Notre Dame Journal of Formal Logic
Volume 30, Number 1, Winter 1989

Notes on Modal Definability

JOHAN van BENTHEM*

1 Introduction This paper contains a few observations on the definability of frame classes in modal logic, utilizing current algebraic methods in the area. For technical background, see [3], [4].

Possible worlds frames induce modal algebras of their subsets, and conversely, modal algebras can be represented as frame-induced set algebras by the Stone ultrafilter representation. This back-and-forth connection allows for a transfer of existing definability results in Universal Algebra to the model theory of possible worlds frames. One notable result is that, in translating the Birkhoff characterization of equational varieties, if a frame F validates the full modal theory of some frame class K, then the following structural connection exists: The 'ultrafilter extension' $\mathrm{ue}(F)$ is a generated subframe of a p-morphic image of an ultrafilter extension of some disjoint union of frames in K. Several well-known theorems on the modal definability of frame classes have been deduced from this and similar observations. Here we shall take a closer look at the structure of the *ultrafilter extensions* involved (Section 2), deriving some additional *definability results* (Section 3). Then we particularize the theory to an important special case, viz. that of *finite frames*, which turns out to require additional techniques (Section 4). Finally, another specialization is considered, to the case of singleton classes K, i.e. to the study of *modal equivalence* between frames (Section 5).

The notion of an ultrafilter extension and its various uses forms a red thread through this report—which is otherwise a loose collection of results 'rounding out' the existing literature.

---

*The contents of this paper form a response to the work of several people. A reading of Fine [13] led to Section 3.1, a review of Sambin & Vaccaro [24] to Section 3.2. Also, notably, involvement with Rodenburg [23] and Doets [9] produced Section 4. And finally, a correspondence with Kees Doets and Dick de Jongh inspired Section 5.2.

I would also wish to thank the referee of this Journal for several valuable suggestions.

Received November 13, 1986; revised March 7, 1988

2 The structure of ultrafilter extensions

2.1 Known results The *ultrafilter extension* $\mathrm{ue}(F)$ of a frame $F = \langle W, R\rangle$ has as its universe the set of all ultrafilters on (the power set of) W, ordered by the following relation R^*:

$$R^* U_1 U_2 \text{ iff for all } X \in U_2,\ \mathrm{m}(X) \in U_1$$

(where $\mathrm{m}(X) =_{def} \{w \in W \mid \text{ for some } v \in X, Rwv\}$).

Occasionally, we shall also need the more general algebraic background of this notion. Ultrafilter extensions $\mathrm{ue}(F)$ may be viewed as being the result of applying a *Stone Representation* construction to the *modal algebra* $\mathrm{A}(F)$ consisting of the subsets of W with their ordinary Boolean structure as well as the new modal operation m. More generally, modal algebras $\mathfrak{A}$ are Boolean algebras having such an additional operation satisfying the basic identities of the 'minimal modal logic' K (in particular, distributivity over disjunctions). The Stone Representation $\mathrm{S}(\mathfrak{A})$ then arises by taking all the ultrafilters on $\mathfrak{A}$ as worlds, and defining an alternative relation among these as above, to obtain a frame. Moreover, this frame comes with a distinguished set of subsets $\mathcal{W}$ (being the canonical images of elements of $\mathfrak{A}$), from which the original algebra can be recovered. Such frames which have a distinguished family $\mathcal{W}$ of subsets closed under Boolean as well as modal operations are called *general frames* $\langle F, \mathcal{W}\rangle$. Evidently, it makes sense to speak of ultrafilter representations of general frames too, being the underlying frames of the Stone Representations $\mathrm{S}(\mathrm{A}(\langle F, \mathcal{W}\rangle))$, where $\mathrm{A}(\langle F, \mathcal{W}\rangle)$ is the set-based modal algebra consisting of $\mathcal{W}$ with its obvious structure. In this perspective, the original ultrafilter extensions $\mathrm{ue}(F)$ derive from the case of $\langle F, \mathcal{W}\rangle$ when $\mathcal{W}$ is the full power set of W.

What is known, in general, about the structure of the $\mathrm{ue}(F)$ frames?

(1) F lies embedded as a *subframe* (not necessarily generated) in $\mathrm{ue}(F)$. (Consider the principal ultrafilters on F.)

(2) $\mathrm{ue}(F)$ is a p-morphic image of some *ultrapower* of F. (For a proof, see [1]. Note that the saturated elementary extension of F employed there can be obtained as a countable ultrapower, by Lemma 6.1.1 in [8]. This observation is due, amongst others, to Goldblatt.) As a corollary, every first-order feature of F which is preserved under p-morphisms also holds in $\mathrm{ue}(F)$. By a preservation theorem in [4], these are precisely those first-order formulas reducible to the form 'one universal quantifier, followed by a formula involving only atomic formulas, falsum, $\wedge$, $\vee$, and restricted quantifiers of the form $\forall y(Rxy \to \ldots$ or $\exists y(Rxy \wedge \ldots$'. (Examples are $\forall x\, Rxx$ (reflexivity), $\forall x \forall y(Rxy \to \forall z(Ryz \to Rxz))$ (transitivity), $\forall x \forall y(Rxy \to \forall z(Rxz \to (Ryz \vee Rzy \vee y = z)))$ (right-linearity), etc.)

(3) The modal theory of $\mathrm{ue}(F)$ is contained in that of F. (This follows by the Stone isomorphism between the modal set algebra of F and some subalgebra of that of $\mathrm{ue}(F)$.)

2.2 Concrete examples Some specific examples will make the notion of ultrafilter extension more concrete.

(1) On *finite* frames, all ultrafilters are principal, and ue(F) remains isomorphic to F itself.

(2) For the *integers* ($\mathbb{Z}$,<), the construction adds points only at infinity. ue($\mathbb{Z}$,<) looks like Figure 1. The key observation here is this. Any free ultrafilter U on $\mathbb{Z}$ contains either the negative integers or the nonnegative ones. Moreover, 'nibbling off' singletons, U either contains all tails $[n, +\infty)$ or their left-hand analogues. By the definition of R^*, U must either succeed all standard numbers, or precede all of them. Moreover, if, say, U lies toward the right, and $X \in U$, then X is cofinal in $\mathbb{Z}$. Hence, m(X) equals $\mathbb{Z}$ itself. It follows that every ultrafilter in ue($\mathbb{Z}$) R^*- precedes U. In particular, the new points at infinity form a cluster of mutually R^*-accessible worlds. (A similar argument shows that the left-infinite points form such a cluster.)

(3) With the *rationals* ($\mathbb{Q}$,<), 'interpolation' occurs, in addition to the above 'extension'. For instance, let U be any ultrafilter on $\mathbb{Q}$ containing all open intervals $(0,q)$, where $q > 0$. Then U becomes an 'infinitesimal', lying to the right of 0, but to the left of all its rational successors. In general, ue($\mathbb{Q}$,<) looks like Figure 2. There are two clusters at infinity, as with $\mathbb{Z}$. But, in addition, each rational lies surrounded by two infinitesimal clusters. And finally, there is a pair of such clusters for each irrational number too.

Sketch of a proof: Let U be any ultrafilter on $\mathbb{Q}$. Consider the case where U contains the nonnegative rationals (the other case being similar). For each $X \in U$, let sup(X) be the supremum of X (in $\mathbb{R}$!) if this exists; sup(X) $= +\infty$, otherwise. *Case 1:* sup(X) $= +\infty$ for all $X \in U$. Then U lies at infinity, containing all right hand tails $[n,+\infty)$, for $n \in IN$. *Case 2:* sup(X) $\in \mathbb{R}$ for some $X \in U$. Then the infimum y exists of all such points sup(X), which must be $\geqq 0$. But now, the $\mathbb{Q}$-interval $[0,y+1]$ must belong to U, and hence so does one of $[0,y)$, $\{y\}$, or $(y,y+1]$. It follows that U is either y or in one of its infinitesimal neighboring clusters. (If y is irrational, the middle possibility cannot occur.)

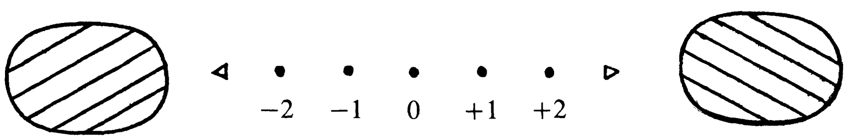


Figure 1.

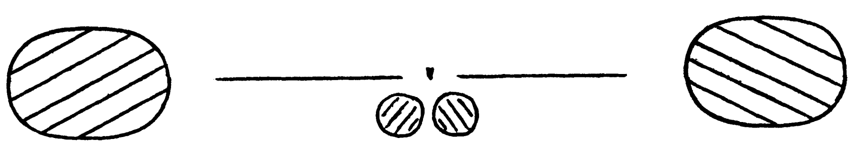


Figure 2.

(4) Finally, for a more complicated example, consider the *binary tree* $(B,<)$ with the relation of succession (see Figure 3). Ultrafilters can now be divided into several kinds:

I. First, there are the principal ultrafilters, i.e., the old nodes.
II. Then, there are free ultrafilters U which are still 'thin', in the following sense.

Definition An ultrafilter U on the binary tree is *thin* if there exists a partial function f: $IN \to B$ picking out at most one node $f(n)$ at each level n of the tree, such that $f[IN] \in U$. (Later on, we shall also use a more general version, where U is to contain some set containing at most one representative from each member of a given disjoint family of sets.)

Prime examples of thin ultrafilters U are those generated by some *branch* τ in B, together with some free ultrafilter V on IN such that τ (when considered as a set of nodes) belongs to U, and more generally, for all subsets X of B,

$$X \in U \text{ iff } \{n \in \mathbb{N} \mid \tau(n) \in X\} \in V.$$

Such ultrafilters again form a cluster of infinite successors of the branch τ (differing only in the choice of the 'index ultrafilter' V). But in this case there is further structure at infinity. For consider the 'immediate successor set' of τ, consisting of all sisters of nodes in τ (see Figure 4). Together with a choice of an index ultrafilter, this successor set generates ultrafilters on B which become (infinitely many!) immediate R^*-successors of the points in the earlier infinite τ-cluster. And so on: at infinity, there lies an infinitely branching infinite tree beyond τ, obtained by considering successor sets of successor sets, etc.

There are also less regular 'thin sets' which can be used, of course—e.g., to produce successors at an infinite distance by choosing ever further successors at consecutive stages. Still, there will always be a connection with one particular branch τ, because of the following sequence of choices which *any* ultrafilter U must make: $\{x \mid 0 \leqq x\}$ or $\{x \mid 1 \leqq x\}$; then, say, $\{x \mid 00 \leqq x\}$ or $\{x \mid 01 \leqq x\}$, etc., which converges toward some specific branch (itself not necessarily a member of U).

III. Finally, there is a third kind of ultrafilter in the binary tree $(B,<)$.

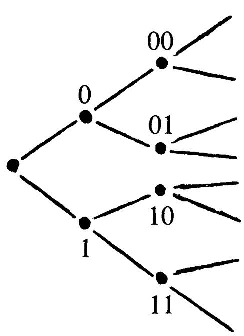


Figure 3.

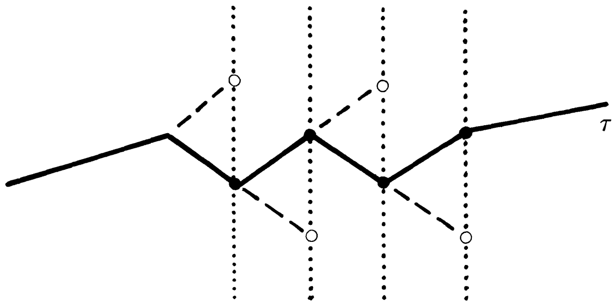


Figure 4.

Definition An ultrafilter is *thick* if it contains all B-complements of 'thin' subsets of B, where the latter contain at most one point at each level. (Note that the set of these complements has the Finite Intersection Property.)

Such thick ultrafilters also lie at infinity, each beyond one particular branch τ.

2.3 Disjoint unions For future purposes, we now look at ultrafilter extensions of *disjoint unions* of frames.

Let $\{F_i \mid i \in I\}$ be a family of frames, with disjoint union $\oplus F_i$. As before, $\mathrm{ue}(\oplus F_i)$ will contain this disjoint union as a subframe. In addition, all *ultrafilter extensions* of these single frames F_j will be included:

Lemma *Every frame* $\mathrm{ue}(F_j)$ *lies embedded as a generated subframe in* $\mathrm{ue}(\oplus F_i)$.

Proof: This can be shown by direct inspection of ultrafilters on the union $\cup W_i$ having W_j as an element. A more elegant argument uses the earlier-mentioned algebraic connection. Recall that any frame F induced a modal algebra $\mathrm{A}(F)$, and the latter induced a Stone representation $\mathrm{S}(\mathrm{A}(F))$ isomorphic to $\mathrm{ue}(F)$. Now, projection onto the j-coordinate is a *modal homomorphism* from $\Pi_{i \in I}\mathrm{A}(F_i)$ onto $\mathrm{A}(F_j)$. By general *duality* then (see [14], [4]), $\mathrm{ue}(F_j) \simeq \mathrm{S}(\mathrm{A}(F_j))$ lies embedded as a *generated subframe* in $\mathrm{S}(\Pi\mathrm{A}(F_i)) \simeq \mathrm{S}(\mathrm{A}(\oplus F_i)) \simeq \mathrm{ue}(\oplus F_i)$.

But $\mathrm{ue}(\oplus F_i)$ also contains other subframes of interest, namely, all *ultraproducts* $\Pi_U F_i$, where U is any (free) ultrafilter on the index set I.

Lemma *Any ultraproduct* $\Pi_U F_i$ *is isomorphic to a subframe of* $\mathrm{ue}(\oplus F_i)$.

Proof: The embedding α is as follows: For any function $f \in \Pi\{W_i \mid i \in I\}$,

$$f_U \overset{\alpha}{\mapsto} \{X \subseteq \cup\{W_i \mid i \in I\} \mid \{n \in I \mid f(n) \in X \cap W_n\} \in U\}.$$

One checks, successively, using the characteristic properties of ultrafilters and ultraproducts, that

(i) $\alpha(f_U)$ is an ultrafilter on $\oplus F_i$ (compare Example 2.2.(4).II),
(ii) this value is the same for all g where $f_U = g_U$,
(iii) α is injective,
(iv) α preserves the relation R both ways.

Ad(iii). Note that f_U is mapped to a 'thin' ultrafilter containing the image of f and all its 'U-sections'. (Actually, on the usual definition of disjoint unions, the functions f in $\Pi_{i \in I} W_i$ *are* subsets of $\oplus W_i$; hence $f \in \alpha(f_U)$.) So if g_U is mapped to the same ultrafilter then $f \in g_U$ and hence, by definition, $\{n \in I \mid g(n) = f(n)\} \in U$; i.e., $f_U = g_U$.

Ad(iv). If $Rf_U g_U$, then $\{n \in I \mid R_n f(n) g(n)\} \in U$. So $\mathrm{m}(g) \in f_U$—and hence $R^* f_U g_U$. (The other direction is similar.)

Thus, $\mathrm{ue}(\oplus F_i)$ contains all frames F_i and also all I-ultraproducts out of these. Moreover, by the definition of the embedding α, no world in any F_i is R-related (either way) to any world in a nontrivial ultraproduct $\Pi_U F_i$. Also, all ultraproducts $\Pi_U F_i, \Pi_{U'} F_i$ with different ultrafilters U, U' are completely unrelated in the same way. (Use the fact that, for some pair X, Y of disjoint subsets of I, $X \in U$, $Y \in U'$, whence $\mathrm{m}(f \cap X) \notin \alpha(g_U)$, for all f, g.)

Still, there may be many other worlds, in general, in $\mathrm{ue}(\oplus F_i)$, viz. those corresponding to 'thick' ultrafilters on $\cup\{W_i \mid i \in I\}$, in the sense of Example 2.2.(4).III. Their location can be R-connected to worlds in the earlier ultraproducts, as in the following illustration.

Example: Let $I = \mathbb{N}$ and F_i be the finite linear order $(\{0, \ldots, i\}, <)$, for $i \in IN$. $\mathrm{ue}(\oplus F_i)$ is a disjoint union of rooted linear orders (compare the observation about the preservation of first-order properties in Section 2.1). Moreover, each F_i lies in it as a *generated* subframe (by the first lemma, using the isomorphism $\mathrm{ue}(F) \cong F$ for finite frames), as shown in Figure 5. Also, as pictured, $\mathrm{ue}(\oplus F_i)$ contains all ultraproducts of the frames F_i, being infinite linear orders of the following form:

$$\text{copy of } (\mathbb{N},<) - \text{copies of } (\mathbb{Z},<) - \text{copy of } (\mathbb{N},>).$$

Finally, the 'thick' ultrafilters in $\mathrm{ue}(\oplus F_i)$, containing all complements of 'thin' sets selecting at most one object in each component W_i, can be located with respect to these as follows.

Let U be a thick ultrafilter. Now, derive an index ultrafilter V as $\{X \subseteq I \mid \cup\{W_j \mid j \in X\} \in U\}$. It is easy to see that U can only be R^*-related to worlds in the ultraproduct $\Pi_V F_i$. Next, by the definition of U, the following thin sets do not belong to U, for fixed $k \in \mathbb{N}$,

- the (images of the) constant functions $\lambda n \cdot k$
- the (images of the) 'diagonal functions' $\lambda n \cdot n - k$.

It follows, by some calculation, that

(i) every world in the initial copy of $(\mathbb{N},<)$ (in $\Pi_V F_i$) has U for an R^*-successor

(ii) no world in the final copy of $(\mathbb{N},>)$ has U for an R^*-successor.

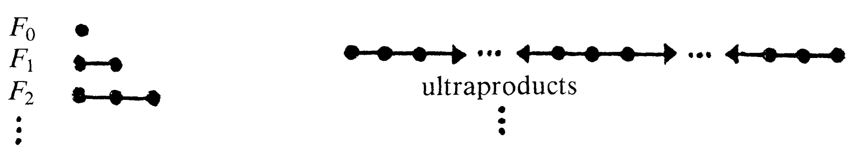


Figure 5.

Therefore, U must have been interpolated somewhere in $\Pi_V F_i$, due to the earlier observation about the linearity of ue($\oplus F_i$).

Finally, a collapse results if the frames F_i all come from some *finite* family of *finite* frames. In that case, all ultrafilters must be 'thin' in the earlier sense, and hence ue($\oplus F_i$) consists of $\oplus\{F_i | i \in I\}$ plus a set of ultrafilters $\Pi_U F_i$, where the latter are actually isomorphic to single frames F_j. So ue($\oplus F_i$) will actually be isomorphic to a mere disjoint union of the original frames.

3 Special logics One way of improving existing definability results in modal logic is by restricting attention to important special kinds of logics. For instance, the usual modal axioms often have special features, syntactic or semantic, which can be exploited.

3.1 Subframe logics In Fine [13], attention is restricted to *subframe logics*, i.e., to modal logics L that are preserved while passing from a frame where they hold to all its subframes. (This holds, e.g., for the usual systems K, T, S4, S5, and also for higher-order systems such as Löb's Logic.) For subframe logics containing K4, and axiomatized by special axiom types, Fine then proves a nice characterization of frame classes definable in terms of such logics. Moreover, he shows that such special logics L are better behaved than modal logics in general. Notably, the following equivalence holds: *L is canonical iff L is elementary and complete.* Here, a modal logic L is *complete* if it coincides with the full modal theory of its associated class of frames, i.e., $L = \{\phi | F \models \phi$ for all F with $F \models L\}$. Next, L is *elementary* if $\{F | F \models L\}$ has a definition in the first-order language of Section 2.1 with R and identity. And finally, L is *canonical* if its truth is preserved in passing from a so-called *descriptive* general frame (i.e., one isomorphic to some Stone Representation S($\mathfrak{A}$)) in which it holds to the underlying full frame. From [10], we know that all complete and elementary modal logics are canonical, but the converse does not hold for modal logics in general. More precisely, it is not hard to see that all canonical logics are complete, but they need not be elementary.

Using the results of Section 2.3, this can be improved somewhat.

Theorem *For all subframe logics L, the following are equivalent:*

(i) *L is elementary*
(ii) *L is preserved under ultrafilter extensions.*

Proof: The entailment from (i) to (ii) was proved in [1] for arbitrary modal logics L. Conversely, it suffices to show that L is preserved under ultraproducts (cf. [3]). So consider $\Pi_U F_i$, with $F_i \models L$ for each $i \in I$. Then $\oplus F_i \models L$, L being modal, and so ue($\oplus F_i$) $\models L$, by assumption. But then, $\Pi_U F_i \models L$, since L is a subframe logic, and $\Pi_U F_i$ lies embedded in ue($\oplus F_i$), by the lemma in Section 2.3.

Corollary *For all subframe logics L, the following are equivalent:*
(i) *L is canonical*
(ii) *L is elementary and complete.*

Proof: By earlier observations, (ii) always implies (i). Conversely, canonicity always implies completeness. Moreover, it also implies preservation under ultrafilter extensions, and so elementarity follows in the present case.

3.2 Natural logics Another important special class of modal formulas, originally studied in [26] and [10], are the *natural* ones, defined as those preserved in passing from a *refined* general frame $\langle F, \mathcal{W}\rangle$ satisfying

$$\forall xy(x = y \leftrightarrow \forall A \in W(Ay \rightarrow Ax)),$$

$$\forall xy(Rxy \leftrightarrow \forall A \in W(Ay \rightarrow x \in \mathrm{m(A)})),$$

to the underlying full frame. Stone representations of modal algebras are refined—and in fact, any general frame can be p-morphically contracted to a refined general frame with the same modal theory. Refined general frames also play a key role in the categorial study of [24].

One useful aspect of refined general frames is that they can be isomorphically embedded in their Stone representations (just like full frames in Section 2.2). Now, call a frame F_2 a *compactification* of a frame F_1 if F_2 is the Stone representation of (F_1, W) for some refined general frame (F_1, W). Thus, F_2 is a more parsimonious version of the full $\mathrm{ue}(F_1)$ (see Section 2). For instance, to continue Example 2.2.(2), the frame consisting of $\mathbb{Z}$ with only two added points at infinity is the Stone representation of the following refined general frame: $(\mathbb{Z},<)$ with the Boolean algebra generated by all finite sets together with the positive integers. In general, there is a whole family of compactifications for F_1, all being p-morphic images of the full frame $\mathrm{ue}(F_1)$.

As we have seen, in general, the truth of modal formulas need not be preserved in going from compactifications to the original frames. But for natural logics L this transition is valid. For let F_2 be the Stone representation of some refined (F_1, W) such that $F_2 \models L$. *A fortiori*, L holds in that general frame on F_2 to which (F_1, W) is isomorphic, and hence $(F_1, W) \models L$. But then, $F_1 \models L$ (as L is natural). This observation leads to the following result, extending one in [1]:

Theorem *A class of frames is definable by means of some natural set of modal formulas iff it satisfies the known closure conditions for definability with canonical sets, but with closure under ultrafilter extensions strengthened to closure under compactifications. (I.e., the class should be closed under the formation of generated subframes, disjoint unions, p-morphic images and compactifications, with the latter requirement also holding for its complement.)*

Proof: If K has a natural definition, then it is closed under generated subframes, p-morphic images, and disjoint unions, because of modal definability as such. Moreover, as natural logics are canonical, K is closed under ultrafilter extensions, and hence under compactifications (the latter being p-morphic images of the former). Finally, the similar closure condition on the complement was proved in the above observation.

Conversely, if K has all these closure properties, then it has at least a canonical modal definition, say L. But this is not all. Let (F_1, W) be any refined frame validating L. Then L also holds in its Stone representation F_2, viewed as a general frame. Since this general frame is *descriptive*, and L is canonical, L must also hold in the full frame F_2. But then, by the closure condition on the complement of K, $F_1 \models L$. So L is in fact natural.

As is well-known, all natural logics are elementary (see [10]). In fact, the above argument showed that all elementary modal logics are preserved under compactifications. The converse fails, however (and thus, the theorem in Section 3.1 does not generalize as might be expected). For instance, all *canonical* logics are preserved in this way (but not all of these are elementary, in general).

Remark: All results in this section, as well as those to come, refer to *global* definability, in terms of truth in all worlds of frames. But *local* versions referring to frames (F, w) with distinguished worlds may be pursued too (cf. [5] or [7]).

4 Special frame classes Another way of improving existing definability results in modal logic is by restricting attention to important special classes of frames, such as *linear* frames (for tense logic), or *well-founded* frames (plausible in general). Perhaps the simplest case is that of *finite frames*. There is a good deal of interest these days in so-called Finite Model Theory (see [16]), refuting the common assumption that this is a marginal, 'easy' area. In modal logic as well there are many interesting questions. These often arise in specializing general concerns to finite frames. Sometimes this leads to a decrease in complexity. For instance, *valid frame consequence* is highly complex in general (cf. [27]), but on the finite frames its complexity goes down to at most Π^0_1. An open question is if it even becomes *decidable*. On the other hand, general issues may also become more complex on finite structures; as general types of argument based on compactness or Löwenheim-Skolem theorems will not transfer to this restricted non-elementarily definable subuniverse. We shall encounter both phenomena in what follows.

4.1 Counterexamples How to characterize the modally definable classes of finite frames? One would like to accomplish this without using the earlier infinitary constructions $\mathrm{ue}(\oplus F_i)$ for infinite index sets I.

One reasonable approach seems to be this: In the underlying algebraic Birkhoff Theorem, if some *finite* algebra $\mathfrak{A}$ validates all identities true in some class K of *finite* algebras, will $\mathfrak{A}$ then be a homomorphic image of some subalgebra of some *finite* product of algebras in K? If so, then we are done, because the transfer to frames will involve only ultrafilter extensions of (finite disjoint unions of) finite frames, i.e., just those frames themselves. But actually this finitized version of the Birkhoff Theorem fails, as is well-known from the algebraic literature on so-called 'pseudo-varieties' (cf. [22]).

Example (in the similarity type $\langle 1 \rangle$): The two-point algebra with a 2-cycle for the unary function is not a homomorphic image of (a subalgebra of) any k-cycle with k an odd prime. Moreover, it is not a homomorphic image of any (subalgebra of a) finite product of such algebras (these will be k-cycles with odd k). But the 2-cycle is indeed a homomorphic image of certain subalgebras of the *infinite* product of all such k-cycles, for the latter contains subalgebras isomorphic to the integers with the successor function.

The finitized Birkhoff Theorem does hold, though for *locally finite* varieties of algebras. But, in the field of modal logic, this only helps in very special cases, e.g., when all frames are restricted to some *fixed* finite *depth.* (In that case, a finite set of formulas $\{\neg, \wedge, \Box\}$-generates only a finite set of formulas again, up to modal equivalence.)

But, in general, there are modal counterexamples to the desired simplification, related to the earlier algebraic one.

Proposition *The reflexive one-point frame F validates the modal theory of the class K of finite linear orders with the relation of immediate succession, without being constructible from these by generated subframes, (finite) disjoint unions, and p-morphisms alone.*

Proof: If $F \nvDash \phi$, then ϕ fails in $(\mathbb{N}, S)$ (as F is a p-morphic image of the latter infinite frame). But then, restricting to a finite subframe of length greater than the *modal* operator *depth* of ϕ, ϕ already fails in some element of K. It follows that F is not constructible as indicated since p-morphic inverses of F must satisfy *succession*: $\forall x \exists y Rxy$.

On the other hand, F can indeed be obtained from the ultrafilter extension of the disjoint union of K, as the latter contains isomorphs of $(\mathbb{N}, S)$. (This may be seen in the examples of Section 2.3.)

4.2 A lead from the transitive case For the moment let us consider an additional restriction. The above counterexample depended heavily on the use of nontransitive frames. By another type of argument, however, we can prove the following result, which is the frame version of a (special case of a) known result from universal algebra on varieties with equationally definable congruences (cf. [6]).

Theorem *On the finite transitive frames, a class of frames is modally definable iff it is closed under the formation of generated subframes, finite disjoint unions, and p-morphic images.*

Proof: Only the 'if'-direction requires proof. Let $F \vDash \mathrm{Th}_{\mathrm{mod}}(K)$, where F is finite. It suffices to show that F is constructible from K by the above three operations. Moreover, as F is a p-morphic image of the disjoint union of its rooted generated subframes, it suffices to consider the latter. Now, for any finite rooted frame (F, w), the *Jankov-Fine formula* $\phi_{F,w}$ (cf. [11]) may be constructed, being a modal formula with the following property: for any frame G and $v \in G$, $\phi_{F,w}$ can be verified in (G, v) iff there exists a p-morphism from the subframe of G generated by v as a root, onto (F, w). In particular, (F, w) verifies $\phi_{F,w}$ with some canonical valuation V, and so $\neg\phi_{F,w}$ cannot belong to the modal theory of K. That is, $\phi_{F,w}$ may be verified somewhere in K: and (F, w) must be a p-morphic image of some generated subframe of an element of K.

This result may yet be extended to so-called *n-transitive* frames, making each node accessible from the root in at most n R-successor steps.

The Jankov-Fine formulas describe (F, w) by means of proposition letters p_x for each world x in F, encoding R-relations by modal operators (e.g., if Rxy, one employs $(p_x \rightarrow \Diamond p_y) \wedge \Box(p_x \rightarrow \Diamond p_y)$). As R is transitive, the combinations

$\psi \wedge \Box\psi$ enforce the truth of subformulas for the whole frame as viewed from w. If R is not transitive, then, in general, one can only enforce this truth up to finite depths, using increasing combinations $\psi \wedge \Box\psi \wedge \Box\Box\psi \ldots$. (For n-transitive frames, of course, one suitably large iteration will still do the job.) The truth of the resulting formulas at locations (G, v) will only enforce the existence of *n-p-morphisms* from (G, v) to (F, w) — where only n iterations are allowed for the relational clauses in the definition of 'p-morphism', starting from the matching pair v, w.

Even so, this is a quite reasonable refined notion, concerning which we make the following general observation. Call a rooted frame F a *local p-morphic image* of K if, for each n, F is an n-p-morphic image of some frame in K. It is easy to see that local p-morphic images validate the full modal theory of K. (For each formula ϕ in that modal theory, consider n-p-morphisms with n equal to the modal operator depth of ϕ, and use the obvious refined version of the standard p-morphism lemma.) But then, combining with the earlier arguments, we have obtained a general result after all:

Theorem *On the finite frames, a class of frames is modally definable iff it is closed under the formation of generated subframes, (finite) disjoint unions, and local p-morphic images.*

This result receives a nice generalization in [20], where it is shown that mere closure under p-morphic images is just enough for the definability of classes of finite frames in terms of 'modal sequents.'

4.3 Elementary classes Finally, we consider an additional restriction which is of general logical interest. Which modal classes of finite frames are *elementary*? Here 'elementary' cannot be taken in the usual liberal sense of being definable by means of some *set* of first-order sentences (EC_Δ). For every class K of finite frames which is closed under isomorphisms is thus definable (relative to the universe of finite frames): take the set of all negations of all complete R-descriptions of the finite frames outside of K. Rather, we are interested in definability by means of a single first-order sentence.

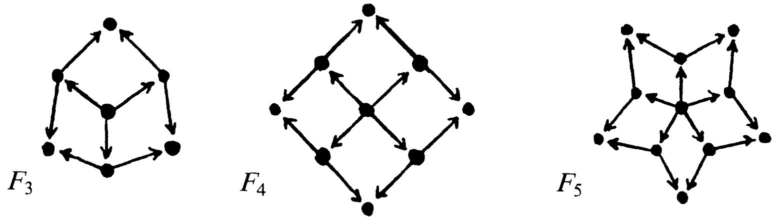
Even so, it might be thought that all modal formulas are elementary in this sense on the finite frames. But a variant of the well-known McKinsey Axiom $\Box\Diamond p \rightarrow \Diamond\Box p$ refutes this:

Theorem *There exist nonelementary modal formulas on the universe of finite frames.*

Proof: Consider the sequence of finite frames shown in Figure 6 (having the transitive closure of the indicated arrows for their alternative relation). The following formula ϕ is valid on precisely those frames F_n with *odd* index n:

$$(\Diamond\Diamond T \wedge \Box(\Diamond T \rightarrow \Diamond p)) \rightarrow \Diamond(\Diamond T \wedge \Box p).$$

(The only relevant check is in the central point, where the antecedent says that each immediate successor sees at least one endpoint with p. Only with odd index n does this imply that at least two adjoining endpoints have p, i.e., that the consequent holds.)



, etc.

Figure 6.

But then, by a Fraïssé-type game argument, it can be shown that no single first-order sentence $\alpha(R,=)$ can define ϕ: as α would also have to hold in (suitably) large *even* F_n. (See [9] for the full argument.)

This counterexample works for all (transitive) frames of fixed finite *depth.* Are all modal formulas elementary on the finite frames, however, if attention is restricted to some fixed finite *width* of branching? (At least for the case of *intuitionistic* propositional formulas an affirmative answer has been provided by P. Rodenburg, via personal communication.)

Remark: The preceding theorem suggests a more hierarchical perspective upon first-order definability. Some modal formulas lack a first-order equivalent on the *finite* frames already. Other modal formulas have one on the finite frames, but not on the *countable* ones. (An example is Löb's Axiom, whose finite frames are the transitive irreflexive ones.) But there are also modal principles which are elementary on the countable frames, though no longer on the *uncountable* ones. (For an example, going back to an axiom in [12], see [9].) What is the smallest cardinality where all nonelementary behavior has become manifest?

There remain several further questions. For instance, what about the first-order definability of the earlier Jankov-Fine formulas, e.g., on the transitive frames? In general, these will not be elementary.

Example: Consider the following transitive frame (F, w):

Its Jankov-Fine formula $\neg\phi_{F,w}$ holds in $(\mathbb{N},<)$: no p-morphism runs from there to (F, w). But it does not hold in the elementary extension $(\mathbb{N},<) \oplus (\mathbb{Z},<)$, which can be mapped p-morphically onto (F, w).

However, first-order definability for such formulas might still hold within the universe of *finite* frames. We will not pursue this, stating only a more restricted result.

Proposition *On the universe of irreflexive transitive frames, the Jankov-Fine formulas are all elementary.*

The reason for this is that finite irreflexive transitive frames have a layered 'inductive' shape, which can be used to describe the necessary relational pattern in p-morphic inverse images.

To conclude, we formulate a more general issue.

Question: How can we find a general structural characterization of first-order definability for modal formulas on the finite frames?

One answer comes immediately from standard model theory: it is necessary and sufficient for such a characterization to be invariant for *Ehrenfeucht-Fraïssé games* over n rounds, for some $n \in \mathbb{N}$. But can this be improved? For instance, if a modal formula ϕ is elementary on the universe of all frames, then it is equivalent to some *restricted* sentence (compare Section 2.1). Does this preservation theorem also hold in the finite case? Then, attention could be restricted to games in which each successive choice must be an R-successor of some earlier one. (For some warning examples of failure of transfer to the finite case, however, see [16].) Moreover, the bounded game approach suggests a further question of syntactic fine-structure, namely: what a priori restrictions can be given on the *quantifier-complexity* of first-order equivalents for given modal formulas? Inspection of the usual examples motivates the following

Conjecture: If ϕ has a first-order definition at all, then it has one of the form $\forall x\, \alpha(x)$, where α has quantifier depth equal to the modal operator depth of ϕ.

Even further syntactic refinements, concerning the number of bound variables employed, might be introduced using the pebbling games of [18].

5 Frame equivalence For a final kind of restriction on general definability theorems, one can make special assumptions on the frame class K involved. One simple case is when K itself consists of a *single frame*: and we arrive at the question which frames are *modally equivalent* ($\equiv_m$), i.e., have the same modal theory, as a given frame.

In general, it follows from earlier results that, if $F_1 \equiv_m F_2$, then ue(F_1) is a generated subframe of a p-morphic image of an ultrafilter extension of some disjoint union of copies of F_2. This is not very informative.

5.1 Finite frames For finite frames F_2, however, more can be said. As was shown in Section 2.3, the frame ue$(\oplus F_2)$ must be isomorphic to some disjoint union of copies of F_2. So F_1 can be obtained from F_2 using only generated subframes, p-morphic images, and disjoint unions.

Moreover, evidently, for *finite* F_1 only finite disjoint unions are needed (compare the discussion in Section 4.1). And for *rooted* finite F_1, generated from some initial world, disjoint unions are not even needed at all:

(*) if F_1, F_2 are rooted finite frames with $F_1 \equiv_m F_2$, then one is obtainable from the other by generated subframes and p-morphic images.

This observation yields a folklore result:

Proposition *For rooted finite frames, modal equivalence coincides with isomorphism.*

Proof: Finite frames related as in (*) must be isomorphic.

This is again the frame version of a well-known algebraic fact: two finite subdirectly irreducible algebras in a congruence distributive variety generate the same variety iff they are isomorphic.

5.2 Well-orders Special classes of *infinite* frames are also worthy of study. For instance, on the universe of *well-orders*, modal equivalence can be characterized completely.

Theorem *The well-orders of ordinal types $\omega \cdot k + n$ ($k \leqq \omega$, $n < \omega$) all have distinct modal theories. But, from then upward, for $k \geqq \omega$, $\omega \cdot k + n \equiv_{\mathrm{m}} \omega \cdot \omega + n$.*

Proof: This theorem can be proved by a *filtration* argument, along the lines of the proof of Theorem II.2.1.6 in [2]. (But the present case is simpler.)

The main steps in the argument are these:

I. $(\alpha,<) \equiv_{\mathrm{m}} (\beta,<)$ implies equality of 'final parts': for some limit ordinals α',β' and some $n < \omega$, $\alpha = \alpha' + n$, $\beta = \beta' + n$.
II. For $k_1 \neq k_2 \leqq \omega$, $n < \omega$, $(\omega \cdot k_1 + n,<)$ and $(\omega \cdot k_2 + n,<)$ can be told apart modally by considering modal formulas exploiting the existence of $\max(k_1,k_2)$ subsequent ω-sequences. Note that this inequality is a one-way affair: e.g., $\omega \cdot 2 + 3$ is isomorphic to a generated subframe of $\omega \cdot 3 + 3$, and hence it validates the complete modal theory of the latter frame. (It is the converse which fails.)
III. For $k \geqq \omega$, however, no such distinctions are possible: if ϕ can be falsified on $\omega \cdot k + n$, then, by filtration, it can be falsified on some finite 'cluster line.' The latter again can be 'unfolded' to transfer the counterexample to any frame with enough room, i.e., enough infinitely ascending sequences for unfolding some arbitrary finite number of clusters: $\omega \cdot \omega$ is the smallest ordinal suitable for this.

Remark: See [25] for a related result. Compare also [21] for a characterization of *first-order* elementary equivalence on well-orders (involving distinguishability up to ω^{ω}). To obtain a characterization of modal equivalence on arbitrary linear frames, one might use the model-theoretic proof of *Bull's Theorem* on the extensions of S4.3 (as presented in [12]). Finally, a more extensive classification for *tense-logical* equivalence of frames in the above spirit may be found in [19].

Could the above result also be derived directly from existing general definability results? Such a derivation would require a study of ultrafilter extensions for well-orders. This is quite feasible. What happens is merely that an ordinal α acquires reflexive clusters (as in Section 2) in front of each *limit* and at the end (if it is a limit itself). Then, again, one can look at the structure of ultrafilter extensions of disjoint unions.

But there is something perverse about such a way of proceeding, i.e., using infinitary constructions where a simple finitary (filtration) argument suffices. Still, there is indeed a connection between the two methods. Forming a filtration may be described as performing a *truncated version* of the ultrafilter representation:

Given a frame F and some Boolean set algebra on F generated by a *finite* number of subsets $X_1, \ldots, X_n$ (admitting only modal operations 'm' to some fixed threshold), one forms the ultrafilters on this finite algebra $\mathfrak{A}$, stipulating that (see Section 2.1):

$U_1 R U_2$ iff, for each set of the form $\mathrm{m}(X) \in \mathfrak{A}$
(with $X \in \mathfrak{A}$), $X \in U_2$ implies $\mathrm{m}(X) \in U_1$.

All the usual facts about filtrations can be restated in this way. Admittedly, nothing would be gained in the above proof by recasting the argument along these lines. Still, it does show that the approach of earlier sections can be 'fine-tuned.'

One possible application is this. If a modal formula is not valid in a frame, it fails already in one of its finite filtrations. So modally definable frame classes K have the following structural *closure property*:

if all finite filtrations of a frame F belong to K, then F itself belongs to K.

Can this be used to improve existing definability theorems?

REFERENCES

[1] van Benthem, J., "Canonical modal logics and ultrafilter extensions," *The Journal of Symbolic Logic*, vol. 44 (1979), pp. 25–37.

[2] van Benthem, J., *The Logic of Time*, Reidel, Dordrecht, 1983.

[3] van Benthem, J., "Correspondence theory," pp. 167–247 in *Handbook of Philosophical Logic*, vol. 2, eds. D. Gabbay and F. Guenthner, Reidel, Dordrecht, 1984.

[4] van Benthem, J., *Modal Logic and Classical Logic*, Bibliopolis, Napoli, 1986.

[5] Blok, W. and P. Köhler, "Algebraic semantics for quasi-classical modal logics," *The Journal of Symbolic Logic*, vol. 48 (1983), pp. 941–964.

[6] Blok, W. and D. Pigozzi, "On the structure of varieties with equationally definable principal congruences," *Algebra Universalis*, vol. 15 (1982), pp. 195–227.

[7] Chagrov, A., "Mnogoobrazija logicheskich matrits," *Algebra i Logika*, vol. 24 (1985), pp. 426–489.

[8] Chang, C. and H. Keisler, *Model Theory*, North Holland, Amsterdam, 1973.

[9] Doets, H., *Completeness and Definability*, Dissertation, Mathematisch Instituut, University of Amsterdam.

[10] Fine, K., "Some connections between elementary and modal logic," pp. 15–31 in *Proceedings of the Third Scandinavian Logic Symposium. Uppsala 1973*, ed. S. Kanger, North Holland, Amsterdam, 1975.

[11] Fine, K., "An ascending chain of S4 logics," *Theoria*, vol. 40 (1974), pp. 110–116.

[12] Fine, K., "Logics extending K4. Part I," *The Journal of Symbolic Logic*, vol. 39 (1974), pp. 31–42.

[13] Fine, K., "Modal logics containing K4. Part II," *The Journal of Symbolic Logic*, vol. 50 (1985), pp. 619–651.

[14] Goldblatt, R., "Metamathematics of modal logic," *Reports on Mathematical Logic*, vol. 6 (1976), pp. 41–77, vol. 7 (1977), pp. 21–52.

[15] Goldblatt, R., "Varieties of complex algebras," Manuscript, Department of Mathematics, Victoria University, Wellington.

[16] Gurevich, Y., "Logic and the challenge of computer science," Report CRL-TR-10-85, Computing Research Laboratory, The University of Michigan, Ann Arbor.

[17] Hughes, G. and M. Cresswell, *A Companion to Modal Logic*, Methuen, London, 1984.

[18] Immerman, N., "Upper and lower bounds for first-order expressibility," *Journal of Computer and Systems Sciences*, vol. 25 (1982), pp. 76–98.

[19] de Jongh, D., F. Veltman, and R. Verbrugge, "All logics for homogeneous linear time," Institute for Language, Logic and Information, University of Amsterdam.

[20] Kapron, B., "Modal sequents and definability," *The Journal of Symbolic Logic*, vol. 52 (1987), pp. 756–762.

[21] Monk, D., Mathematical Logic, Springer-Verlag, Berlin, 1975.

[22] Reiterman, J., "The Birkhoff theorems for finite algebras," *Algebra Universalis*, vol. 14 (1982), pp. 1–10.

[23] Rodenburg, P., *Intuitionistic Correspondence Theory*, Dissertation, Mathematisch Instituut, University of Amsterdam.

[24] Sambin, G. and V. Vaccaro, "Topology and duality in modal logic," *Annals of Pure and Applied Logic*, vol. 37 (1988), pp. 249–296.

[25] Segerberg, K., "Modal logics with linear alternative relations," *Theoria*, vol. 36 (1970), pp. 301–322.

[26] Thomason, S. K., "Semantic analysis of tense logics," *The Journal of Symbolic Logic*, vol. 37 (1972), pp. 150–158.

[27] Thomason, S. K., "Reduction of second-order logic to modal logic," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 21 (1975), pp. 107–114.

Faculty of Mathematics and Computer Science
University of Amsterdam