$			$f\langle 4,3\rangle Apq$			$f\langle 4,3\rangle Aqp$			$f\langle 4,4\rangle Apq$			$f\langle 4,4\rangle Aqp$		
$fp \backslash fq$	$\mathcal{K}$	$\not\mathcal{K}$	$\emptyset$	$\mathcal{K}$	$\not\mathcal{K}$	$\emptyset$	$\mathcal{K}$	$\not\mathcal{K}$	$\emptyset$	$\mathcal{K}$	$\not\mathcal{K}$	$\emptyset$	$\mathcal{K}$	$\not\mathcal{K}$	$\emptyset$	$\mathcal{K}$	$\not\mathcal{K}$	$\emptyset$
$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$
$\mathcal{J}$	$\mathcal{K}$	$\overline{\not\mathcal{K}}$	$\overline{\mathbf{W}}$	$\mathcal{K}$	$\overline{\mathcal{J}}$	$\overline{\mathcal{J}}$	$\mathcal{K}$	$\mathbf{W}$	$\overline{\mathbf{W}}$	$\mathcal{K}$	$\mathbf{W}$	$\mathbf{W}$	$\mathcal{K}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$	$\mathcal{K}$	$\overline{\mathbf{W}}$	$\overline{\mathbf{W}}$
$\emptyset$	$\mathcal{K}$	$\overline{\not\mathcal{K}}$	$\emptyset$	$\mathcal{K}$	$\overline{\mathbf{W}}$	$\emptyset$	$\mathcal{K}$	$\mathbf{W}$	$\emptyset$	$\mathcal{K}$	$\overline{\mathbf{W}}$	$\emptyset$	$\mathcal{K}$	$\overline{\mathbf{W}}$	$\emptyset$	$\mathcal{K}$	$\overline{\mathbf{W}}$	$\emptyset$

These matrices indicate that the only mappings that might preserve semantic equivalence translate 'C' as the $f\langle 3,3\rangle$ and $f\langle 4,4\rangle$ mappings do.[15] Employing the further consideration that $Tp \equiv CCppTp$ in Ł, we narrow the above possible values for f to these: $\langle 1,\langle 3,3\rangle,\langle 1,1,3\rangle\rangle^{\alpha}$, $\langle 3,\langle 3,3\rangle,\langle 1,1,3\rangle\rangle^{\alpha}$, $\langle 1,\langle 4,4\rangle,\langle 2,2,4\rangle\rangle^{\alpha}$, and $\langle 4,\langle 4,4\rangle,\langle 2,2,4\rangle\rangle^{\alpha}$. The possible values for α are easily narrowed by the consideration that where p is an atom, $p \equiv CCppp$ in Ł. Thus if f preserves semantic equivalence, $\alpha = 2$ when the second member of the triple is $\langle 3,3\rangle$, and $\alpha = 3$ when the second member is $\langle 4,4\rangle$. Thus f must be one of the following: $\langle 1,\langle 3,3\rangle,\langle 1,1,3\rangle\rangle^2$,

$$\langle 3,\langle 3,3\rangle,\langle 1,1,3\rangle\rangle^2, \langle 1,\langle 4,4\rangle,\langle 2,2,4\rangle\rangle^3, \text{ or } \langle 4,\langle 4,4\rangle,\langle 2,2,4\rangle\rangle^3.$$

But note that the first and third of these were removed from our official list of mappings in section **2** (the reason, worth noting here, being that they are equivalent to the second and fourth, respectively). The remaining two mappings are of course $\mathbf{F}^2$ and $\mathbf{F}^3$. Hence, if f preserves semantic equivalence, $f = \mathbf{F}^2$ or $f = \mathbf{F}^3$.

NOTES

1. Familiarity with [7] is necessary because we make use here of its vocabulary and its results; but we should not fail to note that [1], because it both opened discussion of the present subject and provided exemplary strategies for obtaining results, remains of paramount importance.

2. See Słupecki's [4]. As was noted in [1], the provision of modal interpretations of this version of Ł yields modal interpretations of every truth-functional three-valued calculus.

3. The modification of S5 that we adopt below was inspired by the remarks of [1], p. 438, n.7. The reader should realize that some modification of S5 is required so we may translate the 'T' operator of Ł into S5. Normally, there is no wff p of S5 such that $\emptyset \neq I(p) \neq \mathcal{K}$ for every interpretation I of S5, and just such a wff is what is needed for our translation.

4. By "adequate translation" we mean translations for which the analogues of Th1-3 of [2] hold.

5. So $I(\Diamond p) = \mathcal{K}$ if $\mathcal{K} \neq I(p) \neq \emptyset$; $I(\Diamond p) = \emptyset$ otherwise.

6. We are referring here to the remark in [7] p. 657, n.8. The restrictions on how the constant is to be interpreted guarantee that the triviality mentioned in [7] will not infect translations of the atoms of Ł that make use of the constant as ours do. We attempt to make the point clearer. Let $\mathbf{S}^A$ be the set of wffs of S5 that translate the atoms of Ł, according to either one

(but not both) of the suggested translations making use of the constant w. Then let $\mathbf{S}$ be the set of wffs of S5 that can be compounded from the members of $\mathbf{S}^A$ according to the usual formation rules, including the members of $\mathbf{S}^A$ themselves. If the fragment $\mathbf{S}$ were synonymous with the whole of S5 in the sense intended in [7], we could do all of the following. We could construct a mapping $\mathbf{m}$ from $\mathbf{S}$ to the set of wffs of S5 that was one-one onto, mapping the members of $\mathbf{S}^A$ onto the atoms and reducing the other members accordingly. And we could then show both that for every interpretation I of S5 there is an interpretation I' such that for every $p \in \mathbf{S}$, $I(p) = I'(\mathbf{m}p)$; and for every I there is an I' such that for every wff p of S5 ($p = \mathbf{m}q$ for some $q \in \mathbf{S}$) $I(p) = I'(q)$. It is on the last mentioned point that the program breaks down, as is readily seen when p is chosen to be atomic, and I is chosen so that $\mathcal{K} \neq I(p) \neq \Phi$ and $\mathbf{W} \neq I(p) \neq \overline{\mathbf{W}}$. Hence the fragment represented by $\mathbf{S}$ is not synonymous with the whole of S5, due to the role of the constant.

7. Proof of this claim can be provided as follows. In L1 of [7] we proved that if an interpretation assigns $\mathscr{P}$ to a wff p of S5, then that interpretation must assign $\mathcal{K}$, Φ, $\mathscr{P}$ or $\overline{\mathscr{P}}$ to any wff compounded from only p. The inductive step we described is equivalent to establishing that $\{\mathcal{K}, \Phi, \mathscr{P}, \overline{\mathscr{P}}\}$ is closed under $\overline{}$, $*$, and $\cup$. This last is equivalent to claiming that any function ψ compounded from $*$, $\overline{}$, and $\cup$ has values in $\{\mathcal{K}, \Phi, \mathscr{P}, \overline{\mathscr{P}}\}$ when its arguments are drawn from that set. Associated with every wff of S5 is such a function ψ whose arguments are fixed on a given interpretation by the values that interpretation assigns the components of the wff. In the f^2 and f^3 mappings, the translation of the atoms of Ł are always assigned values in $\{\mathcal{K}, \Phi, \mathbf{W}, \overline{\mathbf{W}}\}$, and the other translations are compounds of the translations of the atoms. Thus the values an interpretation assigns these translations are all from $\{\mathcal{K}, \Phi, \mathbf{W}, \overline{\mathbf{W}}\}$, and so if the value of a given wff is neither $\mathcal{K}$ nor Φ, it must be $\mathbf{W}$ or $\overline{\mathbf{W}}$.

8. Consider that for such f^2 translations the values for the atoms on any interpretation are always in $\{\mathcal{K}, \Phi, \mathbf{W}\}$. Suppose inductively that where p is a non-atomic wff of Ł, the value of $I(f^2q)$ is in $\{\mathcal{K}, \Phi, \mathbf{W}\}$ if f^2q is less complex than f^2p. The induction runs easily because the relevant matrices show that when $\mathcal{K} \neq I(f^2p) \neq \Phi$, then either $I(f^2p) = I(f^2q)$ where f^2q is a component of (and hence a less complex wff than) f^2p, or else $I(f^2p) = \mathbf{W}$. The same argument with 'f^3' in place of 'f^2' and '$\overline{\mathbf{W}}$' in place of '$\mathbf{W}$' establishes our claim about the f^3 translations whose matrices have the characteristics we described.

9. By "appropriate analogues" is of course meant the result of substituting '@' for 'f' in Th1-3.

10. It might turn out that $\langle r, \langle s, t\rangle, \langle u, v, w\rangle\rangle^{b}$ is one of the mappings we eliminated earlier in section 2, but there is nothing vicious in this since we've retained a mapping to which @ is still sure to be equivalent, equivalence of mappings being a transitive relation.

11. Note that the theorem says more than just that $\mathbf{F}^2$ and $\mathbf{F}^3$ *preserve* semantic equivalence; it also claims that no semantic equivalences are "generated" (i.e., if $p \not\equiv q$ in Ł, $\mathbf{F}^2p \not\equiv \mathbf{F}^2q$ in S5; if $p \not\equiv q$ in Ł, $\mathbf{F}^3p \not\equiv \mathbf{F}^3q$ in S5).

12. We are deliberately assuming here that the variables in the matrices for our 3042 official mappings are *not* just placeholders for only, say, $\mathbf{W}$.

13. Unfortunately, the consideration that the $\mathscr{P}$ in the matrix for the $f1$ translation of 'N' can take on more than one value, is not enough to eliminate the $f1$ mappings on our given premises, because $I(f1Tp) = I(f1NTp)$ according to the matrix, once it is given that $I(f1Tp)$ is a constant other than $\mathcal{K}$ or Φ.

14. See [7], section 4.

15. This again makes use of the assumption mentioned in note 12 above.

REFERENCES

For references [1]-[6], see [7].

[7] Duffy, M. J., "Modal interpretations of three-valued logic. I," *Notre Dame Journal of Formal Logic*, vol. XX (1979), pp. 647-657.

Temple University
Philadephia, Pennsylvania