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NECESSARY AND CONTINGENT DEDUCTION

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1 *Introduction* Mathematical logic originated and developed from the necessity to explain and to found mathematics. It is a discipline aiming, mainly, to explain and to assure the deductive processes in mathematics. This appears definitely even in Gottlob Frege's works and in the development of this science. P. S. Novicov writes[1]: "One of the fundamental problems of mathematical logic remains the analysis of the foundations of mathematics. Now it has gone beyond the frame of this problem and is having an important influence on the development of mathematics itself".

Mostowski showed[2] that the general problems of the foundations of mathematics are:

A. What is the nature of the notions considered in mathematics? To what extent are they formed by man and to what extent are they imposed from outside, and whence do we gain knowledge of their properties?

B. What is *the nature of mathematical proofs* and what are the criteria allowing us to distinguish correct from false proofs?

Considering now mathematical logic itself, as a mathematically well built system, we remark that the central problem of this science is Mostowski's problem B. This problem can be simply defined as follows: *what is the logico-mathematical mechanism of the deductive process and what justifies it?*

That the principal aim of mathematical logic is to make a theory of deduction, is shown explicitly even in *Principia Mathematica* by Whitehead and Russell. The propositional calculus is called in this work "theory of deduction"; introducing further the idea of propositional function and quantification (with apparent variables and type theory), the authors affirm they are making an "extension of the theory of deduction".

More than that, Wittgenstein showed that the true formulae of mathematical logic (tautologies) are only "the forms of a proof"[3], a conclusion taken over by F. Waismann[4], and developed by us[5].

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We shall not go farther to quote other authors on this problem. The main problem of mathematical logic is the problem of deduction, deduction which justifies the mathematical truths, and makes them necessary, a necessity which seems to F. Gonseth to be "*presque divine*".

The mathematical propositions have indeed two specific characters:

1. *they are true*
2. *they are necessary*

The task of a theory of deduction, and therefore of a logico-mathematical theory too, will consist in showing that the formal structures it establishes leads to *true* and *necessary* propositions. We shall show in the following pages that the logico-mathematical deduction can explain only the truth of propositions, not their necessity.

2 *Syllogistic deduction* The first to have examined deduction, under this twofold aspect, of a logical device leading to *true* and *necessary* propositions, was Aristotle. The form of any deduction is, in his conception, the syllogism. Here is what he writes in *Prior Analytics*[6]: "The syllogism is a discourse (λόγος), in which some (things) being given, other (things) than what was given follow *necessarily* (ἐξ ἀνάγκης), by the simple fact of what is given". In this very simple phrase many precisions are made, which generally are not taken into account[7]. We shall only insist on the fact that the conclusion is drawn in a necessary way (*ex anânkes*). Aristotle himself emphasized the *necessary* character that syllogistic deduction implies, and here is how he explained this necessity[8]: "When three terms are having among them such relations such as the minor term is contained in the totality of the middle term, and the middle term is contained, or is not contained, in the totality of the major term, then there is *with necessity* a perfect syllogism between the extreme terms"[9].

The Stagirite calls a *perfect* syllogism that one which needs nothing else than what is given in the premises "for the conclusion to be evident"; the syllogism is imperfect if it needs one or several (things), "what, it is true, follow necessarily from the given terms, but are not explicitly enounced in the premisses". These "things", that are implicitly contained in the premises, are made explicit by immediate inferences (inversion, contraposition), by means of which the syllogisms of the other figures are reduced to the first one (perfect).

Summing up this discussion we see that the mechanism of deduction takes finally the syllogistic forms of the first figure, which the Scholastic logicians have called by the artificial words: *Barbara*, *Celarent*, *Dario*, *Ferio*.

Two problems must be clarified with regard to modes of the first figure:

1. Why their conclusion is *necessary* and in what sense

2. Whether the proof of this *necessity* is not open to criticism.

3 *Necessary deduction* The above problems were already discussed by the old commentators, and are again, in our time, in the attention of logicians. We shall explain them again in order to make some nuances precise.

Regarding the necessity of the conclusion, we must emphasize the distinction Aristotle made between the modality "*necessary*" of a proposition and the *necessary* conclusion of a syllogism. These two kinds of *necessity* were distinguished as being respectively the "*absolute necessity*" and the "*relative necessity*". "The conclusion", he writes, "is not necessary in an absolute manner, but it is necessary due to the given premises"[10]. In other words, if we start the deduction from necessary modal propositions, the deductive device will produce conclusions having the modality "*necessary*". These conclusions will be necessary in an absolute manner, while assertorial premises will produce necessarily assertorial conclusions, without having the modality "*necessary*". Aristotle calls these two necessities[11] *relative necessity*—*τίνων ὄντων ἀναγκαῖον*—(the necessity in respect to something existing); and *absolute necessity*—*ἁπλῶς ἀναγκαῖον*—(simple necessity).

Now, let us see how Aristotle proves the necessity of the conclusion, relative necessity, if following with necessity—*ἐξ ἀνάγκης*—from the premises, but not having itself the modality "necessary"—*ἀνάγκη*. The problem has two aspects: (1) the proof reducing the syllogisms of the other figures to the first; (2) the proof of the validity of the syllogisms in the first figure.

If every demonstration has the form of a syllogism, as Aristotle affirms, then to reduce the syllogisms to the first figure or to prove the validity of the syllogisms of the first figure are circular proofs which demonstrate nothing, and then the conclusion does not follow with necessity from the premises.

This objection was raised by Petrus Ramus[12] and later by Leibniz[13], but it is not right, because it does not take into account the nature of the reasonings Aristotle used in reducing or proving the validity of syllogistic modes, *which are not syllogistic demonstrations*. In effect, to reduce the other syllogisms to the first figure, and to prove the validity of the syllogisms of the first figure, Aristotle uses three devices that are not syllogistic: (1) immediate inferences (direct transformation of the propositions of a syllogism into others, by conversion and transposition), (2) reduction to absurdities, (3) *ecthesis*[14]. Therefore, there is no circularity in the Aristotelian proofs of the validity of the syllogistic forms, and the conclusions of these valid forms follow *necessarily*—*ἐξ ἀνάγκης*—because they are demonstrated.

4 *Non-necessary deduction* In addition to this kind of deduction the Stoics introduced "the non-demonstrated deduction". They divided the types of arguments into two classes[15]:

1. *Demonstrated arguments*—*λόγοι ἀπόδεικτοι*;
2. *Non-demonstrated arguments*-*λόγοι* or *τρόποι ἀναπόδεικτοι*.

The *apodeiktoi arguments* are the various types of syllogisms, which Aristotle demonstrated in the *Prior Analytics*.

The *anapodeiktoi arguments* are specific to Stoic logic, the simplest one being the well known *modus ponens*, amply developed and used by present day mathematical logic.

Diogenes Laertios[16] says that Chrysippos considered there were five types of such reasonings, and Sextus Empiricus[17] confirms this number, which were the basic forms of the Stoic logic, "to which they seem to reduce all the others". Here are these basic types of inference of the non-demonstrated deduction, as quoted by Sextus Empiricus:

1. The first argument is that deducing the consequent from the non-simple hypothetical proposition and antecedent. In the symbols of modern logic this is *modus ponens* and can be written:

$$\begin{array}{l} p \supset q \\ \underline{p\quad\ } \\ q \end{array}$$

or, as scheme of deduction, $p \supset q \,.\, p :\supset .\, q$

2. The second argument is that deducing the opposite of the antecedent from the non-simple hypothetical proposition and the opposite of the consequent:

$$\begin{array}{l} p \supset q \\ \underline{\sim q\quad\ } \\ \sim p \end{array}$$

or the scheme: $p \supset q \,.\sim q :\supset .\sim p$

3. The third argument is that according to which, from the negation of a conjunction and of one of its parts, one concludes the other part:

$$\begin{array}{l} \sim (p \,.\, q) \\ \underline{p\qquad\quad} \\ \sim q \end{array}$$

or: $\sim (p \,.\, q) \,.\, p :\supset .\sim q$

4. The fourth argument is that concluding from a disjunctive proposition and from one of its parts, the contradictory of the other part:

$$\begin{array}{l} p \vee q \\ \underline{p\qquad\quad} \\ \sim q \end{array}$$

or: $p \vee q \,.\, p :\supset .\sim q$

5. The fifth argument concludes from a disjunctive proposition and the contradictory of one of its parts, the other part:

$$\begin{array}{l} p \vee q \\ \underline{\sim p\quad\ } \\ q \end{array}$$

or: $p \vee q \,.\sim p :\supset .\, q$

The Stoics used to formulate these arguments considering only their "form", which for them was the "figure" of reasoning, without any content; it was the logico-grammatical form, the parts being noted with numbers, as for instance: "if the first, then the second"... This fact is confirmed by many texts.[18] Here is what Apuleius writes[19]: "The Stoics, on the other hand, used to replace what was written by numbers as "*if the first, then the second; but the first; therefore the second*"." Here are the five types of hypothetical arguments, expressed schematically with numbers:

1. If the first, then the second
But the first
Therefore the second.

2. If the first, then the second
But not the second
Therefore not the first.

3. Not at the same time the first and the second
But the first
Therefore not the second.

4. Either the first or the second
But the first
Therefore not the second.

5. Either the first or the second
But not the second
Therefore the first.

With these five forms of reasoning, which they called "simple", ἁπλοῖ, they built numerous other arguments, non-simple arguments—οὐχ ἁπλοῖ—all being reduced to the five basic ones, as we are told that Chrysippos reduced them using "formulae of reduction", much simplified by Antipatros[20].

All these forms of hypothetical reasonings, which form the base of mathematical logic, were considered as valid without proof. How did the Stoics explain the fact that they accepted these forms of reasonings without demonstration? Firstly, they thought that a type of reasoning is correct only from the point of view of its formal structure. That is why they believed that the Aristotelian syllogisms, although *materially* correct, were *formally* incorrect[21].

The problem of explaining the necessity of these forms of reasoning must, nevertheless, have been the subject of their researches. It is not admissible that such great logicians as the Stoics were not aware that *the necessity of the conclusion was assured by forms of reasoning whose necessity was not proved.* In this sense must be interpretated their statement, that the non-demonstrated arguments are *undemonstrable* arguments. The term ἀναπόδεικτος (*anapodeiktos*) means "non-demonstrated" but also "undemonstrable"[22]. A text of Sextus Empiricus contains a remark which explains their position[23]: "*These arguments are those which they say do not need demonstration for being established, because they serve as proofs for the conclusiveness of all other arguments*". In other words, they accept the Aristotelian thesis, that to avoid a *regressus in infinitum*, one has to stop somewhere—ἀνάγκη στῆναι—and that not everything can be demonstrated.

Thus they accept axiomatically five forms of hypothetical reasonings, without demonstrating them, and declare them "undemonstrated" and "undemonstrable". In this way, as we said, they succeeded in formulating a huge number of deductive forms.

We shall now emphasize the essential difference between these forms of reasoning and the syllogistic forms. While the syllogistic conclusion follows necessarily—ἐξ ἀνάγκης—the conclusion in the Stoic reasoning does not follow necessarily but only hypothetically—ἐξ ὑποθέσεως. The syllogistic conclusion follows necessarily from the premises, and this necessity is due to the fact that it is demonatrated; in the Stoic reasoning, the conclusion does not follow necessarily, because there is no demonstration to show that it must follow from the premises. This result must not be understood in the sense that the conclusion in the Stoic reasoning has the modality "hypothetical", but only that it is obtained by means of a logical mechanism which is not demonstrated—the Stoics said it is not demonstrable.

As Aristotle himself explained, and as we have shown, the conclusion of a syllogism, though following necessarily, does not have the modality "necessary". Likewise with the conclusion of a Stoic reasoning: the conclusion of a "undemonstrated" reasoning does not have the modality "hypothetical", but it is *true*; only it is not obtained necessarily.

5 *Deduction by accident* The results we reached so far, in a very general manner, can be better explained, if we take into account some texts in Aristotle's works. The Stoic deduction supposes that in the logical mechanism of reasoning the premises can be false. This simple fact shows that this deductive mechanism has a peculiarity, which we shall explain further.

Aristotle dealt with the cases when we draw a false conclusion from true premises, or when we draw a true conclusion from false premises, studying these problems in the three figures[24]. Here is what he writes[25]: "It is possible for the premises of a syllogism to be true; it is also possible for both of them to be false, or one to be true and the other false. From true premises it is not possible to draw a false conclusion, but from false premises it is possible to draw a true conclusion, with this reservation: *it will not refer to ⟨⟨for what reason⟩⟩, but to what there is in fact. This is because ⟨⟨for what reason⟩⟩ cannot make the object of a syllogism with false premises*".

We shall not reproduce the whole argument of Aristotle, which is to be found (completed) in his other works, such as *Metaphysics*. The explanation of these two kinds of syllogism (with true or false premises) is summed up by Waitz[26], Trendelenburg[27] and Tricot[28] as follows. Aristotle starts from the difference that exists between explaining ⟨⟨what there is in fact⟩⟩, indicated by the conjunction ὅτι (since), and the explanation by ⟨⟨the cause that determines⟩⟩ indicated by the conjunction διότι (for what reason or cause)[29]. He says that, since from false premises can be derived truth and as well falseness, it follows that these kinds of premises (false) give us only an explanation *in fact* of the conclusion, and not a *causal* explanation of it. Tricot is in agreement with Waitz and Trendelenburg: "Since the conclusion, although being true, does not follow from true premises, there is no possibility of an explanation *by cause*, but only of the enunciation of a simple fact".

Examining various types of syllogism, Aristotle concludes[30]: "It is clear that, if the conclusion is false, the propositions from where starts the reasoning must be *necessarily* false, all or only some of them; on the contrary, when the conclusion is true, it is *necessary* for the premises to be true; but it is possible that none of the parts of syllogism be true and nevertheless the conclusion is true, only it is not (true) *necessarily*. The reason of this fact is that two things being in such a relation that the existence of one of them entails necessarily the existence of the other, the non-existence of the last will entail the non-existence of the first, while the existence of the last will not entail necessarily the existence of the first. But it is impossible that the existence and the non-existence of the same thing entail necessarily the existence of one and the same thing".

Tricot[31] sums up this argumentation: "It is impossible for a consequent (B), which derives necessarily from an antecedent (A), to be derived necessarily from the contradictory (proposition) of the same antecedent (i.e., non-A). If it is true that *si est homo, est animal* (if somebody is a man, he is an animal) one cannot say *si non est homo, est animal* (if somebody is not a man, he is an animal)". We shall add to Tricot's explanation, that such a conclusion can sometimes be true, *but not necessarily*. In other words, we can say, in some cases, that, in fact, "if somebody is not a man, it is an animal"; but this is true only by chance. This result is stated by Tricot in these terms[32]: "It is impossible to draw a conclusion from false premises otherwise than *per accidens*".

We shall only add that these results are totally in accord with Aristotle's concept of the nature of syllogism, of demonstration, and of science in general. Indeed he writes[33]: "The demonstrated knowledge must result from *true* premises, prime, immediate, better and prior known than the conclusion, *whose cause they are* (αἰτίων τοῦ συμπεράσματα)."

6 *Conclusion* The above considerations lead us to the conclusion that reasonings of the Stoic type allow us to reach conclusions not necessarily but as *a matter of fact*. More precisely, the results obtained by such logical proceedings are *simply true*, but *not necessarily* true.

This consequence was very well known to the Scholastic logicians, who taking over and developing these logical deductive mechanisms, called them "consequences"—*consequentiae*—showed that there were two kinds of such consequences: (1) *consequentia formalis*; (2) *consequentia materialis*[34]. The valid consequence (*bona*) by virtue of its form—*de forma*—is that one which, if it signifies something adequately by its antecedent, signifies it also adequately by its consequent. The valid material consequence—*materialis bona*—is that one whose consequent is not a consequent by virtue of the formal meaning of the antecedent. An example of such a *consequentia bona materialis* is: *Homo est asinus, ergo baculus stat in angulo* ("Man is an ass, then the stick is in the corner").

Any formal consequence is also valid materially, but not vice-versa (Scholastic rule). The main rule of material consequence is the following: *Ex falsis verum, ex veris nil nisi verum* (from false (propositions) the truth, from true (propositions) nothing but truth).

Accepting this kind of deduction, the logico-mathematical forms of deduction can reach queer results. Let us consider the above example given by the Scholastic logicians: *Homo est asinus, ergo baculus stat in angulo.* This implication is true since its first number is false; but from false the truth can be deduced; therefore *baculus stat in angulo,* a materially true proposition (if it is true in fact) is a conclusion drawn from false propositions. *But it is drawn by accident.*

The first to have drawn the attention on these queer and arbitrary ways of concluding was C. I. Lewis[35]. He says that all these curious consequences are due to the fact that the implication of Russell is an implication in extension, without any analogous relation in intension. The inference depends, in his opinion, on the *meaning* of the propositions; therefore it is a relation in intension. Russell's implication appears to Lewis as a *material* implication, since only in materially given cases can a real inference be established, otherwise it says nothing and for this reason it is *contingent.* This was just Aristotle's opinion, as we have seen, only it was more systematically exposed.

The solution Lewis proposed was to introduce modalities as values for propositions: possible, impossible, necessary etc., in order to obtain conclusions having the modality *necessary.* But he, as well as other logicians who constructed formal systems by means of modalities, did not take into account the precise difference Aristotle made between the necessary proposition (ἀνάγκη) and those that obtained necessarily (ἐξ ἀνάγκης). By introducing modalities in the development of deduction, one can show only the relations among modal propositions; but mathematical truth is never expressed through modal propositions. On the other hand, deduction in modal systems is made by virtue of the same schemes of deduction, which are not demonstrated, and thus this deduction is *contingent.* That is evident, once the implication of Lewis, as well as other implications, can lead to a conclusion starting from false propositions. Aristotle's deduction imposed, as we have seen, the condition that the premises be *true*; in this case the syllogistic conclusion necessarily was true. In the contrary case, the conclusion cannot be necessarily true.

What is interesting, and what ought to have struck anyone studying the logical mechanism of mathematics, is the fact that all—absolutely all—propositions of a mathematical theory are *true*. There is no false proposition in such a theory. That situation has been emphasized by the Roumanian mathematician Octav Onicescu[36], who has consequently built up a logical system with a single value, *the truth*, which is able, in his opinion, to explain the real deductive articulation of the propositions of a mathematical theory.

Mathematical logic did not renounce the use of material implication, and numerous forms of it developed. It developed in this sense with the logistic mechanism, built up in this way, i.e., transforming the whole theory of deduction into a formal deductive system (using, or not using, modalities). It tried to establish the logical foundations of mathematics. Putting aside other problems raised by this construction, we remark that mathematical logic, considered only as the deductive tool of the mathematical sciences, is able to deduce the *truth* of mathematical propositions, but has no capacity to deduce *necessarily* this truth—necessity which appeared to Gonseth, as we have said, "almost divine". In other words mathematical logic can show that a mathematical proposition is *true in fact*, but cannot give the reason (διότι) that makes it *necessarily true*.

In this way, *necessary deduction* was transformed into *contingent deduction*. This means, however, that the deductive device of symbolic logic can not explain what is most essential to the nature of mathematical truths: their necessity.

NOTES

1. *Elements of Mathematical Logic*, translated from Russian into Roumanian, p. 9, "Editura ştiinţifică," Bucarest (1966).

2. A. Mostowski et al., *The Present State of Investigations on the Foundations of Mathematics*, Polska Akademia Nauk, Warszaw (1955).

3. L. Wittgenstein, *Tractatus Logico-Philosophicus*, Paul Kegan, London (1933), Prop. 6.1262.

4. F. Waismann, "Ist die Logik eine deduktive Theorie?," *Erkenntnis*, vol. 7, The Hague (1938).

5. A. Dumitriu, "La science de la logique," *Notre Dame Journal of Formal Logic*, vol. XII (1971), pp. 385-405.

6. *Prior Analytics*, I, 1, 24b. The Greek text is: *συλλογισμὸς δὲ ἐστὶ λόγος ἐν ὣς τεθέντον τιvῶν τὶ τῶν κειμένον ἐξ ἀνάγκης συμβαίνει τῶ ταῦτα εἶναι*. The Scholastic translation (Peter of Spain, *Summulae Logicales*, IV) is: *Syllogismus est oratio in qua quibusdam positis necesse est aliud accidere per ea quae posita sunt*. In both texts the expressions *ἐξ ἀνάγκης* (necessarily) and *necesse* (necessarily) appear.

7. We have insisted on these questions in our work *Istoria Logicii*, 8.8, "Editura didactică şi pedagogică," Bucarest (1969).

8. *Prior Analytics*, I, 4, 25b.

9. *Op. cit.*, I, 1, 24b, 20-25.

10. *Op. cit.*, I, 10, 30b.

11. *Prior Analytics*, I, 10, 30, b. *Second Analytics*, II, 11, 94a. This distinction was made precise, for instance, by Alexander of Aphrodisia, re-discussed in our times by H. Maier in *Die Syllogistik des Aristoteles*, and others. A critical analysis of this problem can be found in G. Patzig's book, *Die aristotelische Syllogistik* (Göttingen, 1958). He makes a strict interpretation of the difference existing between "necessary proposition" and "proposition from necessity" (*op. cit.*, ch. II).

12. Petrus Ramus, *Animadversiones in dialecticam Aristotelis*, Paris (1543).

13. Leibniz, *Nouveaux Essais sur l'Entendement Humain*, IV, II, 1, Amsterdam (1765).

14. J. Łukasiewicz in *Aristotle's Syllogistic*, §19, 3rd edition, Oxford (1958), examined closely the proof called *ecthesis*—*ἔκθεσις*—by the help of which Aristotle demonstrated the laws of conversion. The ecthesis consists in adding to the given terms another new term, which is "extracted" out of those given. This is also the meaning of the word *ἔκθεσις*—"extraction".

15. Sextus Empiricus, *Adversus Mathematicos*, VIII, 223 and *Pyrrhonienses Hypotyposes*, II, 156; Diogenes Laertios, *The lives and doctrines of philosophers*, VII, 79.

16. *Op. cit.*, VII, 79.
17. Sextus Empiricus, *Pyrrhonienses Hypotyposes*, II, 157.
18. Diogenes Laertios, *op. cit.*, VII, 80; Apuleius, *De Dogmate Platonis*, III; Boethius, *De Syllogismo Hypothetico*, etc.
19. *Op. cit.*, III. The Latin text is: *Stoici porro pro litteris numeros usurpant ut "si primum, secundum; atqui primum; secundum igitur"*.
20. Plutarch, in *The contradictions of Stoics*, c. 1047, affirmed that Chrysippos said he could make with ten simple propositions more than a million such forms of reasoning.
21. Alexander of Aphrodisia, *Ad Analytica Priora Commentarium*.
22. See the note of Aram Frenkian in the translation into Roumanian of *Pyrrhonienses Hypotyposes*, Bucarest (1965), p. 93.
23. *Op. cit.*, 156.
24. *Prior Analytics*, II, 2, 3, 4.
25. *Op. cit.*, IV, 2, 53b.
26. Th. Waitz, *Aristotelis Organon graece*, 2 vol., Berlin (1844-1846), I, p. 483.
27. A. Trendelenburg, *Elementa logicae Aristotelis*, Berlin (1836), p. 107.
28. J. Tricot, translation of the *Prior Analytics*, Paris (1936), pp. 209-233.
29. *Metaphysics*, A, 1, 981a.
30. *Prior Analytics*, II, 4, 57a-57b.
31. *Op. cit.*, p. 232.
32. *Op. cit.*, p. 233. Tricot shows, in the same place, quoting Julius Pacius' In *Porphirii Isagogen et Aristotelis Organum Commentarium*, 215, that if we suppose the conclusions from false premises to be necessarily true, then we affirm a contradiction.
33. *Second Analytics*, I, 2, 71b.
34. The Scolastic logicians dedicated ample treatises to the theory of consequences, the best known of them being that of Radulphus Strodus, bearing the title *Consequentiae* (printed in Venice, 1488).
35. C. I. Lewis, *A Survey of Symbolic Logic*, Berkeley and Los Angeles (1918).
36. Octav Onicescu, *Principes de Logique et de Philosophie Mathêmatique*, Bucharest (1971); see also his study "Les Catêgories Logiques Mathêmatiques," *International Logic Review*, vol. 8 (1973), pp. 176-200.

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