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A NEW METHOD FOR DERIVING ADJACENT APPROXIMANTS IN THE PADÉ TABLE

DAVID A. FIELD

ABSTRACT. This paper derives formulas for adjacent approximants in the Padé table by using the Focal Point method of solving systems of linear equations. In addition to presenting a new method to derive formulas, this paper gives inner products to detect blocks in a nonnormal Padé table. These inner products lead to characterizations of the corner entries of the blocks.

1. Introduction. In the spirit of Baker's [**1**] recursion formulas for entries in the Padé table this paper derives additional formulas via a novel method. Other and more numerous recursion formulas exist; for instance, see Gragg [**5**], McCabe [**6**] and Wynn [**9**]. The latter formulas all rely on auxiliary tables. The new approach relies instead on the Focal Point method [**2**] for solving systems of linear equations.

The second section of this paper defines nomenclature and notations used for the Padé table. The third section presents short summaries of published recursion formulas. The fourth section gives a very brief encounter with the Focal Point method. The fifth section demonstrates the derivation of adjacent approximants via the Focal Point method. The sixth section shows how the FP-method can detect blocks in a nonnormal Padé table.

2. The Padé table. The power series representation of $f(x)$,

$$f(x) = \sum_{i=0}^{\infty} c_i x^i\,, \quad c_0 \neq 0\,, \tag{2.1}$$

determines an infinite two dimensional table of rational functions of $f(x)$, called $[n/m]$-Padé approximants. The entries in the n^{th}-column of this table have numerator polynomials of degree n. The horizontal

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rows have denominator polynomials with fixed degree m. This section assumes that $f(x)$ has the necessary derivatives so that the $[n/m]$-entry in this Padé table,

$$r_{n/m}(x) = \frac{P_{n/m}(x)}{Q_{n/m}(x)}\,, \tag{2.2}$$

exists where $P_{n/m}(z)$ and $Q_{n/m}(z)$ denote polynomials of degree n and m, respectively. A normal Padé table has unique $[n/m]$-entries which satisfy the following criteria. Nonunique entries will be discussed in Section 6.

According to convention $r_{n/m}(x)$ has been reduced to lowest terms. Let

$$P_{n/m}(x) = a_0^{(n/m)} + \cdots + a_n^{(n/m)} x^n$$

and

$$Q_{n/m}(x) = b_0^{(n/m)} + \cdots + b_m^{(n/m)} x^m\,. \tag{2.3}$$

Consequently, convention imposes the condition that $b_0^{(n/m)} = 1$. This condition makes all the denominators in the [n/0]-entries in the first row equal one so that the first row of the Padé table contains the Taylor polynomials of $f(x)$. The additional criterion that the Maclaurin expansion of $r_{n/m}(x)$ agrees with the series in (2.1) as far as possible determines the coefficients $a_0^{(n/m)}, \ldots, a_n^{(n/m)}, b_0^{(n/m)}, \ldots, b_m^{(n/m)}$. This criterion yields

$$f(x) - \frac{P_{n/m}(x)}{Q_{n/m}(x)} = \frac{f(x)Q_{n/m}(x) - P_{n/m}(x)}{Q_{n/m}(x)}\,, \tag{2.4}$$

which has a power series expansion whose first $m+n+1$ terms must equal zero. The numerator in (2.4) produces the two sets of linear equations in (2.5a) and (2.5b). Solutions to these equations determine the values of $a_k^{(n/m)}$, $0 \le k \le n$ and $b_k^{(n/m)}$, $0 \le k \le m$. For notational convenience only, temporarily assume $m < n$. In general $0 \le n$, $0 \le m$,

$$\begin{aligned} c_0 b_0^{(n/m)} - a_0^{(n/m)} &= 0 \\ c_1 b_0^{(n/m)} + c_0 b_1^{(n/m)} - a_1^{(n/m)} &= 0 \end{aligned} \tag{2.5a}$$

$$\cdots$$

$$c_n b_0^{(n/m)} + \cdots + c_0 b_n^{(n/m)} - a_n^{(n/m)} = 0\,,$$

and

$$\text{(2.5b)}\qquad \begin{aligned} c_{n+1} b_0^{(n/m)} + \cdots + c_{n+1-m} b_m^{(n/m)} &= 0 \\ &\cdots \\ c_{n+m} b_0^{(n/m)} + \cdots + c_n b_m^{(n/m)} &= 0\,. \end{aligned}$$

The solution of the equations in (2.5a) depend on the solutions of the equations in (2.5b). Singular systems in (2.5b) lead to degeneracies in the Padé table; that is, repeated adjacent entries in the Padé table. A *normal* Padé table has sets of nonsingular linear equations in (2.5b) for $n > 0$ and $m > 0$ and unique corresponding entries in the table. Solving these equations with standard linear equation solvers can be avoided because formulas provide alternative recursive methods to populate the Padé table [**1, 5, 6, 9**].

3. Recurrence relations and the Padé table. In a normal Padé table known recurrence relations use two adjacent entries to calculate a third adjacent entry. This section reviews recurrence relations associated with this method for constructing the Padé table. The fourth section will present the Focal Point method and the fifth section will use the Focal Point method for deriving recurrence relations.

Starting with successive $(n-1)$-th- and n-th-Taylor Polynomials, the $[(n-1)/0]$- and $[n/0]$-entries in the Padé table, Baker [**1**] supplies recursions for $n > j \geq 1$;

$$\text{(3.1a)}\quad \frac{P_{(n-j)/j}(x)}{Q_{(n-j)/j}(x)} = \frac{a_{n-j}^{((n-j)/(j-1))} P_{(n+1-j)/(j-1)}(x) - x a_{n+1-j}^{((n+1-j)/(j-1))} P_{(n-j)/(j-1)}(x)}{a_{n-j}^{((n-j)/(j-1))} Q_{(n+1-j)/(j-1)}(x) - x a_{n+1-j}^{((n+1-j)/(j-1))} Q_{(n-j)/(j-1)}(x)}$$

and

$$\text{(3.1b)}\quad \frac{P_{(n-j-1)/j}(x)}{Q_{(n-j-1)/j}(x)} = \frac{a_{n-j}^{((n-j)/j)} P_{(n-j)/(j-1)}(x) - a_{n-j}^{((n-j)/(j-1))} P_{(n-j)/j}(x)}{a_{n-j}^{((n-j)/j)} Q_{(n-j)/(j-1)}(x) - a_{n-j}^{((n-j)/(j-1))} Q_{(n-j)/j}(x)}\,.$$

[(n-j)/(j-1)]	[(n-j+1)/(j-1)]		[(n-j)/(j-1)]
[(n-j)/j]		[(n-j-1)/j]	[(n-j)/j]

FIGURE 3.1. Baker's recursion formulas derive the shaded entries in the Padé table.

The configurations in Figure 3.1 display the use of Baker's recursions to determine the unknown shaded entries associated with two known adjacent and contiguous entries. Baker uses these recursions in a staircase progression from the first row toward approximants on the main diagonal, the usual location of the most accurate approximations relative to the total number of coefficients used by the numerator and denominator polynomials.

Note that equations (3.1) use the coefficients $a_*^{(*/*)}$ of the numerator polynomials, $P_{*/*}$. Computationally, the recursions start with the coefficients of a power series and subsequent approximants require the coefficients of the four polynomials on the right hand of a formula. A comparable recursion derived by the FP-method would not, as in (3.1), explicitly use the highest coefficients from polynomials in the numerator but instead exhibit coefficients from polynomials in the denominator.

Baker refers to Wynn's extensive work on the qd-algorithm [**9**]. Wynn's work includes determining the coefficients in the numerators of the continued fraction in (5.13) whose approximants lie on the staircase diagonals progressing down and parallel to the main downward diagonal of the Padé table. Computationally, the qd-algorithm requires two adjacent columns in a qd-table to compute the next column which contains a coefficient of a partial numerator of the continued fraction whose recurrence relation in (5.14) generates adjacent and contiguous Padé approximants. The evaluation of rhombus rules determines the entries in the qd-table.

Gragg's comprehensive survey of the Padé table [**5**] also includes recursive formulas similar to (3.1). Gragg uses a $\pi\zeta$-table which, in similar fashion to the qd-algorithm, requires evaluating a rhombus rule to compute the entries in the $\pi\zeta$-table. The evaluation of Gragg's recursion formulas have computational complexity similar to the qd-algorithm.

To produce approximants in the Padé table, McCabe [**6**] also uses an auxiliary rectangular table whose entries contain coefficients of continued fractions. The entries depend on a somewhat different application of the rhombus rules found in the qd-algorithm. McCabe's method has the significant advantage of determining Padé approximants toward the right across rows, down columns as well as down staircase diagonals and up skew diagonals. The method depends on a normal Padé table. To produce Padé approximants McCabe's work uses the recursion formulas in (5.14) and has computational requirements similar to the qd-algorithm.

The Focal Point method will produce entries in the Padé table directly from the power series and adjacent entries without using auxiliary qd-type algorithms and tables. The recurrence formulas proceed in any direction as long as two adjacent and unequal Padé approximants in a row, a column, a diagonal or a skew diagonal have been calculated in any manner whatsoever. Although the derivations of recurrence relations for the Padé approximants assume normality, Padé tables that do not satisfy normality still benefit from the derivations. In particular, the Focal Point method can detect a block in a nonnormal Padé table without evaluating a determinant. Section 6 contains theorems that characterizes the upper left corner of a block in a nonnormal Padé table. Computationally, evaluating the recursion formulas derived via the FP-method requires, as in all the previous methods, the coefficients of two adjacent Padé approximants and, commensurate with entries in an auxiliary table, coefficients of the power series.

4. The focal point method. The FP-method solves the nonsingular set of linear equations in (4.1) without changing any coefficient a_{ij} nor any value b_i. For simplicity, assume that in (4.1) $a_{1i} \neq 0$, $1 \leq i \leq n$, and $b_1 \neq 0$.

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\
&\cdots \\
a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n\,.
\end{aligned}
\tag{4.1}
$$

The FP-method begins by calculating n independent vectors $\mathbf{v}_1^{(1)}, \ldots,$ $\mathbf{v}_n^{(1)}$ where all but one of the components of each $\mathbf{v}_i^{(1)}$, $1 \leq i \leq n$, equals

0 and the i-th-component of $\mathbf{v}_i^{(1)}$ equals b_1/a_{1i}. Clearly these vectors are independent solutions of the first equation in (4.1). The next step in the FP-method relies on the following lemma.

Lemma. *If* $\mathbf{w}_1$ *and* $\mathbf{w}_2$ *are solutions to the* i*-th-equation in* (4.1)*, then for any real number* t*,* $(1-t)\mathbf{w}_1 + t\mathbf{w}_2$ *is also a solution to the* i*-th-equation.*

In its second step, the FP-method applies the Lemma $n-1$ times by substituting the vector $(1-t)\mathbf{w}_1 + t\mathbf{w}_2$ into the second equation in (4.1), solving for t with $\mathbf{w}_1 = \mathbf{v}_1^{(1)}$ and $\mathbf{w}_2 = \mathbf{v}_i^{(1)}$ for $2 \leq i \leq n$ and thereby calculating $n-1$ vectors $\mathbf{v}_i^{(2)}$, $1 \leq i \leq n-1$, as solutions to the first two equations in (4.1). Iterating on the second step for the successive $n-2$ equations produces a solution vector to all the linear equations in (4.1).

Relaxing the previous assumption that $a_{1i} \neq 0$, $1 \leq i \leq n$, and $b_1 \neq 0$ and dealing with possible degeneracies when applying the lemma in successive equations have been addressed in [**2**]. The FP-method requires approximately $n^2/2$ storage registers. The number of multiplications and divisions in the FP-method is, as with Gaussian elimination [**4**], a cubic polynomial in n with a leading term $n^3/3$.

The FP-method essentially produces sets of $n+1-i$ vectors, S_i, $1 \leq i \leq n$, which generate solution spaces for the first i equations in (4.1). In [**3**] the solution spaces have been exploited in solving heat equations with high gradients near the boundary. In this case the FP-method solves linear equations evolving from finite differences and reduces nonlinear multivariate boundary conditions to a system of two polynomial equations in two unknowns. The numerical characteristics of the FP-method have interesting geometric interpretations [**2**].

A key aspect of the FP-method applies to the Padé table. To solve the last linear equation in (4.1), the FP-method produces two linearly independent solutions to the first $n-1$ equations. To derive adjacent approximants in the Padé table the FP-method will produce two vectors which take advantage of the fact that the linear equations corresponding to nearby Padé approximants table have much in common.

5. The Padé table, FP-method and recurrence relations. Formulas for deriving Padé approximants based on two known adjacent entries in the Padé table motivates the use of the FP-method. Their derivations are conceptually simple and often lead to easily implemented expressions involving inner products. A derivation of a recursion for a staircase progression down diagonals parallel to the main diagonal will illustrate the FP-method. As a byproduct this application will offer an alternative method for computing the coefficients of the s-fraction in (5.13).

Given the $[(n-1)/m]$*-th and* $[n/m]$*-th Padé approximants.* First, restructure the linear equations in (2.5b) for the $[n/m]$-th entry in the Padé table into

$$\begin{aligned} c_n b_1^{(n/m)} + \cdots + c_{n+1-m} b_m^{(n/m)} &= -c_{n+1} \\ &\cdots \\ c_{n+m-1} b_1^{(n/m)} + \cdots + c_n b_m^{(n/m)} &= -c_{n+m}\,. \end{aligned} \tag{5.1}$$

Next, introduce the notation $\mathbf{c}_{j:k} = (c_j, \ldots, c_{j+1-k})$ and denote the solution of (5.1) by

$$\mathbf{b}_{n/m} = (b_1^{(n/m)}, \ldots, b_m^{(n/m)})^T\,. \tag{5.2}$$

Observe that the equations in (5.1) can be expressed as $\mathbf{c}_{j:m} \cdot \mathbf{b}_{n/m} = -c_{j+1}$ for $n \le j \le n+m-1$.

Now consider the corresponding linear equations for the $[(n-1)/m]$- and $[n/(m+1)]$-entries; namely, the m equations

$$\begin{aligned} c_{n-1} b_1^{((n-1)/m)} + \cdots + c_{n-m} b_m^{((n-1)/m)} &= -c_n \\ &\cdots \\ c_{n+m-2} b_1^{((n-1)/m)} + \cdots + c_{n-1} b_m^{((n-1)/m)} &= -c_{n+m-1}\,, \end{aligned} \tag{5.3}$$

and the $m+1$ equations

$$\begin{aligned} c_n b_1^{(n/(m+1))} + \cdots + c_{n-m} b_{m+1}^{(n/(m+1))} &= -c_{n+1} \\ &\cdots \\ c_{n+m-1} b_1^{(n/(m+1))} + \cdots + c_{n-1} b_{m+1}^{(n/(m+1))} &= -c_{n+m} \\ c_{n+m} b_1^{(n/(m+1)} + \cdots + c_n b_{m+1}^{(n/(m+1)} &= -c_{n+m+1}\,. \end{aligned} \tag{5.4}$$

Note that in (5.1), (5.3) and (5.4), $c_k = 0$ whenever $k < 0$.

Append a null component to $\mathbf{b}_{n/m}$ with an *append operator* α_0,

$$\alpha_0(\mathbf{b}_{n/m}) = (b_1^{(n/m)}, \ldots, b_m^{(n/m)}, 0)^T \,. \tag{5.5}$$

$\alpha_0(\mathbf{b}_{n/m})$ satisfies the first m equations in (5.4). Obtain another vector that satisfies the homogeneous form of these m equations by prepending to the solution of (5.3) a unit first component. Use the *prepend operator* ρ_1,

$$\rho_1(\mathbf{b}_{(n-1)/m}) = (1, b_1^{((n-1)/m)}, \ldots, b_m^{((n-1)/m)})^T \,. \tag{5.6}$$

Define $\mathbf{w}$, a new solution to the first m equations in (5.4) and a solution clearly independent from $\alpha_0(\mathbf{b}_{n/m})$, by

$$\mathbf{w} = \alpha_0(\mathbf{b}_{n/m}) + \rho_1(\mathbf{b}_{(n-1)/m}) \,. \tag{5.7}$$

Since the linearity of the equations in (5.4) implies that

$$\mathbf{w}_{n/(m+1)} = (1-t)\alpha_0(\mathbf{b}_{n/m}) + t\mathbf{w} = \alpha_0(\mathbf{b}_{n/m}) + t\rho_1(\mathbf{b}_{(n-1)/m}) \tag{5.8}$$

must also satisfy the first m equations in (5.4) for all values of t, substitute $\mathbf{w}_{n/(m+1)}$ into the last equation in (5.4) and solve for the unknown t. Denote the solution by

$$t_{n/(m+1)} = \frac{-c_{n+m+1} - \mathbf{c}_{n+m:m+1} \cdot \alpha_0(\mathbf{b}_{n/m})}{\mathbf{c}_{n+m:m+1} \cdot \rho_1(\mathbf{b}_{(n-1)/m})} \,. \tag{5.9}$$

Associate the components of the vectors in (5.8) with the numerators and denominators of the entries of the Padé table to yield

$$\frac{P_{n/(m+1)}(x)}{Q_{n/(m+1)}(x)} = \frac{P_{n/m}(x) + t_{n/(m+1)}xP_{(n-1)/m}(x)}{Q_{n/m}(x) + t_{n/(m+1)}xQ_{(n-1)/m}(x)} \,. \tag{5.10}$$

Given the $[(n-1)/m]$*-th and* $[(n-1)/(m+1)]$*-th Padé approximants.* The companion recurrence formula for the other step in the staircase progression has a similar derivation which yields

$$\frac{P_{n/(m+1)}(x)}{Q_{n/(m+1)}(x)} = \frac{P_{(n-1)/(m+1)}(x) + t_{n/(m+1)}xP_{(n-1)/m}(x)}{Q_{(n-1)/(m+1)}(x) + t_{n/(m+1)}xQ_{(n-1)/m}(x)} \,. \tag{5.11}$$

where,

$$t_{n/(m+1)} = \frac{-c_{n+m+1} - \mathbf{c}_{n+m:m+1} \cdot \mathbf{b}_{(n-1)/(m+1)}}{\mathbf{c}_{n+m:m+1} \cdot \rho_1(\mathbf{b}_{(n-1)/m})}\,. \tag{5.12}$$

The following theorem justifies using (5.10) and (5.11) as an alternative to the qd-type of algorithms that calculate coefficients of S-fractions.

Theorem 5.1. *The entries in a normal Padé table corresponding to the series in* (2.1) *satisfy the recurrence relations in* (5.10) *and* (5.11). *Furthermore,* $t_{n/(m+1)}$ *in* (5.9) *and* (5.12) *give the partial numerators of the corresponding* S*-fraction*

$$\frac{a_0}{1-}\ \frac{a_1 x}{1-}\ \frac{a_2 x}{1-}\cdots\,. \tag{5.13}$$

Proof. In his last chapter [**8**], Wall shows that the staircase entries down the main diagonal in a normal Padé table correspond to the power series in (2.1). Wall also shows that these entries agree with the n^{th}-approximants, A_n/B_n, of the S-fraction. A_n/B_n satisfies the recurrence relations

$$A_n = A_{n-1} - a_n x A_{n-2} \quad \text{and} \quad B_n = B_{n-1} - a_n x B_{n-2}\,, \tag{5.14}$$

for $n \geq 2$, $A_1 = a_0$, $A_0 = 0$, $B_1 = 1$ and $B_0 = 1$. $a_0 = c_0$ and [**8**, p. 382] the remaining a_i satisfy equations now known as the rhombus rules found in the qd-algorithm. The formulas in (5.10) and (5.11) correspond to (5.14); see references [**8, 9**] for the S-fractions corresponding to other parallel staircase diagonals.

Among the four entries of the Padé Table shown in Figure 5.1, the two recurrence formulas in (5.10) and (5.11) use two horizontal and two vertical entries respectively to derive the $[n/(m+1)]$-th Padé approximant. The FP-method applies to any two of the four $[(n-1)/m]$-, $[(n-1)/(m+1)]$-, $[n/m]$- or $[n/(m+1)]$-th Padé approximants, from which the remaining two can be derived. Consequently, for two generic pairs of these approximants Figure 5.2 displays the footprint of all contiguous adjacent approximants derivable via the FP-method.

[(n-1)/m]	[n/m]
[(n-1)/(m+1)]	[n/(m+1)]

FIGURE 5.1. Among the four approximants any pair can be derived from remaining pair.

The derivations use combinations of an *append operator* which appends a zero component to a vector, a *prepend operator* which prepends a unit or a zero component to a vector and a *truncation operator* which truncates the first or last component of a vector.

The footprints indicate that the FP-method allows progression toward, in the sense of the Manhattan metric, the beginning of the Padé Table. Even though none of the references in Section 3 show the development of a recurrence formula in this direction, such derivations exist. The data and arithmetic operations required for these derivations and for the FP-method have the same orders of magnitude. The FP-method offers a distinctly different alternative derivation that sometimes yields striking simplicity. For instance, in Figure 5.2 the $[(n-1)/(m-1)]$-th approximant satisfies

$$\frac{P_{(n-1)/(m-1)}(x)}{Q_{(n-1)/(m-1)}(x)} = \frac{P_{n/m}(x) - P_{(n-1)/m}(x)}{Q_{n/m}(x) - Q_{(n-1)/m}(x)}\,. \tag{5.15}$$

In Figure 5.2 the footprint of adjacencies for the pair of diagonal approximants, $[(n-1)/m]$ and $[n/(m+1)]$, does not include the continuation of the diagonal approximants; namely, the $[(n-2)/(m-1)]$-th and $[(n+1)/(m+2)]$-th approximants. It should be noted that a recurrence formula can be derived directly from the linear equations associated with the adjacent, but not contiguous, $[(n-1)/m]$-th and $[n/(m+1)]$-th approximants and thereby avoid staircase stepping along the diagonal. Invoking the lemma in Section 4 for derivations of approximants down staircase diagonals will produce inner products that differ substantially from those produced by derivations that proceed directly down a diagonal. Indeed, intermediate steps in deriving an approximant in a footprint will occasionally produce an approximant well outside the footprint. This behavior shows that the FP-method

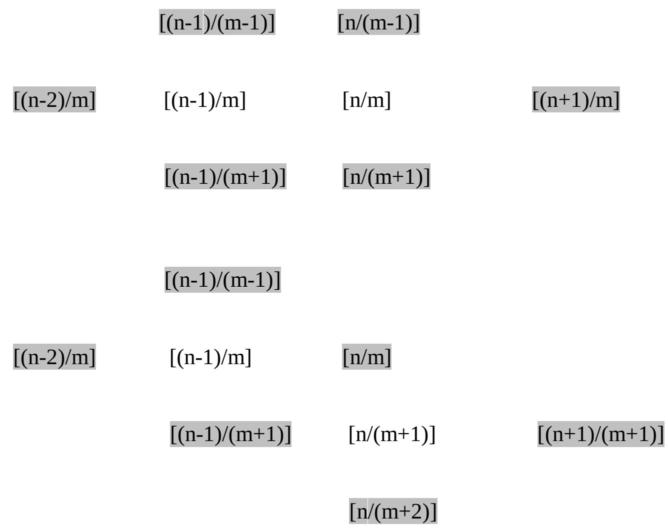

FIGURE 5.2. The generic footprints of contiguous adjacent approximants derivable via the FP-method from a horizontal or a diagonal pair of Padé approximants. Vertical and skew pairs have similar footprints.

differs fundamentally from published methods to add approximants in partially filled Padé tables.

6. Nonnormal Padé tables. This section describes another use of the FP-method that stems from intermediate steps in the derivation of approximants in typical footprints of contiguous adjacencies shown in Figure 5.2. Wall [**8**] defines a *square block of order* r as a square array of r^2 identical approximants in a nonnormal Padé table. In Wall's Theorem 98.2, 5 determinants characterize blocks of order r. This section contains theorems that give a simpler way to identify blocks in nonnormal Padé tables.

All four approximants in Figure 5.1 may all be different even though one or all may be a corner of a block within a nonnnormal Padé table. A block that has its upper left corner at the $[n/(m+1)]$-th can be characterized by different $[n/m]$-th, $[(n-1)/(m+1)]$-th and $[n/(m+1)]$-th approximants and the inability of the FP-method to produce unique $[(n+1)/(m+1)]$-th and $[n/(m+2)]$-th approximants. The proof of the following theorem makes this observation more precise for the case of an isolated block.

Theorem 6.1. *Let the* $[(n-1)/(m+1)]$*-th and* $[n/m]$*-th Padé approximants be unique. If the* $[n/(m+1)]$*-th Padé approximant satisfies*

$$-c_{n+m+2} - \mathbf{c}_{n+m+1:m+2} \cdot \alpha_0(\mathbf{b}_{n/(m+1)}) = 0\,, \tag{6.1}$$

then the $[n/(m+1)]$*-th approximant lies at the upper left hand corner of a block in a nonnormal Padé table.*

Proof. To derive the $[n/(m+2)]$-th approximant from the $[(n-1)/(m+1)]$-th and $[n/(m+1)]$-th approximants use the derivation in Section 5 for the $[n/(m+1)]$-th approximant with changes to account for the $(m+2)$-th row instead of the $(m+1)$-th row in the Padé Table. Equation (5.8) becomes

$$\mathbf{w}_{n/(m+2)} = \alpha_0(\mathbf{b}_{n/(m+1)}) + t_{n/(m+2)}\rho_1(\mathbf{b}_{(n-1)/(m+1)})\,. \tag{6.2}$$

Taking into account (6.1) and substituting $\mathbf{w}_{n/(m+2)}$ to solve the last linear equation for the $[n/(m+2)]$-th approximant implies that $t_{n/(m+2)}$ or $\mathbf{c}_{n+m+1:m+2}\cdot\rho_1(\mathbf{b}_{(n-1)/(m+1)})$ equal zero. Since $t_{n/(m+2)}$ is arbitrary, choose $t_{n/(m+2)}$ to be zero. The uniqueness of the $[(n-1)/(m+1)]$-th and $[n/m]$-th approximants proves the theorem.

Remarks. If instead of the hypothesis in Theorem 6.1

$$\mathbf{c}_{n+m+1:m+2} \cdot \rho_1(\mathbf{b}_{(n-1)/(m+1)}) = 0\,, \tag{6.3}$$

then the linear equations $\mathbf{c}_{j:m+2}\cdot\mathbf{b}_{n/(m+2)} = -c_{j+1}$ for $n \le j \le n+m+1$ corresponding to the $[n/(m+2)]$-th approximant have a nonzero null space. Furthermore, (6.3) implies that $\mathbf{b}_{(n-1)/(m+1)}$ is also a solution to the linear equations in (5.4) so that the equality of the $[(n-1)/(m+1)]$-th and $[n/(m+1)]$-th approximants is equivalent to (6.3).

Suppose that the $[(n-1)/(m+1)]$-th and $[n/(m+1)]$-th approximants are given and unequal. (6.1) implies that the $[(n-1)/(m+1)]$-th approximant lies to the left boundary of a block that contains the

$[n/(m+1)]$-th approximant. To determine whether the upper left hand corner of this block occurs at the $[n/(m+1)]$-th approximant, use the FP-method to derive the $[n/m]$-th approximant. The FP-method produces a vector

$$\mathbf{w}_{n/m} = (1 - t_{n/m})\tau_{m+1}(\mathbf{b}_{(n-1)/(m+1)}) + t_{n/m}\tau_{m+1}(\mathbf{b}_{n/(m+1)}), \tag{6.4}$$

where τ_{m+1} truncates the last component of the vectors $\mathbf{b}_{(n-1)/(m+1)}$ and $\mathbf{b}_{n/(m+1)}$. Substituting $\mathbf{w}_{n/m}$ into (5.1) produces on the right hand side of each equation a term with the factor

$$(1 - t_{n/m})b_{m+1}^{((n-1)/(m+1)))} + t_{n/m}b_{m+1}^{(n/(m+1))}. \tag{6.5}$$

which must be set to zero. This factor can be set to zero whenever

$$b_{m+1}^{((n-1)/(m+1)))} \neq b_{m+1}^{(n/(m+1))} \tag{6.6}$$

in which case an upper left hand corner of a block occurs at the $[n/(m+1)]$-th approximant. The following theorem summarizes some of these remarks.

Theorem 6.2. *If the* $[(n-1)/(m+1)]$*-th and* $[n/(m+1)]$*-th Padé approximants are unequal and if (6.1) and (6.6) are satisfied, then the* $[n/(m+1)]$*-th approximant lies at the upper left hand corner of a block in the Padé table.*

REFERENCES

1. G.A. Baker, *The Padé approximent method: Related generalizations*, in *The Padé approximant in theoretical physics* (G.A. Baker and J.L. Gammel, eds.), Academic Press, New York, 1970.

2. D.A. Field, *The focal point method for solving systems of linear equations*, J. Applied Math. and Physics (ZAMP) **36** (1985), 24–46.

3. ———, *The focal point method and almost linear systems of equations*, Contemp. Math., vol. 47, 1985, pp. 121–130.

4. C.-E. Fröberg, *Introduction to numerical analysis*, Addison-Wesley, Reading, MA, 1965.

5. W.B. Gragg, *The Padé table and its relation to certain algorithms of numerical analysis*, SIAM Review **14** (1972), 1–62.

6. J.H. McCabe, *The quotient-difference algorithm and the Padé table: An alternative form and a general continued fraction*, Math. Comp. **41** (1983), 183–197.

7. H. Rutishauser, *Der Quotient-Differenzen-Algorithmus*, Mittlg. Inst. Angew. Math. ETH, Nr. **7**, Birkhäuser Verlag, Basel, 1956.

8. H.S. Wall, *Analytic theory of continued fractions*, D. Van Nostrand Company, New York, 1948.

9. P. Wynn, *The rational approximation of functions which are formally defined by a power series expansion*, Math. Comp. **14** (1960), 147–186.

General Motors Research, Development & Planning Center, 30500 Mound Road, Warren, MI 48090-9055
E-mail address: dfield@gmr.com