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Duality and Completeness for US-Logics

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Abstract The semantics of e-models for tense logics with binary operators for ‘until’ and ‘since’ (US-logics) was introduced by Bellissima and Bucalo in 1995. In this paper we show the adequacy of these semantics by proving a general Henkin-style completeness theorem. Moreover, we show that for these semantics there holds a Stone-like duality theorem with the algebraic structures that naturally arise from US-logics.

1 Introduction In 1968 Kamp [5] introduced tense logics with binary operators U and S (‘until’ and ‘since’). Interest and research about these logics have rapidly grown since then, particularly in connection with computer science. The operators U and S arose from semantical intuitions connected with the concept of Kripke model: thus their semantics preceded the syntactical aspects. But the traditional relational semantics, notwithstanding the validity of general completeness theorems (see Xu [8]), turned out to be strongly inadequate for a global and systematic treatment of US-logics. From our point of view, the heart of this inadequacy is the fact that, when Kripke models are employed for US-logics, the distinguishable model theorem fails, in the sense that there exist models in which the presence of equivalent points cannot be eliminated. In other words, such models are not equivalent to any distinguishable (i.e., indeed, without equivalent points) model. This fact prevents the usual construction of canonical models and also prevents a Stone-like duality theory between frames and Boolean algebras with operators, as it holds instead for modal and tense logics with unary operators (see, e.g., Bull and Segerberg [2]). In fact, both in canonical models and in models arising from algebras, the points (which are maximal consistent extensions of a logic in the former case and ultrafilters of the originating algebra in the latter one) are taken exactly once.

In Bellissima and Bucalo [1] a new kind of semantics was introduced based on the notion of *e-model* which is a generalization of that of a Kripke model. For these models the filtration theorem and, consequently, the distinguishable model theorem have been proved. In the present paper we show that the concept of e-model and the

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related ones of e-frame and general e-frame can be legitimately considered the right ones for US-logics, in the sense that there holds a Stone-like duality theorem between general e-frames and algebras with operators u and s, that is, a theorem which respects the classical duality theory for Boolean algebras. Such a theorem is achieved by determining the class of general e-frames which can be obtained as duals of algebras, namely, the *descriptive* general e-frames. By using e-models it is thus possible to construct canonical models and to obtain in such a way a general Henkin-style completeness theorem.

2 *Preliminaries and basic results* The US-language consists of a denumerable set of propositional variables (which we indicate with $p, q, r, \ldots$), the usual Boolean connectives (including $\top$ and $\bot$, i.e., constants for truth and falsehood, respectively), and two binary operators U and S. Formulas are defined as usual: the clauses for U and S state that, if φ and ψ are formulas, then $U(\varphi, \psi)$ and $S(\varphi, \psi)$ are formulas. The intended meaning of $U(\varphi, \psi)$ $[S(\varphi, \psi)]$ is that φ will be true at some future time [was true at some past time], and until then, that is, at any time between now and that moment [since then, i.e., at any time between that moment and now], ψ will be true [was true]. According to this interpretation, traditional unary tense operators are defined as follows: $F\varphi$ ('it will be the case that φ') stands for $U(\varphi, \top)$, $P\varphi$ ('it was the case that φ') for $S(\varphi, \top)$, $G\varphi$ ('it is always going to be the case that φ') for $\neg F\neg\varphi$, and $H\varphi$ ('it has always been the case that φ') for $\neg P\neg\varphi$. Now we define US-logics.

Definition 2.1 A *US-logic* is a set of formulas of the US-language which contains all classical tautologies together with the following axioms:

1. $G(p \to q) \to (U(p,r) \to U(q,r)) \wedge (U(r, p) \to U(r,q))$,
2. $H(p \to q) \to (S(p,r) \to S(q,r)) \wedge (S(r, p) \to S(r,q))$,
3. $p \wedge U(q,r) \to U(q \wedge S(p,r), r)$,
4. $p \wedge S(q,r) \to S(q \wedge U(p,r), r)$,

and is closed under uniform substitution, modus ponens, and temporal generalization.

If we denote by K_{US} the minimal US-logic, we have that $\mathsf{K}_{US} \vdash \varphi$ if and only if φ is true in all Kripke models (see [8]; K_{US} is called $TL_{US}(\varnothing)$ there).

We now define the semantics of e-frames, e-models, and so on, following [1] (we remark that we shall never omit the "e" when referring to these semantics, hence when we write just "frame," "model," and so on, it means that we are referring to the standard relational semantics). Given two points x, y of a Kripke frame $\langle W, R\rangle$ such that xRy, we write $[x, y]$ for the set $\{z : xRzRy\}$ (note that R does not need to be a reflexive relation, so it may happen that $x, y \notin [x, y]$; moreover, to be pedantic, we should write $[x, y]_R$, but we shall drop the subscript to simplify our notation).

Definition 2.2 An *e-frame* $\mathcal{F}$ is a triple $\langle W, R, \beta\rangle$, where $\langle W, R\rangle$ is a Kripke frame, and β is a function from R into $\mathcal{P}(\mathcal{P}(W))$ such that, for all $(x, y) \in R$:

1. $\beta(x, y) \neq \varnothing$;
2. if $Z \in \beta(x, y)$, then $Z \subseteq [x, y]$.

An *e-model* over an e-frame $\mathcal{F} = \langle W, R, \beta\rangle$ is a pair $\langle \mathcal{F}, V\rangle$ where V is a function mapping propositional variables to subsets of W. The definition of truth of a formula in a point x of an e-model is standard for propositional variables and Boolean connectives. Furthermore,

1. $x \models U(\varphi, \psi)$ iff there exists a point y such that xRy and $y \models \varphi$, and there exists $Z \in \beta(x, y)$ such that $z \models \psi$, for each $z \in Z$,
2. $x \models S(\varphi, \psi)$ iff there exists a point y such that yRx and $y \models \varphi$, and there exists $Z \in \beta(y, x)$ such that $z \models \psi$, for each $z \in Z$.

Thus the elements of $\beta(x, y)$ are sets Z of points between x and y such that it is "enough" for ψ to hold in Z to have $U(\varphi, \psi)$ true at x. Naturally, if for any $(x, y) \in R$ we have $\beta(x, y) = \{[x, y]\}$, then the concept of e-model coincides with that of Kripke model. Therefore the semantics of e-models extends Kripke semantics. Observe that in any e-model the truth definition for F, P, G, and H coincides with the usual one for unary tense operators because β plays no part in it.

We recall that a *distinguishable* model is a model which has no equivalent points. We extend this terminology to e-models.

Proposition 2.3 *For each e-model* $\mathbf{M}$ *there exists a distinguishable e-model* $\mathbf{M}'$ *such that* $\mathbf{M} \equiv \mathbf{M}'$.

Proof: See [1], Theorem 3.7. □

Definition 2.4 A *general e-frame* is a four-tuple $\langle W, R, \beta, \Pi \rangle$, where $\langle W, R, \beta \rangle$ is an e-frame and Π is a subset of $\mathcal{P}(W)$ containing $\varnothing$ and W, and closed under Boolean operations and under the operators

$$u(B, C) = \{x \in W : \exists y \in B : xRy \text{ and } \exists Z \in \beta(x, y) : Z \subseteq C\}$$

and

$$s(B, C) = \{x \in W : \exists y \in B : yRx \text{ and } \exists Z \in \beta(y, x) : Z \subseteq C\}.$$

Clearly, an e-model over a general e-frame $\mathcal{F} = \langle W, R, \beta, \Pi \rangle$ is a pair $\langle \mathcal{F}, V \rangle$ where V is a function mapping propositional variables to elements of Π. We say that a general e-frame $\langle W, R, \beta, \Pi \rangle$ is *β-upward closed* if, for any $(x, y) \in R$, the set $\beta(x, y)$ is upward closed with respect to set-theoretical inclusion, relatively to the set $[x, y]$.

Remark 2.5 Any general e-frame $\mathcal{F} = \langle W, R, \beta, \Pi \rangle$ is equivalent to the β-upward closed general e-frame $\mathcal{F}' = \langle W, R, \beta', \Pi \rangle$ where for any $(x, y) \in R$, $\beta'(x, y)$ is the set-theoretical upward closure of $\beta(x, y)$ (i.e., $\beta'(x, y) = \{Z' : Z \subseteq Z' \subseteq [x, y]$ for some $Z \in \beta(x, y)\}$). Therefore we may always consider a general e-frame as β-upward closed.

3 Canonical e-models Given any US-logic L, we define its *canonical e-model* $\mathbf{M}_L$ as $\langle W_L, R_L, \beta_L, V_L \rangle$ where

1. W_L is the set of all maximal consistent extensions of L,
2. xR_Ly iff $U(\varphi, \top)$ (i.e. $F\varphi$) belongs to x for any $\varphi \in y$,
3. let xR_Ly and $Z \subseteq [x, y]$; then $Z \in \beta_L(x, y)$ iff for any φ, ψ such that $\varphi \in y$ and, for each $z \in Z$, $\psi \in z$, it holds $U(\varphi, \psi) \in x$,
4. $V_L(p) = \{x \in W_L : p \in x\}$, for any variable p.

Lemma 3.1 (Fundamental Lemma) *For any formula χ and any $x \in W_L$, it holds $\mathbf{M}_L \models \chi[x]$ if and only if $\chi \in x$.*

To prove Lemma 3.1, we need some preliminary definitions and results, already used by Xu in his completeness proof (see [8]; the technique was used first in Burgess [3]).

Definition 3.2 Let $x, y \in W_L$, and let ψ be a formula. We write $\rho(x, \psi, y)$ to indicate that $U(\varphi, \psi) \in x$ for every $\varphi \in y$.

Remark 3.3 It is straightforward to see that xR_Ly if and only if $\rho(x, \top, y)$ and therefore that if $\rho(x, \psi, y)$ for some ψ, then xR_Ly.

Proposition 3.4

(i) Let $x, y \in W_L$, and let ψ be a formula. Then $\rho(x, \psi, y)$ iff $S(\varphi, \psi) \in y$ for every $\varphi \in x$.

(ii) Let $x \in W_L$, and let $U(\varphi, \psi) \in x$. Then there is a y such that $\rho(x, \psi, y)$ and $\varphi \in y$.

(iii) Suppose that $\rho(x, \psi, y), \neg U(\vartheta, \xi) \in x$ and $\vartheta \in y$. Then there is a $t \in W_L$ such that $\rho(x, \top, t)$, $\rho(t, \top, y)$, $\psi \in t$, and $\neg\xi \in t$.

Proof: For (i) and (ii), see [3], Lemma 2.3 and 2.4, respectively. For (iii), see [8], Lemma 2.4. □

Proof of Lemma 3.1: By induction on the complexity of χ. The only nontrivial cases are $\chi = U(\varphi, \psi)$ and $\chi = S(\varphi, \psi)$. We only consider the case of U (the one for S being proved in the same way, thanks to Proposition 3.4(i)). Suppose then that $\mathbf{M}_L \models U(\varphi, \psi)[x]$. Then by definition of truth in an e-model we have that there exists $y \in W_L$ such that xR_Ly and $\mathbf{M}_L \models \varphi[y]$, and there exists a $Z \in \beta_L(x, y)$ such that, for any $z \in Z$, $\mathbf{M}_L \models \psi[z]$. By induction hypothesis we get $\varphi \in y$ and, for any $z \in Z$, $\psi \in z$. But $Z \in \beta_L(x, y)$, and so by definition of β_L we obtain $U(\varphi, \psi) \in x$. Conversely, suppose $U(\varphi, \psi) \in x$. By Proposition 3.4(ii) there exists a y such that $\varphi \in y$ and $\rho(x, \psi, y)$, and by Remark 3.3 we have xR_Ly. Now let $Z = \{z : \mathbf{M}_L \models \psi[z]\} \cap [x, y]$. Suppose by contradiction that $Z \notin \beta_L(x, y)$. Again by definition of β_L, this means that there exists a ϑ and a ξ such that $\vartheta \in y$, $\xi \in z$ for any $z \in Z$, and $\neg U(\vartheta, \xi) \in x$. From Proposition 3.4(iii) it follows that there exists a $t \in W_L$ such that $\rho(x, \top, t)$, $\rho(t, \top, y)$, $\psi \in t$, and $\neg\xi \in t$. This implies $t \in Z$ and $\xi \notin t$, a contradiction. Therefore $Z \in \beta_L(x, y)$ and, by definition of truth, $\mathbf{M}_L \models U(\varphi, \psi)[x]$. □

Lemma 3.1 immediately leads to the following result, which in turn yields a Henkin-style proof of the completeness of any US-logic with respect of the class of its e-models.

Theorem 3.5 (Fundamental Theorem) *Let L be a US-logic and φ a formula. Then $L \vdash \varphi$ if and only if $\mathbf{M}_L \models \varphi$.*

Corollary 3.6 (Completeness for e-models) *A formula φ is true in every e-model of a US-logic L if and only if $L \vdash \varphi$.*

Also, completeness can be extended to general e-frames just as in unary modal logics. In fact, define $\Pi_L = \{X \subseteq W_L : X = \{x : \varphi \in x\}$ for a formula $\varphi\}$. It is easy to see that $\mathcal{F}_L = \langle W_L, R_L, \beta_L, \Pi_L \rangle$ is a general e-frame (the *canonical general e-frame* of L).

Corollary 3.7 (Completeness for general e-frames) *Let φ be a formula. Then $\mathbf{M}_L \models \varphi$ if and only if $\mathcal{F}_L \models \varphi$.*

Theorem 3.5, together with the filtration theorem for e-models (see [1], Theorem 3.6), yields the finite e-model property for K_{US}. In fact, we have the following result (note that any e-model is an e-model of K_{US}).

Theorem 3.8 *If $\mathsf{K}_{US} \nvdash \varphi$, then there exists a finite e-model which falsifies φ.*

Proof: Suppose $\mathsf{K}_{US} \nvdash \varphi$. Write $\mathbf{M}_K$ for the canonical e-model of K_{US}. By Theorem 3.5, $\mathbf{M}_K \not\models \varphi$. Let Σ be the (finite) set of all subformulas of φ and let $\mathbf{M}'$ be the finest filtration of $\mathbf{M}_K$ through Σ: such a filtration exists by Lemma 3.5 of [1] and is clearly finite. Now the filtration theorem for e-models implies $\mathbf{M}' \not\models \varphi$. But $\mathbf{M}'$ is a finite e-model: hence, the theorem is proved. $\square$

Since K_{US} is finitely axiomatizable, as an immediate consequence of the finite e-model property we get its decidability.

Corollary 3.9 *The logic K_{US} is decidable.*

4 Duality In this section we examine the duality between general e-frames and algebras. We follow the presentation of the duality for unary modal logics given in [2] (which can be extended to unary tense logics, see, for example, Thomason [7]).

Definition 4.1 Let $\langle A, \vee, \wedge, {}', 0, 1\rangle$ be a Boolean algebra, and let u and s be binary operations on A. For any $b \in A$, define $f(b) = u(b, 1)$, $p(b) = s(b, 1)$, $g(b) = (f(b'))'$, $h(b) = (p(b'))'$. We say that $\mathcal{A} = \langle A, \vee, \wedge, {}', 0, 1, u, s\rangle$ is a *US-algebra* if the following axioms, which are of course the algebraic version of those for US-logics, are satisfied for any b, c, d in A:

1. $g(b' \vee c) \leq (u(b, d))' \vee u(c, d)$,
2. $g(b' \vee c) \leq (u(d, b))' \vee u(d, c)$,
3. $h(b' \vee c) \leq (s(b, d))' \vee s(c, d)$,
4. $h(b' \vee c) \leq (s(d, b))' \vee s(d, c)$,
5. $b \wedge u(c, d) \leq u(c \wedge s(b, d), d)$,
6. $b \wedge s(c, d) \leq s(c \wedge s(b, d), d)$,
7. $g(1) = 1$,
8. $h(1) = 1$.

A valuation v on a US-algebra $\mathcal{A}$ is a function from the formulas to the elements of $\mathcal{A}$ such that the following conditions are satisfied for any formulas φ, ψ:

1. $v(\neg\varphi) = (v(\varphi))'$,
2. $v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$,
3. $v(U(\varphi, \psi)) = u(v(\varphi), v(\psi))$,
4. $v(S(\varphi, \psi)) = s(v(\varphi), v(\psi))$.

An *algebraic model* $\langle \mathcal{A}, v\rangle$ is a US-algebra equipped with a valuation, and a formula φ is verified in such a model if and only if $v(\varphi) = 1$.

Definition 4.2 Given a general e-frame $\mathcal{F} = \langle W, R, \beta, \Pi\rangle$, we define its *dual US-algebra* $\mathcal{F}^+$ as follows: $\mathcal{F}^+ = \langle \Pi, \cup, \cap, {}', \varnothing, W, u, s\rangle$, where

$$u(B, C) = \{x \in W : \exists y \in B : xRy \text{ and } \exists Z \in \beta(x, y) : Z \subseteq C\},$$

$$s(B, C) = \{x \in W : \exists y \in B : yRx \text{ and } \exists Z \in \beta(y, x) : Z \subseteq C\}.$$

Incidentally, we observe that the definition of $\mathcal{F}^+$ may be given also for (standard) general frames, simply by setting $u(B, C) = \{x \in W : \exists y \in B : xRy \text{ and } [x, y] \subseteq C\}$ and $s(B, C) = \{x \in W : \exists y \in B : yRx \text{ and } [x, y] \subseteq C\}$. This definition coincides with Definition 4.2, when considering general frames as particular general e-frames. But if one limits oneself to general frames, it is not possible to achieve a Stone-like duality: we show this by a simple example. Consider the two-point Boolean algebra **2**. On **2** it is possible to define three distinct US-algebras, $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, by setting, in $\mathcal{A}_1$, $u(1, 1) = 1$ and $u(1, 0) = 0$, in $\mathcal{A}_2$, $u(1, 1) = 0 = u(1, 0)$, and in $\mathcal{A}_3$, $u(1, 1) = 1 = u(1, 0)$. (It is easy to see that the remaining conditions on u, and those on s, are implied by these identities and by the axioms of US-algebras.) These algebras can actually be obtained as duals of general frames $\langle W, R, \Pi\rangle$: namely, $\mathcal{A}_1$ is obtained as dual by setting $W = \{x\}$, $R = \{(x, x)\}$ and $\Pi = \mathcal{P}(W)$, $\mathcal{A}_2$ by $W = \{x\}$, $R = \varnothing$ and $\Pi = \mathcal{P}(W)$, and $\mathcal{A}_3$ by $W = \{x, y\}$ (with $x \neq y$), $R = \{(x, y), (y, x)\}$ and $\Pi = \{\varnothing, W\}$. Nevertheless, if we want to go back from US-algebras to general frames in a Stone-like manner, that is, considering the set of ultrafilters of the algebra, then from **2** we always obtain a one-point frame. But, up to isomorphism, there are only two one-point general frames, because there are only two ways to define R and one to define Π (clearly, it is $\mathcal{A}_3$ that can not correspond to any one-point frame). As we will show, this problem does not occur with general e-frames.

Definition 4.3 Given a US-algebra $\mathcal{A} = \langle A, \vee, \wedge, {}', 0, 1, u, s\rangle$, we define its *dual general e-frame* $\mathcal{A}_+$ as follows: $\mathcal{A}_+ = \langle W_A, R_A, \beta_A, \Pi_A\rangle$, where W_A is the set of ultrafilters of $\mathcal{A}$, xR_Ay if and only if $u(b, 1) \in x$ for any $b \in y$, $\Pi_A = \{\{x : b \in x\} : b \in A\}$, and, for $(x, y) \in R_A$ and $Z \subseteq [x, y]$, $Z \in \beta_A(x, y)$ if and only if $u(b, c) \in x$ for any $b, c \in A$ such that $b \in y$ and $c \in z$ for each $z \in Z$ (the closure properties of Π_A, required by Definition 2.4, will follow from next results, as we shall see).

As an example, we can consider the algebras $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ again and construct their duals. $\mathcal{A}_{1+}$ is obtained by setting $W_{A_1} = \{x\}$, $R_{A_1} = \{(x, x)\}$, $\Pi_{A_1} = \mathcal{P}(W_{A_1})$, and $\beta_{A_1}(x, x) = \{\{x\}\}$. $\mathcal{A}_{2+}$ is obtained by setting $W_{A_2} = \{x\}$, $R_{A_2} = \varnothing$, $\Pi_{A_2} = \mathcal{P}(W_{A_2})$, and $\beta = \varnothing$. Finally, $\mathcal{A}_{3+}$ is obtained by setting $W_{A_3} = \{x\}$, $R_{A_3} = \{(x, x)\}$, $\Pi_{A_3} = \mathcal{P}(W_{A_3})$, and $\beta_{A_3}(x, x) = \{\varnothing, \{x\}\}$.

As in unary modal and tense logics, if $\mathcal{A}$ is finite, then Π_A coincides with $\mathcal{P}(W_A)$, and therefore $\mathcal{A}_+$ is just an e-frame. Moreover, it is possible to show that if L is a US-logic and $\mathcal{A}$ is the free US-algebra on ω generators in the variety generated by the algebras for L, then $\mathcal{A}_+$ is isomorphic to the canonical general e-frame of L (we say that Φ is an isomorphism from $\langle W, R, \beta, \Pi\rangle$ onto $\langle W', R', \beta', \Pi'\rangle$ if and only if it is an isomorphism of general frames, and moreover, for any $(x, y) \in R$ and any $Z \subseteq [x, y]$ it holds $Z \in \beta(x, y)$ if and only if $\Phi[Z] \in \beta'(\Phi(x), \Phi(y))$; by $\Phi[Z]$ we mean the set $\{\Phi(z) : z \in Z\}$).

A valuation V on a general e-frame $\mathcal{F}$ determines an algebraic model $\langle \mathcal{F}^+, v\rangle$ in a natural way, by setting $v(\varphi) = \{x : \langle \mathcal{F}, V\rangle \models \varphi[x]\}$. It is then straightforward to see that

a formula χ is true in $\langle \mathcal{F}, V\rangle$ if and only if it is true in $\langle \mathcal{F}^+, v\rangle$,

and therefore χ is true in $\mathcal{F}$ if and only if it is true in $\mathcal{F}^+$. Moreover, if $\mathcal{A}$ is a US-algebra and v is a valuation on it, one can define a valuation on $\mathcal{A}_+$ by setting, for any propositional variable p, $V(p) = \{x : v(p) \in x\}$, and it holds that

a formula χ is true in $\langle \mathcal{A}, v\rangle$ if and only if it is true in $\langle \mathcal{A}_+, V\rangle$.

This last result is proved by showing, by induction on the complexity of χ, that $x \in V(\chi)$ if and only if $v(\chi) \in x$ (the crucial points being of course the cases $\chi = U(\varphi, \psi)$ and $\chi = S(\varphi, \psi)$). The proof is analogous to that of Lemma 3.1, by considering that Definiton 3.2, Remark 3.3, and Proposition 3.4 still hold if points of canonical models (i.e., maximal consistent extensions of a logic) are replaced by ultrafilters of a US-algebra, and formulas are replaced by points of the algebra. As an example, we show the induction step for U. Suppose that $x \in V(U(\varphi, \psi))$. Then by definition of truth we have that there exists a $y \in W_A$ such that xR_Ay and $\langle \mathcal{A}_+, V\rangle \models \varphi[y]$, and there exists a $Z \in \beta_A(x, y)$ such that, for any $z \in Z$, $\langle \mathcal{A}_+, V\rangle \models \psi[z]$. By induction hypothesis we get $v(\varphi) \in y$ and, for any $z \in Z$, $v(\psi) \in z$. But $Z \in \beta_A(x, y)$, and so by definition of β_A we obtain $u(v(\varphi), v(\psi)) = v(U(\varphi, \psi)) \in x$. Conversely, suppose $u(v(\varphi), v(\psi)) = v(U(\varphi, \psi)) \in x$. By Proposition 3.4(ii) there exists a $y \in W_A$ such that $v(\varphi) \in y$ and $\rho(x, v(\psi), y)$, and by Remark 3.3 we have xR_Ay. Now let $Z = \{z : \langle \mathcal{A}_+, V\rangle \models \psi[z]\} \cap [x, y]$. Suppose by contradiction that $Z \notin \beta_A(x, y)$. By definition of β_A, this means that there exist a b and a c in $\mathcal{A}$ such that $b \in y$, $c \in z$ for any $z \in Z$, and $(u(b, c))' \in x$. From Proposition 3.4(iii) it follows that there exists a $t \in W_A$ such that $\rho(x, 1, t)$, $\rho(t, 1, y)$, $v(\psi) \in t$, and $c' \in t$. This implies $t \in Z$ and $c \notin t$, a contradiction. Therefore $Z \in \beta_A(x, y)$. By definition of truth, it follows $x \in V(U(\varphi, \psi))$.

As announced in Definition 4.3, from this proof one gets also the closure properties of Π_A required by Definition 2.4. Similarly, one can show that the function Φ from the points of a US-algebra $\mathcal{A}$ to Π_A, defined by $\Phi(b) = \{x : x$ an ultrafilter on $\mathcal{A}$ with $b \in x\}$ is an isomorphism from $\mathcal{A}$ to $(\mathcal{A}_+)^+$ (consider, for example, the case of the operator u: from the definitions, we have that $u(\Phi(b), \Phi(c)) = u(\{t \in W_A : b \in t\}, \{t \in W_A : c \in t\}) = \{x \in W_A : \exists y : b \in y$ and xR_Ay, and $\exists Z \in \beta_A(x, y) : \forall z \in Z, c \in z\}$; now, proceeding as in the above proof, one can show that this set coincides with $\{x \in W_A : u(b, c) \in x\}$, which by definition of Φ is just $\Phi(u(b, c))$). Therefore it holds that

any US-algebra $\mathcal{A}$ is isomorphic to $(\mathcal{A}_+)^+$.

As in unary modal and tense logics, problems arise when one starts from frames: in our case, it does not hold for any general e-frame $\mathcal{F}$ that $\mathcal{F}$ is isomorphic to $(\mathcal{F}^+)_+$. In unary logics, on that account, those general frames for which such an isomorphism holds have been characterized (the *descriptive* general frames; see Goldblatt [4]). We recall that a general frame $\mathcal{F} = \langle W, R, \Pi\rangle$ is descriptive if it satisfies

(i) $(\forall B \in \Pi)(x \in B \Longleftrightarrow y \in B) \Longrightarrow x = y$,

(ii) $(\forall B \in \Pi)(y \in B \Longrightarrow x \in f(B)) \Longrightarrow xRy$,

where $f(B) = \{x : \exists y(xRy \text{ and } y \in B)\}$ is the operator of $\mathcal{F}^+$, that is, the dual (modal) algebra of $\mathcal{F}$, and

(iii) for any ultrafilter D of $\mathcal{F}^+$, there exists $x \in W$ such that $D = \{B \in \Pi : x \in B\}$.

Clearly, since US-logics extend unary logics, these conditions are still necessary to obtain isomorphism between $\mathcal{F}$ and $(\mathcal{F}^+)_+$ (note that in this case $f(B)$ coincides with $u(B, W)$, where u is the operator of the US-algebra $\mathcal{F}^+$). But they are not sufficient: in fact, one has also to determine opportune conditions on β.

Definition 4.4 A general e-frame $\mathcal{F} = \langle W, R, \beta, \Pi\rangle$ *satisfies condition* Π-β if and only if, for any $(x, y) \in R$ and any $Z \subseteq [x, y]$ it holds that, if

(1) for all $B, C \in \Pi$ such that $y \in B$ and $Z \subseteq C$ there exists a $w \in B$ such that xRw, and a $T \in \beta(x, w)$ such that $T \subseteq C$,

then $Z \in \beta(x, y)$.

Definition 4.5 A general e-frame is *descriptive* if and only if it is descriptive as a general frame and satisfies condition Π-β.

Condition Π-β says, essentially, that the behavior of the elements of Π with respect to β forces that of other sets of points with respect to β. Its role will be made clearer by the proof of Theorem 4.7.

Now we show that being descriptive is a necessary and sufficient condition for a general e-frame $\mathcal{F}$ to be isomorphic with $(\mathcal{F}^+)_+$. We treat the two directions separately.

Theorem 4.6 *For any US-algebra $\mathcal{A}$, the general e-frame $\mathcal{A}_+$ is descriptive.*

Proof: From a well-known result of unary modal and tense logics we have that the general frame underlying $\mathcal{A}_+$ is descriptive. Thus we have to check only condition Π-β. Assuming condition 1 of Definition 4.4, we want to show that $Z \in \beta_A(x, y)$ (here x and y are points of $\mathcal{A}_+$, that is, ultrafilters on $\mathcal{A}$). By definition of β_A, this means that, for b, c points of $\mathcal{A}$, if $b \in y$ and $c \in z$ for any $z \in Z$, then $u(b, c) \in x$. Now let us call B (resp. C) the point of $\mathcal{A}_+$ corresponding to b (resp. c), that is, the set of ultrafilters to which b (resp. c) belongs. Clearly, $B, C \in \Pi_A$, and moreover, $b \in y$ translates to $y \in B$, and $c \in z$ for any $z \in Z$ translates to $Z \subseteq C$. Suppose then $y \in B$ and $Z \subseteq C$. By hypothesis, it follows that there exists a $w \in B$ such that xR_Aw and a $T \subseteq C$ such that $T \in \beta_A(x, w)$. But $w \in B$ means $b \in w$, and $T \subseteq C$ means that $c \in t$, for any $t \in T$. Thus from $T \in \beta_A(x, w)$ it follows that $u(b, c) \in x$, by definition of β_A. □

Theorem 4.7 *If $\mathcal{F} = \langle W, R, \beta, \Pi\rangle$ is a descriptive general e-frame, then the function Φ from $\mathcal{F}$ to $(\mathcal{F}^+)_+$ defined by $\Phi(x) = \{B \in \Pi : x \in B\}$ is an isomorphism.*

Proof: It is known from unary modal and tense logics that Φ is an isomorphism of general frames. Therefore we only have to check that $Z \in \beta(x, y)$ if and only if $\Phi[Z] \in \beta_A(\Phi(x), \Phi(y))$, where A is the carrier of $\mathcal{F}^+$. Now, from the definitions it follows that $\Phi[Z] \in \beta_A(\Phi(x), \Phi(y))$ means that for any $B, C \in \Pi$, if $y \in B$ and $Z \subseteq C$, then there exists a $w \in B$ such that xRw and a $T \subseteq C$ such that $T \in \beta(x, w)$ (in other words, $\Phi[Z] \in \beta_A(\Phi(x), \Phi(y))$ is just condition 1 of Definition 4.4). Suppose then $Z \in \beta(x, y)$, and take $B, C \in \Pi$ such that $y \in B$ and $Z \subseteq C$. Clearly we may take $w = y$ and $T = Z$, thus obtaining $\Phi[Z] \in \beta_A(\Phi(x), \Phi(y))$. Conversely, assume $\Phi[Z] \in \beta_A(\Phi(x), \Phi(y))$. Then, using condition Π-β, we obtain $Z \in \beta(x, y)$. □

Again in analogy with the unary case, we have the following result.

Theorem 4.8 *Every finite general e-frame $\mathcal{F} = \langle W, R, \beta, \Pi\rangle$ which is descriptive as a general frame is descriptive (as a general e-frame).*

Proof: Since by Remark 2.5 any e-frame can be considered as β-upward closed, it is enough to show that if $\mathcal{F}$ is finite and descriptive as a general frame, then condition Π-β is equivalent to β-upward closedness (we recall that a finite general frame is descriptive if and only if $\Pi = \mathcal{P}(W)$). It is clear that every general e-frame which satisfies condition Π-β is β-upward closed. For the converse, assume β-upward closedness and condition 1 of Definition 4.4. Since $\Pi = \mathcal{P}(W)$, we may take $B = \{y\}$ and $C = Z$, and it follows that there exists $T \subseteq Z$ such that $T \in \beta(x, y)$. Applying β-upward closedness, we obtain $Z \in \beta(x, y)$, and thus condition Π-β is satisfied. □

For the infinite case, β-upward closedness does not imply condition Π-β, as the following counterexample shows.

Let X be a denumerable set and define $W = X \cup \{-\infty, \infty\}$ (where $-\infty, \infty \notin X$). Set $R = \{(-\infty, x) : x \in X\} \cup \{(x, \infty) : x \in X\} \cup \{(-\infty, \infty)\}$ and $\Pi = \mathcal{P}(W)$. Now $\mathcal{F} = \langle W, R, \Pi\rangle$ is a general frame and we can consider its bidual $(\mathcal{F}^+)_+$, in the sense of unary modal and tense logics. As is well known, $(\mathcal{F}^+)_+$ is a descriptive general frame, and it is straightforward to verify that $(\mathcal{F}^+)_+ \cong \langle W', R', \Pi'\rangle$, where $W' = W \cup Y$ for a set Y of cardinality $2^{\aleph_0}$ disjoint with W, $R' = R \cup \{(-\infty, y) : y \in Y\} \cup \{(y, \infty) : y \in Y\}$, and Π' contains all finite subsets of W, but if $B \in \Pi'$ is infinite then $B \cap Y \neq \varnothing$. Now define a general e-frame $\mathcal{G}$ over $(\mathcal{F}^+)_+$, by setting $\beta(-\infty, \infty) = \{Z \subseteq X \cup Y : Z \cap Y \neq \varnothing\}$ (in all other cases the only possibility is $\beta(x, y) = \{\varnothing\}$). Clearly, $\mathcal{G}$ is β-upward closed. But $\mathcal{G}$ does not satisfy condition Π-β. In fact, let Z be any infinite subset of X, and let $B, C \in \Pi'$ be such that $\infty \in B$ and $Z \subseteq C$. Then, since C is infinite, we have $C \cap Y \neq \varnothing$. Thus $C \cap Y \in \beta(-\infty, \infty)$, and since $\infty \in B$ and $C \cap Y \subseteq C$, condition 1 of Definition 4.4 is fulfilled for $x = -\infty$, $y = \infty$ and our Z with $w = \infty$ and $T = C \cap Y$. But $Z \notin \beta(-\infty, \infty)$, because $Z \cap Y = \varnothing$, so condition Π-β is not satisfied.

5 *Morphisms*

Now, by analogy with the unary case, we want to give a definition of *general e-frame morphism*, corresponding (contravariantly) to that of US-algebra morphism. In such a way, we obtain a category-theoretic contravariant duality between descriptive general e-frames and US-algebras.

Definition 5.1 Let $\mathcal{F}_1 = \langle W_1, R_1, \beta_1, \Pi_1\rangle$ and $\mathcal{F}_2 = \langle W_2, R_2, \beta_2, \Pi_2\rangle$ be general e-frames, and let Φ be a function from W_1 to W_2. We say that Φ is a *general e-frame morphism* if Φ is a general frame morphism, and furthermore

(i) if $Z \in \beta_1(x, y)$, then $\Phi[Z] \in \beta_2(\Phi(x), \Phi(y))$;

(ii) for any $x \in W_1$ and $B, C \in \Pi_2$, if there exists a $\bar{y} \in B$ such that $\Phi(x) R_2 \bar{y}$ and a $\bar{Z} \in \beta_2(\Phi(x), \bar{y})$ such that $\bar{Z} \subseteq C$, then there exists a $v \in \Phi^{-1}[B]$ such that $x R_1 v$ and a $Z \in \beta_1(x, v)$ such that $Z \subseteq \Phi^{-1}[C]$;

(iii) for any $y \in W_1$ and $B, C \in \Pi_2$, if there exists a $\bar{x} \in B$ such that $\bar{x} R_2 \Phi(y)$ and a $\bar{Z} \in \beta_2(\bar{x}, \Phi(y))$ such that $\bar{Z} \subseteq C$, then there exists a $w \in \Phi^{-1}[B]$ such that $w R_1 y$ and a $Z \in \beta_1(w, y)$ such that $Z \subseteq \Phi^{-1}[C]$.

Note that conditions (ii) – (iii) involve Π_1 and Π_2. Thus our definition of general e-frame morphism recalls that of weak contraction (see, e.g., Sambin and Vaccaro [6]) more than the usual one of p-morphism of the unary case. To achieve a stronger analogy with the unary case, one should find a notion of general e-frame morphism which does not involve Π_1 and Π_2 and for descriptive general e-frames is equivalent to the one we have given. A good candidate may be the definition obtained from 5.1 by replacing (ii) – (iii) with the following ones.

(ii)′ for any $x \in W_1$, $\bar{y} \in W_2$, and $\bar{Z} \subseteq W_2$, if $\Phi(x) R_2 \bar{y}$ and $\bar{Z} \in \beta_2(\Phi(x), \bar{y})$, then there exists a $v \in W_1$ such that $x R_1 v$ and $\Phi(v) = \bar{y}$, and a $Z \in \beta_1(x, v)$ such that $Z \subseteq \Phi^{-1}[\bar{Z}]$;

(iii)′ for any $y \in W_1$, $\bar{x} \in W_2$, and $\bar{Z} \subseteq W_2$, if $\bar{x} R_2 \Phi(y)$ and $\bar{Z} \in \beta_2(\bar{x}, \Phi(y))$, then there exists a $w \in W_1$ such that $w R_1 y$ and $\Phi(w) = \bar{x}$, and a $Z \in \beta_1(w, y)$such that $Z \subseteq \Phi^{-1}[\bar{Z}]$.

In fact, it is easy to verify that (ii)′ – (iii)′ are equivalent to (ii) – (iii) when $\mathcal{F}_1$ and $\mathcal{F}_2$ are *finite* descriptive general e-frames (i.e., W_1 and W_2 are finite sets, and $\Pi_1 = \mathcal{P}(W_1)$, $\Pi_2 = \mathcal{P}(W_2)$). Moreover, if we call *e-frame morphism* a function which is a frame morphism and satisfies conditions (i), (ii)′, and (iii)′, we have that an e-frame morphism Φ preserves the truth of formulas (that is, $x \models \varphi$ if and only if $\Phi(x) \models \varphi$), as can be shown by an easy induction on the complexity of the formula. But whether the equivalence of (ii) – (iii) and (ii)′ – (iii)′ holds also for infinite descriptive general e-frames is an open question.

The following two theorems show the announced correspondence between general e-frame morphisms and US-algebra morphisms.

Theorem 5.2 *Let Φ be a general e-frame morphism from $\mathcal{F}_1 = \langle W_1, R_1, \beta_1, \Pi_1 \rangle$ to $\mathcal{F}_2 = \langle W_2, R_2, \beta_2, \Pi_2 \rangle$. Then the function Ψ from $\mathcal{F}_2^+$ to $\mathcal{F}_1^+$ defined by $\Psi(B) = \Phi^{-1}[B]$ is a US-algebra morphism.*

Proof: It is straightforward to verify that Ψ preserves Boolean operators. As for u, using the definitions and conditions (i) and (ii) of Definition 5.1, we obtain $\Psi(u(B, C)) = \Phi^{-1}[u(B, C)] = \Phi^{-1}[\{\bar{x} \in W_2 : \exists \bar{y} \in B : \bar{x} R_2 \bar{y} \text{ and } \exists \bar{Z} \in \beta_2(\bar{x}, \bar{y}) : \bar{Z} \subseteq C\}] = \{x \in W_1 : \exists \bar{y} \in B : \Phi(x) R_2 \bar{y} \text{ and } \exists \bar{Z} \in \beta_2(\Phi(x), \bar{y}) : \bar{Z} \subseteq C\} = \{x \in W_1 : \exists v \in \Phi^{-1}[B] : x R_1 v \text{ and } \exists Z \in \beta_1(x, v) : Z \subseteq \Phi^{-1}[C]\} = \{x \in W_1 : \exists v \in \Psi(B) : x R_1 v \text{ and } \exists Z \in \beta_1(x, v) : Z \subseteq \Psi(C)\} = u(\Psi(B), \Psi(C))$ (note: we are using u for both the operator of $\mathcal{F}_1^+$ and that of $\mathcal{F}_2^+$). The case of s is proved analogously, using conditions (i) and (iii) of Definition 5.1. □

Theorem 5.3 *Let Ψ be a US-algebra morphism from $\mathcal{A}_2$ to $\mathcal{A}_1$. Then the function Φ from $\mathcal{A}_{1+}$ to $\mathcal{A}_{2+}$ defined by $\Phi(x) = \{b \in A_2 : \Psi(b) \in x\}$ (where A_2 is the carrier of $\mathcal{A}_2$; we use the analogous notation for $\mathcal{A}_1$) is a general e-frame morphism.*

Proof: Since x is an ultrafilter on $\mathcal{A}_1$ and Ψ is a US-algebra morphism, $\Phi(x)$ is an ultrafilter on $\mathcal{A}_2$. It is clear from well-known results on unary logics that Φ is a general frame morphism: thus we have to check only conditions (i) – (iii) of Definition 5.1. As regards (i), let $xR_{A_1}y$ and $Z \in \beta_{A_1}(x, y)$. By definition of β_{A_1}, this means that for any $b, c \in A_1$, if $b \in y$ and $c \in z$ for any $z \in Z$, then $u(b, c) \in x$ (again, we shall use u for both the operator of $\mathcal{A}_1$ and that of $\mathcal{A}_2$). Now, take $\bar{b}, \bar{c} \in A_2$, and assume that $\bar{b} \in \Phi(y)$ and $\bar{c} \in \bar{z}$, for any $\bar{z} \in \Phi[Z]$ (that is, $\bar{c} \in \Phi(z)$, for any $z \in Z$). By definition of Φ, it follows that $\Psi(\bar{b}) \in y$ and $\Psi(\bar{c}) \in z$ for any $z \in Z$, whence $u(\Psi(\bar{b}), \Psi(\bar{c})) \in x$. But Ψ is a US-algebra morphism, so we have $\Psi(u(\bar{b}, \bar{c})) \in x$ and therefore $u(\bar{b}, \bar{c}) \in \Phi(x)$. By definition of β_{A_2}, it follows that $\Phi[z] \in \beta_{A_2}(\Phi(x), \Phi(y))$. As regards (ii), let $B^*, C^* \in \Pi_{A_2}$, and suppose that there exists a $\bar{y} \in B^*$ such that $\Phi(x)R_{A_2}\bar{y}$ and a $\bar{Z} \in \beta_2(\Phi(x), \bar{y})$ such that $\bar{Z} \subseteq C^*$. Recall that B^* and C^* are sets of ultrafilters on $\mathcal{A}_2$ which correspond to points b^*, c^* of $\mathcal{A}_2$, so that $\bar{y} \in B^*$ is equivalent to $b^* \in \bar{y}$, and $\bar{Z} \subseteq C^*$ is equivalent to $c^* \in \bar{z}$ for any $\bar{z} \in \bar{Z}$. Now, $\bar{Z} \in \beta_2(\Phi(x), \bar{y})$ means that for any $b, c \in A_2$ with $b \in \bar{y}$ and $c \in \bar{z}$ for any $\bar{z} \in \bar{Z}$, it holds $u(b, c) \in \Phi(x)$. Thus, in particular, $u(b^*, c^*) \in \Phi(x)$, which, by definition of Φ, implies $\Psi(u(b^*, c^*)) = u(\Psi(b^*), \Psi(c^*)) \in x$. Let $\bar{B}$ and $\bar{C}$ be the points of Π_{A_1} corresponding to $\Psi(b^*)$ and $\Psi(c^*)$. Then from $u(\Psi(b^*), \Psi(c^*)) \in x$ we obtain that $x \in u(\bar{B}, \bar{C})$, where the latter u is the operator of the algebra $(\mathcal{A}_{1+})^+$ (which is isomorphic to $\mathcal{A}_1$). By Definition 4.2, this means that there exists a $v \in \bar{B}$ such that $xR_{A_1}v$ and a $Z \in \beta_{A_1}(x, v)$ such that $Z \subseteq \bar{C}$. But $v \in \bar{B}$ means $\Psi(b^*) \in v$, hence $b^* \in \Phi(v)$ and $v \in \Phi^{-1}[B^*]$, and moreover, $Z \subseteq \bar{C}$ means $\Psi(c^*) \in z$ for any $z \in Z$, hence $c^* \in \Phi(z)$ for any $z \in Z$ and therefore $Z \subseteq \Phi^{-1}[C^*]$: thus condition (ii) holds. Condition (iii) is proved similarly. □

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