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A PRIORI TRUNCATION ERROR BOUNDS FOR CONTINUED FRACTIONS

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Dedicated to W.B. Jones on his 70th birthday

ABSTRACT. Most of the known continued fraction expansions of special functions are limit periodic. This means that the classical approximants $S_n(0)$ are normally not the best ones to use for approximations. In this paper we suggest a number of approximants $S_n(w_n)$ which converge faster. The estimation of the improvement and bounds for the error $|f - S_n(w_n)|$ (which we still call the truncation error) are mainly obtained by means of Thron's parabola sequence theorem and the oval sequence theorem.

1. Introduction. A number of special functions have nice, well-known continued fraction expansions $K(a_n/1)$, such as, for instance,

(1.1)
$$\arctan z \quad \text{where } a_1 := z, \quad a_{n+1} := \frac{n^2 z^2}{4n^2 - 1} \quad \text{for } n \geq 1,$$

$$\tan z \quad \text{where } a_1 := z, \quad a_{n+1} := -\frac{z^2}{4n^2 - 1} \quad \text{for } n \geq 1, \tag{1.2}$$

the incomplete gamma function

(1.3)
$$\Gamma(a, z)$$
$$\text{where } a_1 := \frac{e^{-z} z^a}{1+z-a}, \quad a_{n+1} := \frac{-n(n-a)}{(2n-1+z- va)(2n+1+z-a)},$$

and the complementary error function

$$\operatorname{erfc}(z) \quad \text{where } a_1 := \frac{e^{-z^2}}{2z}, \quad a_{n+1} := \frac{n}{2z^2}. \tag{1.4}$$

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(For these and more examples, see for instance [**16**, pp. 560–597].) It is known that these expansions converge locally uniformly to the corresponding functions in $Z_1 := \{z \in \mathbf{C};\ |\arg(1+z^2)| < \pi\}$, $Z_2 := \mathbf{C}$, $Z_3 := \{z \in \mathbf{C};\ |\arg z| < \pi\}$ as long as $(a-z)$ is not a positive, odd integer, and $Z_4 := \{z \in \mathbf{C};\ \operatorname{Re} z > 0\}$, respectively. As often happens, these expansions have regularly varying coefficients in the sense that $a_n \to a \in \widehat{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$, monotonically, at least, in the first four cases. We distinguish between the four types:

- Type 1: $a \in A_1 := \{a \in \mathbf{C};\ |\arg(a+1/4)| < \pi\}$. (1.1) is of this type.
- Type 2: $a = 0$. (1.2) is of this type.
- Type 3: $a = -\frac{1}{4}$. (1.3) is of this type.
- Type 4: $a = \infty$. (1.4) is of this type.

The elliptic case $a < -\frac{1}{4}$ is left out, since our continued fractions normally diverge in this situation.

The purpose of this paper is to show how one can approximate functions by means of their continued fraction expansions. We want the approximations

- to have small pointwise errors,
- to be computed by easy, fast and stable algorithms,
- to have reliable and tight truncation error bounds.

In addition, we want to have bounds for the roundoff errors, but that is beyond the scope of this paper.

The approximation we consider is based on "modified" approximants. The truncation error bounds are of the a priori type. Our main tools are Thron's parabola sequence theorem and the oval sequence theorem. These are presented in Section 2 along with the basic ideas. In Section 3 we describe how we can obtain truncation error bounds, and Section 4 gives some bounds which work for Stieltjes fractions in general. Section 5 contains some technical details. In Section 6 we suggest a number of ways to construct approximations from limit periodic continued fraction expansions, and in Sections 7–10 we give some numerical illustrations and examples of truncation error bounds for the five continued fractions (1.1)–(1.4). Since the purpose is to demonstrate general techniques, we are not concerned with functional relationships which may enlarge the z-domain for our approximations.

Section 11 contains some concluding remarks.

Throughout this paper we consider convergent continued fractions $K(a_n/1)$ which satisfy the conditions of Thron's parabola sequence theorem, see Section 2. In particular this implies that the values of $K(a_n/1)$ and all its tails and critical tails, see Section 2, are *finite*. The ideas can be generalized to more general continued fractions. (See Section 5.) The computation of the approximants is done using Maple with 40 digits of precision.

2. The tool box. It is standard to write the approximants

$$S_n(w_n) := \frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_n}{1+w_n} \tag{2.1}$$

of the continued fraction

$$K(a_n/1) := \frac{a_1}{1} + \frac{a_2}{1} + \frac{a_3}{1} + \ldots = \cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\ldots}}}; \quad a_n \in \mathbf{C} \setminus \{0\} \tag{2.2}$$

as compositions of linear fractional transformations

$$S_n(w_n) = s_1 \circ s_2 \circ \cdots \circ s_n(w_n); \quad s_k(w) := \frac{a_k}{b_k + w} \tag{2.3}$$

in w. It is easy to prove that

$$S_n(w) = \frac{A_n + A_{n-1}w}{B_n + B_{n-1}w} \tag{2.4}$$

where A_n and B_n, the *canonical numerators and denominators*, are solutions of the recurrence relation

$$X_n = X_{n-1} + a_n X_{n-2} \quad \text{for } n = 1, 2, 3, \ldots \tag{2.5}$$

with $A_{-1} = 1$, $A_0 = 0$, $B_{-1} = 0$ and $B_0 = 1$. Clearly, $S_n(w_n)$ is formed by replacing the value $f^{(n)}$ of the nth *tail*

$$K_{m=n+1}^{\infty}(a_m/1) := \frac{a_{n+1}}{1} + \frac{a_{n+2}}{1} + \frac{a_{n+3}}{1} + \ldots \tag{2.6}$$

of $K(a_n/1)$ by w_n. Since $K(a_n/1) = S_n(K_{m=n+1}^{\infty}(a_m/1))$, it follows that each one of its tails converges if and only if $K(a_n/1)$ converges. (We allow convergence to ∞.)

It was already suggested by Sylvester [**17**] in 1869 that $S_n(w_n)$ normally approximates f rather well when w_n is chosen close to $f^{(n)}$. This is an important point. In computing approximants, it is always a trade-off between rounding errors and truncation errors. To minimize rounding errors, we want the computation to be stable and n to be small. To minimize truncation errors, we want n to be large. In addition we want the computation to be fast, which again means n small. So reducing n without loss of accuracy is a very good thing, and this is what we obtain when w_n is chosen close enough to $f^{(n)}$.

But we gain more! In addition to the

- convergence acceleration

we also obtain

- more stable computation and
- better truncation error bounds.

Indeed, in some cases we really need to use modifications in order to find a priori truncation error bounds which are useful at all.

Another point in favor of modifications is that given w_n, the work of computing $S_n(w_n)$ is essentially equivalent to computing the classical approximant $S_n(0)$. Indeed, we may set $x_n := w_n$, instead of $x_n := 0$, and then use the backwards algorithm

$$x_{k-1} := s_k(x_k) = a_k/(1 + x_k) \quad \text{for } k = n, n-1, \dots, 1$$

as usual. This returns $x_0 = S_n(w_n)$. Hence, it pays off to put in a little effort up front to find good values for w_n. Then a combination of a priori truncation error bounds and a priori rounding error bounds gives the size of n needed to obtain the wanted accuracy. The extreme case where $f^{(n)}$ is known, leads to the extreme "convergence acceleration" $f = S_n(f^{(n)})$.

For convenience we shall let $S_k^{(n)}(w_{n+k})$ denote the approximants of (2.6). Then $s_n \circ S_k^{(n)}(w_{n+k}) = S_{k+1}^{(n-1)}(w_{n+k})$ and $S_n \circ S_k^{(n)}(w_{n+k}) = S_{n+k}(w_{n+k})$.

We say that $\{V_n\}_{n=0}^{\infty}$; $\varnothing \neq V_n \subset \widehat{\mathbf{C}}$ (proper subset) is a sequence of *value sets* for $K(a_n/1)$ if $s_n(V_n) \subseteq V_{n-1}$ for all $n \in \mathbf{N}$. This leads to the nested sets $S_n(V_n) \subseteq S_{n-1}(V_{n-1}) \subseteq V_0$ which were the basis for Thron's celebrated parabola sequence theorem:

Thron's parabola sequence theorem [**19**]. *Let* $0 < g_n < 1$ *for* $n \geq 1$ *and* $-(\pi/2) < \alpha < (\pi/2)$ *be given, and let*

$$\begin{aligned} P_{\alpha,n} &:= \{a \in \mathbf{C};\ |a| - \operatorname{Re}(a\, e^{-i2\alpha}) \\ &\qquad \leq 2g_{n-1}(1-g_n)\cos^2\alpha\} \quad \text{for } n \geq 2, \\ V_{\alpha,n} &:= \{w \in \mathbf{C};\ \operatorname{Re}(w\, e^{-i\alpha}) \geq -g_n \cos\alpha\} \quad \text{for } n \geq 1, \\ W_{\alpha,n} &:= V_{\alpha,n} \cup \{\infty\} \quad \text{for } n \geq 1. \end{aligned} \tag{2.7}$$

Further, let $K(a_n/1)$ *have* $a_n \in P_{\alpha,n}$ *for all* $n \geq 2$. *Then the following hold.*

A. $\{V_{\alpha,n}\}$ *and* $\{W_{\alpha,n}\}$ *are sequences of value sets for* $K(a_n/1)$ *for appropriately chosen* $V_{\alpha,0}$ *and* $W_{\alpha,0}$ *(for instance,* $V_{\alpha,0} := a_1/(1+V_{\alpha,1})$, $W_{\alpha,0} := a_1/(1+W_{\alpha,1})$*).*

B. *The radius of the circular disk* $S_n(W_{\alpha,n})$ *is bounded by*

$$T_n := \frac{|a_1|}{2(1-g_1)\cos\alpha \prod_{\nu=2}^{n}\left(1 + \frac{g_{\nu-1}(1-g_\nu)(1-k_\nu+d_{\nu-1})\cos^2\alpha}{|a_\nu|}\right)} \tag{2.8}$$

where

$$d_n := \frac{G_n}{\sum_{k=0}^{n-1} G_k}; \qquad G_k := \prod_{\nu=1}^{k} \frac{1-g_\nu}{g_\nu} \tag{2.9}$$

and

$$k_n := \frac{|a_n| - \operatorname{Re}(a_n\, e^{-2i\alpha})}{2g_{n-1}(1-g_n)\cos^2\alpha}. \tag{2.10}$$

C. *If* $K(a_n/1)$ *converges, then its value* $f \in V_{\alpha,0}$ *and its tail values* $f^{(n)} \in V_{\alpha,n}$ *for all* n. *In particular this means that all* $f^{(n)}$ *and* f *are finite.*

D. *The Euclidean distance between the boundary* $\partial V_{\alpha,n}$ *of* $V_{\alpha,n}$ *and the critical tail sequence*

$$(2.11)\qquad -h_n := S_n^{-1}(\infty) = -\frac{B_n}{B_{n-1}} = -1 - \frac{a_n}{1} + \frac{a_{n-1}}{1} + \cdots + \frac{a_2}{1}$$

satisfies $\operatorname{dist}(-h_n, \partial V_{\alpha,n}) \geq g_n d_n \cos\alpha$ *for* $n \geq 1$.

Remarks 2.1.

1. Thron had another formulation for $P_{\alpha,n}$: $a_n \in P_{\alpha,n}$ if and only if

$$(2.12)\qquad a_n = e^{2i\alpha}\hat{k}_n g_{n-1}(1-g_n)(u_n + iv_n)\cos^2\alpha, \quad u_n, v_n \in \mathbf{R}$$

where $v_n^2 \leq 4u_n + 4$ and $0 \leq \hat{k}_n \leq 1$. (We have shifted the numbering of g_n.) Thron then proved that the bound T_n in (2.8) holds with $k_n = \hat{k}_n$. We shall prove that T_n still is a bound for the radius of $S_n(W_{\alpha,n})$ if k_n is given by (2.10).

Let a_n, $0 \neq a_n \in P_{\alpha,n}$, be given, where $P_{\alpha,n}$ is given by (2.7). Let first $v_n \neq 0$ or $v_n = 0$ with $u_n < 0$. Then a_n can be written as in (2.12) for some $0 < \hat{k}_n \leq 1$, where

$$v_n^2 = 4u_n + 4$$

and

$$c_n = e^{2i\alpha} g_{n-1}(1-g_n)(u_n + iv_n)\cos^2\alpha$$

is the point of intersection between the parabola $\partial P_{\alpha,n}$ and the ray from the origin through a_n. This follows since $P_{\alpha,n}$ is a convex set. Indeed,

$$\begin{aligned}|a_n| - \operatorname{Re}(a_n e^{-2i\alpha}) &= \hat{k}_n\{|c_n| - \operatorname{Re}(c_n e^{-2i\alpha})\}\\ &= \hat{k}_n g_{n-1}(1-g_n)\{(u_n+2) - u_n\}\cos^2\alpha\end{aligned}$$

which shows that $\hat{k}_n = k_n$ as given by (2.10). In particular, $0 < k_n \leq 1$.

Next, let $v_n = 0$ and $u_n > 0$. Then a_n lies on the axis of the parabola $\partial P_{\alpha,n}$, and k_n given by (2.10) is equal to 0. The expression (2.12) for a_n can then be written $a_n = \hat{k}_n \hat{c}_n$ where $\hat{c}_n = e^{2i\alpha} g_{n-1}(1-g_n)\hat{x}_n \cos^2\alpha$ for an arbitrarily large $\hat{x}_n > 0$. Thron's bound works for all $\hat{x}_n > 0$.

Hence $\lim_{\hat{x}_n\to\infty} T_n$ must also work, that is, we can use $k_n = \hat{k}_n = 0$ in the formula for T_n.

What we gain by our formulation of Thron's parabola sequence theorem is an optimal choice for k_n to give a minimal value for T_n. (See also [**15**].)

2. If $T_n \to 0$, then the nestedness $S_n(W_{\alpha,n}) \subseteq S_{n-1}(W_{\alpha,n-1})$ shows that $S_n(w_n)$ converges uniformly with respect to $w_n \in W_{\alpha,n}$ to its value f. From (2.8) we see that $T_n \to 0$ if $\sum g_{\nu-1}(1-g_\nu)(1-k_\nu+d_{\nu-1})/|a_\nu| = \infty$. If $\limsup k_n < 1$, then $T_n \to 0$ if $\sum g_{\nu-1}(1-g_\nu)/|a_\nu| = \infty$. In particular, $\limsup k_n < 1$ if $\arg a_n = 2\alpha$ from some n on.

3. $2T_n$ is an upper bound for $|f - S_n(w_n)|$ whenever $a_k \in P_{\alpha,k}$ for $2 \le k \le n$ and $w_n \in W_{\alpha,n}$.

4. We emphasize again that if $0 \ne a_n \in P_{\alpha,n}$ for all $n \ge 2$, then $h_n \ne \infty$ for all $n \ge 1$ and $S_n(w_n) \ne \infty$ when $w_n \in W_{\alpha,n}$. Moreover, if $T_n \to 0$, then $K(a_n/1)$ converges, and all $f^{(n)} \ne \infty$ and $f \ne \infty$.

The oval sequence theorem (OST) [**16**, p. 145] is similar to Thron's parabola sequence theorem, but the value sets V_n are circular disks instead of half planes $V_{\alpha,n}$. (There is an unfortunate misprint in [**16**, p. 146]. The first factor, $2R_0$, in the truncation error bound (5.4.17) should be replaced by $2R_n$. If we use $w_n = C_n$, we can even replace it by R_n as we do in this paper. We have also replaced $|w_0| + R_0$ by $|a_1|/(|1+w_1| - R_1)$ which is a better bound for $|f|$.)

The oval sequence theorem. *Let $w_n \in \mathbf{C}$ and $0 \le R_n < |1+w_n|$ be given for $n = 1, 2, 3, \ldots$ such that*

$$\begin{aligned} E_n := \{a \in \mathbf{C};\ &|a(1+\overline{w}_n) - w_{n-1}(|1+w_n|^2 - R_n^2)| + |a|R_n \\ &\le R_{n-1}(|1+w_n|^2 - R_n^2)\} \ne \varnothing \quad \textit{for } n = 2, 3, 4, \ldots . \end{aligned} \tag{2.13}$$

Let, further, $K(a_n/1)$ be a continued fraction with $a_n \in E_n$ for all $n \ge 2$. Then the following hold.

A. $V_n := \{w \in \mathbf{C};\ |w - w_n| \le R_n\}$ *for $n = 0, 1, 2, \ldots$ is a sequence of value sets for $K(a_n/1)$ for appropriately chosen V_0.*

B. $|S_{n+m}(w) - S_n(w_n)| \le Q_n := \frac{|a_1|}{|1+w_1|-R_1} \cdot \frac{R_n}{|1+w_n|} \prod_{k=1}^{n-1} M_k$ *for $n \ge 1$ and $m \ge 0$ when $w \in V_{n+m}$, where $M_k := \max_{u \in V_k} |u/(1+u)|$.*

Proof. A. Let $w \in V_n$. Then $w = w_n + r\,e^{i\theta}$ for some $r \in [0, R_n]$ and $\theta \in \mathbf{R}$. We need to show that $|\frac{a_n}{1+w} - w_{n-1}| \le R_{n-1}$. We have

$$\left|\frac{a_n}{1+w} - w_{n-1}\right| = \left|\frac{a_n - w_{n-1}(1 + w_n + r\,e^{i\theta})}{1 + w_n + r\,e^{i\theta}}\right|$$
$$\le \frac{|a_n - w_{n-1}(1+w_n)| + |w_{n-1}|R_n}{|1+w_n| - R_n} \le R_{n-1}$$

where the last inequality follows from the fact that $a_n \in E_n$.

B. Clearly,

$$|S_{n+m}(w) - S_n(w_n)| \le |S_m^{(n)}(w) - w_n| \cdot \max_{u \in V_n} |S_n{}'(u)| \le R_n \cdot \max_{u \in V_n} |S_n{}'(u)|$$

where

$$S_n'(u) = \{s_1 \circ s_2 \circ \cdots \circ s_n\}'(u)$$
$$= \prod_{k=1}^{n} \frac{-a_k}{(1+u_k)^2} = \prod_{k=1}^{n} \frac{-u_{k-1}}{1+u_k} = \frac{-u_0}{1+u} \prod_{k=1}^{n-1} \frac{-u_k}{1+u_k}$$

where $u_m := s_{m+1} \circ s_{m+2} \circ \cdots \circ s_n(u) \in V_m$ for $m = 0, 1, 2, \ldots, n-1$ and $u_n := u$. Since $|u_0| = |a_1/(1+u_1)| \le |a_1|/(|1+w_1| - R_1)$, this proves the assertion. □

We shall apply these two theorems in the following way:

- Given $K(a_n/1)$ with $0 \ne a_n \in P_{\alpha,n}$ for $n \ge 2$ for a fixed $\alpha \in (-\pi/2, \pi/2)$, where $T_n \to 0$.
- Choose approximants $S_n(w_n)$, i.e., choose w_n, with $w_n \in V_{\alpha,n}$.
- Let w_n be the center of V_n and choose R_n such that $a_n \in E_n$ for all $n \ge 2$.
- Then $|f^{(n)} - w_n| \le R_n$ and $|f - S_n(w_n)| \le Q_n$ for $n \ge 1$.
- To prove that $a_n \in E_n$ we shall compare a_n to $\hat{a}_n := w_{n-1}(1 + w_n)$, see Remark 2.2.2 below. Therefore we try to make $|a_n - \hat{a}_n|$ small when we choose $\{w_n\}$.

To measure the improvement obtained by using $S_n(w_n)$ instead of $S_n(0)$ to approximate our finite function values f, we shall use the ratio

$$\Phi_n(w_n, u) := \frac{f - S_n(w_n)}{f - S_n(u)} = \frac{h_n + u}{h_n + w_n} \cdot \frac{f^{(n)} - w_n}{f^{(n)} - u} \tag{2.14}$$

with $u := 0$. This expression for Φ_n is easily derived from the fact that $f = S_n(f^{(n)})$. The factor $h_n/(h_n + w_n)$ is normally bounded (but not always, as we shall see).

Remarks 2.2. 1. OST, as given in [**16**, p. 145], also gives convergence criteria for $K(a_n/1)$, but we shall not use them here. As described above, we apply Thron's parabola sequence theorem to prove convergence of $K(a_n/1)$. That $K(a_n/1)$ converges to the "right value" is a consequence of the correspondence, [**11**, p. 176], [**16**, p. 271].

2. For given values of w_n and R_n, the sets E_n are bounded by Cartesian ovals, symmetric about the axis $z = t\, e^{2i\alpha_n}$, $v_{1,n} \le t \le v_{2,n}$, where $2\alpha_n := \arg \hat{a}_n$, $\hat{a}_n := w_{n-1}(1 + w_n)$, and

$$\begin{aligned} v_{1,n} &:= (|w_{n-1}| - R_{n-1})(|1 + w_n| + \varepsilon_n R_n), \\ v_{2,n} &:= (|w_{n-1}| + R_{n-1})(|1 + w_n| - R_n), \end{aligned}$$

where $\varepsilon_n := 1$ if $R_{n-1} \le |w_{n-1}|$ and $\varepsilon_n := -1$, otherwise [**16**, formulas (4.4.19), (4.4.20)]. If, in particular, $r_n := R_{n-1}|1 + w_n| - R_n|w_{n-1}| - R_n R_{n-1} > 0$, then the circular disk $D_n := \{a \in \mathbf{C};\ |a - \hat{a}_n| \le r_n\}$ is contained in E_n, where $\hat{a}_n := w_{n-1}(1 + w_n)$. It is often easier to check if $a_n \in D_n$.

3. The linear fractional transformation $t(u) := u/(1 + u)$ maps the circular disk V_k onto a circular disk with center c_k and radius ρ_k given by

$$c_k := 1 - \frac{1 + \overline{w}_k}{|1 + w_k|^2 - R_k^2} \quad \text{and} \quad \rho_k := \frac{R_k}{|1 + w_k|^2 - R_k^2}.$$

Hence

$$M_k = |c_k| + \rho_k = \frac{|w_k + |w_k|^2 - R_k^2| + R_k}{|1 + w_k|^2 - R_k^2}.$$

If $w_k > 0$ and $w_k(1 + w_k) \ge R_k^2$, this reduces to $M_k = (w_k + R_k)/(1 + w_k + R_k)$, and if $w_k < 0$ and $1 + w_k > 0$, then $M_k = (|w_k| + R_k)/(1 - |w_k| - R_k)$.

Notation. We shall use the notation introduced so far throughout the paper. That is,

- Z_k for $k = 1, \ldots, 4$ denotes the convergence sets for (1.1)–(1.4).

- $\mathcal{A}_1 := \{a \in \mathbf{C};\ |\arg(a+1/4)| < \pi\}$.
- S_n, s_n, $S_n^{(m)}$ are the linear fractional transformations, and $S_n(w_n)$ are the approximants of $K(a_n/1)$.
- A_n and B_n are the canonical numerators and denominators of $K(a_n/1)$.
- f, $f^{(n)}$ are the values of $K(a_n/1)$ and its tails, and $h_n := -S_n^{-1}(\infty)$ gives its critical tail sequence $\{-h_n\}$.
- α, g_n, d_n, $P_{\alpha,n}$, $V_{\alpha,n}, W_{\alpha,n}$, T_n are as given in Thron's parabola sequence theorem.
- E_n, V_n, R_n, M_n, Q_n are as given in OST.
- $\hat{a}_n$, D_n, r_n, $\Phi_n(w_n, u)$ are as given in Remark 2.2.2 and formula (2.14).

In addition, we shall use

- $\Delta_n := |1+w_n| - |w_{n-1}|$ and $\Delta := \liminf_{n\to\infty} \Delta_n$.
- $\psi_n = \mathcal{O}(\varphi_n)$ as $n \to \infty$ to mean that $\limsup_{n\to\infty} |\psi_n/\varphi_n| < \infty$.
- $\psi_n \sim \varphi_n$ as $n \to \infty$ to mean that $\lim_{n\to\infty} \psi_n/\varphi_n = 1$.
- $\delta_n := \sup_{m\geq n} |a_m - \hat{a}_m|$.

3. Truncation error bounds. It goes without saying that finding useful w_n and estimating their effect $\Phi_n(w_n, 0)$ is easier than to come up with good, reliable truncation error bounds. It is also a question of what we mean by "good" error bounds. Should they be small, or easy to compute, or valid for a large z-region? It may be hard to achieve all this in one go. In practice it is useful to have different bounds for different purposes.

3.1. The bounds $2T_n$ and Q_n. The bound

$$|f - S_n(w_n)| \leq 2T_n \quad \text{when } w_n \in W_{\alpha,n} \quad \text{and} \quad a_k \in P_{\alpha,k} \quad \text{for all } k \geq 2 \tag{3.1}$$

from Thron's parabola sequence theorem works for all $z \in Z_i$ in our examples. Normally it is not hard to determine what $\{g_n\}$ and α to use for a given continued fraction expansion. Hence this bound is simple to use. It takes some effort to compute, but in Section 3.3 we show

how the product may be replaced by a power at a low cost of accuracy under proper conditions. Hence $2T_n$ is a very nice bound.

The drawback is that it is not small, although it was obtained by very careful estimation. Since T_n is a bound for the radius of $S_n(W_{\alpha,n})$, it is valid also if w_n is chosen far away from $f^{(n)}$. One might say that $V_{\alpha,n}$ is too large to give good truncation error bounds in general. In particular, the bound does not pick up the convergence acceleration we obtain by using good modifying factors w_n in our approximants $S_n(w_n)$.

This is compensated for in OST, where V_n may be chosen small. Hence the bound

$$|f - S_n(w_n)| \le Q_n \quad \text{when } a_k \in E_k \quad \text{for all } k \ge 2 \tag{3.2}$$

is normally much smaller. Indeed, since we aim at $R_n \to 0$, we normally get $Q_n/T_n \to 0$. The bound Q_n reduces to the well known result below when we set $w_n := 0$ and $R_n := g_n$ for all n.

Theorem 3.1. *Let $|a_n| \le g_{n-1}(1 - g_n)$ for all $n \ge 2$, where $0 < g_n < 1$ for all n. Then $K(a_n/1)$ converges, $|f^{(n)}| \le g_n$ for $n \ge 1$ and*

$$|f - S_n(0)| \le \frac{|a_1|}{1 - g_1}\, g_n \prod_{k=1}^{n-1} \frac{g_k}{1 - g_k} \quad \text{for } n \ge 2. \tag{3.3}$$

Also the product in Q_n can be replaced by a power under proper conditions. But both in (3.2) and (3.3) the hard part remains: finding values for $R_n = g_n$ that work for a given continued fraction. In Section 4 we shall see some techniques for picking R_n. It turns out that the closer our choice of w_n is to the actual tail value $f^{(n)}$, the easier it normally is to come up with useful values for R_n, and thus to find good error bounds.

3.2. A useful trick. We want R_n to be small to make M_k small, but it has to be large enough to include a_n in E_n for all $n \ge 2$. This can sometimes only be achieved from some larger n on. Then the following formulas may be useful:

Lemma 3.2.

(3.4)
$$f-S_n(w_n)=\frac{(f_{N-1}-f_N)h_N}{(h_N+f^{(N)})(h_N+S_{n-N}^{(N)}(w_n))}\,(f^{(N)}-S_{n-N}^{(N)}(w_n))$$
$$=\frac{(f_{N-1}-f_N)(1+f^{(N+1)})(1+S_{n-N-1}^{(N+1)}(w_n))}{h_N(h_{N+1}+f^{(N+1)})(h_{N+1}+S_{n-N-1}^{(N+1)}(w_n))}$$
$$\cdot\,(f^{(N)}-S_{n-N}^{(N)}(w_n))$$

for $n>N$.

Here h_N and h_{N+1} are given by (2.11), and $f_{N-1}:=S_{N-1}(0)$ and $f_N:=S_N(0)$ can be computed, at least for reasonably small N. If $a_n\in E_n$ for $n\geq N+2$, then $S_{n-N-1}^{(N+1)}(w_n)$ and $f^{(N+1)}$ are elements in V_{N+1}. Moreover, OST gives a bound for $|f^{(N)}-S_{n-N}^{(N)}(w_n)|$. This gives us a bound for the second expression in (3.4). If we also know that $a_{N+1}\in E_{N+1}$, then also $f^{(N)}$ and $S_{n-N}^{(N)}(w_n)$ are elements in V_N, and we can use the first expression in (3.4).

Proof of Lemma 3.2. The first expression for $f-S_n(w_n)$ follows, since $f=S_N(f^{(N)})$ and $S_n(w_n)=S_N(S_{n-N}^{(N)}(w_n))$ where $S_N(w)=(A_N+A_{N-1}w)/(B_N+B_{N-1}w)$ and $h_N=B_N/B_{N-1}$.

The second expression follows from the first one since
$$h_N=\frac{a_{N+1}}{h_{N+1}-1},\qquad f^{(N)}=\frac{a_{N+1}}{1+f^{(N+1)}}$$
and
$$S_m^{(N)}(w)=\frac{a_{N+1}}{1+S_{m-1}^{(N+1)}(w)}.\qquad\square$$

For $N=1$ we have $h_1=1$, $f_0=0$ and $f_1=a_1$, and thus

(3.5)
$$|f-S_n(w_n)|\leq|a_1|\cdot H_2^2\cdot\frac{|a_2|}{|1+w_2|-R_2}\cdot\frac{R_n}{|1+w_n|}\prod_{k=2}^{n-1}M_k\quad\text{for } n\geq 2$$

if $a_n \in E_n$ for $n \geq 3$, where

$$H_k := \max_{u \in V_k} |(1+u)/(h_k+u)|. \tag{3.6}$$

Similarly, since $h_2 = 1 + a_2$, $f_1 = a_1$ and $f_2 = a_1/(1+a_2)$, $N = 2$ gives

$$\begin{aligned} |f - S_n(w_n)| \leq \frac{|a_1 a_2|}{|1+a_2|^2} \cdot H_3^2 \cdot \frac{|a_3|}{|1+w_3| - R_3} \\ \cdot \frac{R_n}{|1+w_n|} \prod_{k=3}^{n-1} M_k \quad \text{for } n \geq 3 \end{aligned} \tag{3.7}$$

if $a_n \in E_n$ for $n \geq 4$. More generally

$$\begin{aligned} |f - S_n(w_n)| \leq \left| \frac{f_{N-1} - f_N}{h_N} \right| \cdot H_{N+1}^2 \cdot \frac{|a_{N+1}|}{|1+w_{N+1}| - R_{N+1}} \\ \cdot \frac{R_n}{|1+w_n|} \prod_{k=N+1}^{n-1} M_k \end{aligned}$$

for $n \geq N+1$ if $a_n \in E_n$ for $n \geq N+2$.

Formula (3.4) is also useful if we know that $a_n \in P_{\alpha,n}$ for $n \geq N+2$, where $P_{\alpha,n}$ is as in Thron's parabola sequence theorem. Then $f^{(N+1)} \in V_{\alpha,N+1}$ and $S_{n-N-1}^{(N+1)}(w_n) \in V_{\alpha,N+1}$ if $w_n \in V_{\alpha,n}$. Therefore,

$$|f - S_n(w_n)| \leq \left| \frac{f_{N-1} - f_N}{h_N} \right| \cdot \frac{\tilde{H}_{N+1}^2 |a_{N+1}|}{2(1-g_{N+1})\cos\alpha} \Big/ \prod_{\nu=N+2}^{n} \tilde{M}_\nu \tag{3.8}$$

for $n \geq N+2$, where

$$\tilde{H}_\nu := \sup_{u \in V_{\alpha,\nu}} |(1+u)/(h_\nu+u)| \tag{3.9}$$

and

$$\tilde{M}_\nu := 1 + \frac{1}{|a_\nu|} \Big(g_{\nu-1}(1-g_\nu)(1-k_\nu+d_{\nu-1})\cos^2\alpha \Big). \tag{3.10}$$

Remarks 3.1.

1. We may want to choose N larger than necessary to get the bound smaller.

2. By the same kind of argument as in Remark 2.2.3 we find that

$$\begin{aligned}H_k = \max_{u\in V_k}\left|\frac{1+u}{h_k+u}\right| &= \frac{(|\overline{h_k}+\overline{w_k})(1+w_k)-R_k^2|+|h_k-1|R_k}{|h_k+w_k|^2-R_k^2}\\ &\le \frac{|1+w_k|+R_k}{|h_k+w_k|-R_k}\end{aligned}$$

with equality if $h_k < 1$ and $(h_k+w_k)(1+w_k) > R_k^2$. If $h_k \ge 1$, $h_k+w_k+R_k > 0$ and $(h_k+w_k)(1+w_k) \ge R_k^2$, then H_k reduces to $(1+w_k+R_k)/(h_k+w_k+R_k)$.

3. Similarly, we find that

$$\tilde{H}_k = \sup_{u\in V_{\alpha,k}}\left|\frac{1+u}{h_k+u}\right| = |c_k|+\rho_k$$

where

$$c_k := 1+\frac{(1-h_k)e^{-i\alpha}}{2\mathrm{Re}\,(h_ke^{-i\alpha}-g_k)\cos\alpha},\qquad \rho_k := \frac{|1-h_k|}{2\mathrm{Re}\,(h_ke^{-i\alpha}-g_k)\cos\alpha}$$

since $-h_k \notin V_{\alpha,k}$, and so $t_k(u) := (1+u)/(h_k+u)$ maps $W_{\alpha,k} := V_{\alpha,k}\cup\{\infty\}$ onto a circular disk with center at c_k and radius ρ_k.

4. Since

$$h_N = \frac{B_N}{B_{N-1}} \quad\text{and}\quad f_{N-1}-f_N = \frac{\prod_{k=1}^N(-a_k)}{B_NB_{N-1}},$$

it follows from the second expression in (3.4) that

$$|f-S_n(w_n)| \le \Big|\prod_{k=1}^N(-a_k)\Big|\cdot \widehat{H}_{N+1}^2|f^{(N)}-S_{n-N}^{(N)}(w_n)|$$

when $a_n\in E_n$ for $n\ge N+2$, where

$$\widehat{H}_k = \max_{u\in V_k}|(1+u)/(B_k+B_{k-1}u)| = |B_{k-1}|\cdot H_k.$$

3.3. Simplifications and other ideas.

1. The product $\prod_{k=N+1}^{n-1} M_k$ in Q_n involves a lot of computation and makes it harder to determine in advance what n to use in $S_n(w_n)$ to reach the wanted accuracy, something we need in order to estimate the roundoff error a priori. However, we often have that M_k decreases monotonically with k. In such cases we may replace this product by a power of M_{N+1} (or by $M_{N+1}M_{N+2}\dots M_{N+m-1}M_{N+m}^{n-1-N-m}$ for some $m \in \mathbf{N}$). A similar trick can be used to simplify the product in T_n.

2. The bound

$$
\begin{aligned}
|h_n+\zeta| &\geq \operatorname{dist}(-h_n, \partial V_{\alpha,n}) + \operatorname{dist}(\zeta, \partial V_{\alpha,n}) \\
&\geq d_n g_n \cos\alpha + \operatorname{Re}(\zeta e^{-i\alpha}) + g_n \\
&\geq d_n g_n \cos\alpha \quad \text{for } \zeta \in V_{\alpha,n}
\end{aligned}
\tag{3.11}
$$

from Thron's parabola sequence theorem, can be used to simplify the first expression in (3.4) for large N. Since $h_N = 1 + a_{N+1}/h_{N+1}$, it gives

$$
\begin{aligned}
|f - S_n(w_n)| \leq{} & \frac{(1 + |a_{N+1}|/d_{N+1}g_{N+1}\cos\alpha)|f_{N-1} - f_N|}{(d_N g_N \cos\alpha)^2} \\
& \cdot \frac{|a_{N+1}|}{|1+w_{N+1}| - R_{N+1}} \cdot \frac{R_n}{|1+w_n|} \prod_{k=N+1}^{n-1} M_k
\end{aligned}
\tag{3.12}
$$

for $n \geq N+1$ if $a_k \in E_k$ for $k \geq N+2$, $a_k \in P_{\alpha,k}$ for $2 \leq k \leq n$, $w_N \in V_{\alpha,N}$ and $w_n \in V_{\alpha,n}$. This simplification may be useful if $a_k \in E_k$ only for large indices.

3. Probably a better idea is to combine Thron's parabola sequence theorem with (3.11) and the bound $|f^{(n)} - w_n| \leq R_n$ from OST:

$$
\begin{aligned}
|f - S_n(w_n)| &= |\Phi_n(w_n, \infty)| \cdot |f - S_n(\infty)| \\
&\leq |\Phi_n(w_n, \infty)| \cdot 2T_n \\
&= \left| \frac{f^{(n)} - w_n}{h_n + w_n} \right| \cdot 2T_n \\
&\leq \frac{2R_nT_n}{|h_n + w_n|} \leq \frac{2R_nT_n}{d_n g_n \cos\alpha + \operatorname{Re}(w_n e^{-i\alpha}) + g_n}
\end{aligned}
\tag{3.13}
$$

when $a_k \in P_{\alpha,k}$ for $k \geq 2$, $w_n \in V_{\alpha,n}$ and $a_k \in E_k$ for all $k \geq n+2$.

4. Yet another bound can be obtained by using Thron and Waadeland's bounds [**20**] for $|\Phi_n(w,0)|$ which work under special conditions for $K(a_n/1)$ of type 1 or 3. Then

$$|f - S_n(w)| = |\Phi_n(w,0)| \cdot |f - S_n(0)| \le |\Phi_n(w,0)| \cdot 2T_n. \tag{3.14}$$

We shall mainly examine bounds of the types (3.1), (3.2) combined with (3.4) and (3.13) in this paper.

4. General truncation error bounds for Stieltjes fractions. The continued fractions $K(a_n/1)$ in our examples (1.1), (1.2), (1.4) and (1.5) are of Stieltjes type. In particular this means that $\arg a_n$ is independent of n for $n \ge 2$. For such continued fractions the bound from Thron's parabola sequence theorem can be made smaller and simpler. Clearly we may choose $2\alpha := \arg a_n$ for $n \ge 2$ except if $a_n < 0$, in which case we choose $\alpha := 0$. Assume that $2\alpha = \arg a_n$ for $n \ge 2$. Then $k_n = 0$, and we may choose $g_n = g$ for all n. Any $g \in (0,1)$ may be used. Used in (2.9) this gives

$$d_n = \frac{G^n(1-G)}{1-G^n} = \frac{G-1}{1-1/G^n} \quad \text{where} \quad G := \frac{1-g}{g}.$$

Since $k_\nu = 0$, this gives

$$\begin{aligned} g_{\nu-1}(1-g_\nu)(1-k_\nu+d_{\nu-1}) &= g(1-g)\,\frac{G-1/G^{\nu-1}}{1-1/G^{\nu-1}} \\ &= (1-g)^2\,\frac{1-1/G^\nu}{1-1/G^{\nu-1}} \to 1 \end{aligned}$$

as $g \to 0$. Hence we can use the bound

$$|f - S_n(w_n)| \le 2T_n := \frac{|a_1|}{\cos\alpha} \Big/ \prod_{\nu=2}^{n} \left(1 + \frac{\cos^2\alpha}{|a_\nu|}\right) \tag{4.1}$$

for $\operatorname{Re}(w_n e^{-i\alpha}) \ge 0$ and $n \ge 2$ when $\arg a_n = 2\alpha$ for all $n \ge 2$ and $|\alpha| < \pi/2$. In particular this works for $w_n = 0$. Notice also that $g_n d_n = g(G-1)/(1-1/G^n) \to 0$ as $g \to 1$. This means that the bound (3.13) takes the form

$$|f - S_n(w_n)| \le \frac{2R_nT_n}{1 + \operatorname{Re}(w_n e^{-i\alpha})} \tag{4.2}$$

if $\alpha = \frac{1}{2}\arg a_n \in [(\pi/2),(\pi/2)]$ and $a_k \in E_k$ for all $k > n$, where $2T_n$ is given by (4.1).

If $a_n < 0$ we have $\alpha = 0$ and $\cos\alpha = 1$. For this case we have $a_n \in P_{0,n}$ if and only if $|a_n| \leq g_{n-1}(1-g_n)$, and we are normally better off using Theorem 3.1. The problem of finding suitable $g_n = R_n$ is treated in the next section.

Gragg and Warner [**3**] derived the bound

$$|f - S_n(0)| \leq \frac{2|a_1|}{\cos\alpha}\prod_{\nu=2}^{n}\frac{\sqrt{1+4|a_\nu|/\cos^2\alpha}\,-1}{\sqrt{1+4|a_\nu|/\cos^2\alpha}\,+1} \quad \text{for } n \geq 1 \tag{4.3}$$

for continued fractions of $K(a_n/1)$ with $\arg a_n = 2\alpha$ for $n \geq 2$ and $|\alpha| < \pi/2$. (They had a slightly different form, but setting $\zeta := e^{-i\alpha}$ in their expression, brings it over to (4.3).) The factor in front of the product in (4.3) is twice the size of the corresponding factor in (4.1). On the other hand, (4.3) probably wins in the long run since

$$\frac{1}{1+1/\mu} > \frac{\sqrt{1+4\mu}\,-1}{\sqrt{1+4\mu}\,+1} \quad \text{for all } \mu > 0.$$

But keep in mind that (4.3) only works for classical approximants $S_n(0)$.

5. The choice of R_n. OST is designed to estimate the effect of choosing w_n close to $f^{(n)}$. Since $f^{(n)} \in V_n$, we have $|f^{(n)} - w_n| \leq R_n$. However, the radii R_n have to be chosen or guessed in advance. If Δ_n is positive and not too small, the following rather coarse lemma is of help to guess suitable values for R_n.

Lemma 5.1.

A. *If $\Delta_n > 0$, $R_n \leq R_{n-1}$, $0 < R_n \leq \frac{1}{2}\Delta_n$ and $|a_n - \hat{a}_n| \leq \frac{1}{2}\Delta_n R_n$ for a fixed $n \in \mathbf{N}$, then $a_n \in D_n \subseteq E_n$.*

B. *Let $K(a_n/1)$ and $\{w_n\}$ be such that all $\Delta_n > 0$ and*

$$R_n := \sup_{m \geq n} \frac{2|a_m - \hat{a}_m|}{\Delta_m} \tag{5.1}$$

satisfies $R_n \leq \frac{1}{2}\Delta_n$ for all $n \in \mathbf{N}$, and set $R_0 := R_1$. Then $a_n \in E_n$ for all $n \in \mathbf{N}$.

C. *Let* R_n *be as in part B with* $R_n \le \frac{1}{2}\Delta_n$ *for* $n > n_0$. *If there exists an* $\alpha \in (-\pi/2, \pi/2)$ *such that* $a_n \in P_{\alpha,n}$ *and* $V_n \cap V_{\alpha,n} \neq \varnothing$ *from some* n *on, and* $\lim_{n\to\infty} T_n = 0$, *then* $K(a_n/1)$ *converges,* $|f^{(n)} - w_n| \le R_n$ *for* $n > n_0$, *and* $|f^{(n)} - w_n| \le R_{n+1}$ *for* $n = n_0$. *If, moreover,* $\liminf \Delta_n > 0$, *then* $f^{(n)} - w_n = \mathcal{O}(\delta_n)$ *as* $n \to \infty$.

Proof. A. We have

$$\begin{aligned} r_n &:= R_{n-1}|1+w_n| - R_n|w_{n-1}| - R_nR_{n-1} \\ &\ge R_n(\Delta_n - R_n) \ge \frac{1}{2}\,\Delta_n R_n > 0. \end{aligned}$$

By Remark 2.2.2 it therefore suffices to prove that $|a_n - \hat{a}_n| \le r_n$, which clearly holds under our conditions.

B. Since $|a_n - \widehat{a}_n| \le \frac{1}{2}\Delta_n R_n$ where $R_n \le R_{n-1}$, the result follows from part A.

C. Without loss of generality (we may look at a tail of $K(a_n/1)$) we may assume that the conditions hold for all $n > n_0 = 0$, and set $R_0 := R_1$. Since $T_n \to 0$, we know that $K(a_n/1)$ converges to some $f \in V_{\alpha,0}$ and $S_n(u_n) \to f$ uniformly with respect to $u_n \in V_{\alpha,n}$. Since $a_n \in P_{\alpha,n} \cap E_n$ for all n, we have $S_n(V_{\alpha,n} \cap V_n) \subseteq S_{n-1}(V_{\alpha,n-1} \cap V_{n-1})$ for all n, and thus $f^{(n)} \in V_{\alpha,n} \cap V_n$ for all n. That is, $|f^{(n)} - w_n| \le R_n$.

If $\Delta := \liminf \Delta_n > 0$, it is evident from (5.1) that $R_n = \mathcal{O}(\delta_n)$. $\square$

Remarks 5.1.

1. If $\Delta_n > 0$ is small, then $R_n \le \frac{1}{2}\Delta_n$ is a severe restriction. By Remark 2.2.2 it is clear that we only need $\frac{1}{2}\Delta_n R_n \le r_n$, i.e., $(2R_{n-1} - R_n)|1+w_n| - R_n|w_{n-1}| - 2R_nR_{n-1} \ge 0$ when R_n is given by (5.1) to conclude that $a_n \in E_n$.

2. If $\frac{1}{2}\Delta_n R_n \not\le r_n$, then one may try to choose $\{R_n\}$ which converges monotonically, but slower than (5.1), to 0.

3. If $|a_n - \hat{a}_n|/\Delta_n$ decreases monotonically towards 0, then (5.1) reduces to $R_n := 2|a_n - \hat{a}_n|/\Delta_n$.

In [**5**, Theorem 4.1] the conditions on Δ_n were modified by means of a sequence $\{t_n\}$ of positive numbers to be freely chosen. It was sufficient that

$$\tilde{\Delta}_n := \frac{|1+w_n|}{t_n} - \frac{|w_{n-1}|}{t_{n-1}}$$

(or more generally

$$\tilde{\Delta}_n := \frac{|b_n+w_n|}{t_n} - \frac{|w_{n-1}|}{t_{n-1}}$$

for continued fractions $K(a_n/b_n)$) satisfied conditions as above to obtain expressions for R_n. However, this just amounts to using an equivalence transformation [**11**, p. 31], [**16**, p. 72] on $K(a_n/1)$ or $K(a_n/b_n)$. That is, if $\{\rho_n\}_{n=0}^{\infty}$ is a sequence of nonzero numbers with $\rho_0 := 1$, then $K(\rho_{n-1}\rho_n a_n/\rho_n b_n)$ has the same classical approximants as $K(a_n/b_n)$. Moreover, if $K(a_n/b_n)$ converges and has tail values $f^{(n)}$, then $K(\rho_{n-1}\rho_n a_n/\rho_n b_n)$ converges and has tail values $\rho_n f^{(n)}$. We say that $K(\rho_{n-1}\rho_n a_n/\rho_n b_n)$ is *equivalent* to $K(a_n/b_n)$. This equivalence transformation has some nice consequences:

- Thron's parabola sequence theorem and OST can be formulated for continued fractions $K(a_n/b_n)$. The multiple parabola theorem [**10**, Theorem 5.2] and [**5**, Theorem 4.1] show examples of how these versions may look.
- $\tilde{\Delta}_n$ is what we get for $K(\rho_{n-1}\rho_n a_n/\rho_n b_n)$ with $|\rho_n| := 1/t_n$.

The trick with the introduction of t_n is still important, though. It shows that one should choose an equivalent form of the continued fraction with care. For most of the ideas in this paper we want $\{a_n\}$ and $\{b_n\}$ to converge monotonically in $\widehat{\mathbf{C}}$. Still, for our particular examples, already $\{a_n\}$ in $K(a_n/1)$ has this property.

6. Choosing $\{w_n\}$ for limit periodic continued fractions. As always we assume that the continued fraction has the form $K(a_n/1)$ where $a_n \to a \in \widehat{\mathbf{C}}$, and that it converges to a finite value. Let us first examine the effect $\Phi_n(w_n, 0)$ of some different choices of w_n for our four types of continued fractions:

Type 1. $a \in \mathcal{A}_1$. Then $K(a_n/1)$ converges, $f^{(n)} \to w := (q-1)/2$ where $q := \sqrt{1+4a}$ with $\operatorname{Re} q > 0$, and thus $\Delta_n \to \Delta > 0$ if $w_n \to w$. (See for instance [4].) Moreover, $h_n \to 1 + w$, and thus $h_n/(h_n + w_n)$ is bounded and $\Phi_n(w_n, 0) = \mathcal{O}(\delta_n)$ if $w_n \to w$. (See (2.14).)

Type 2. $a = 0$. $K(a_n/1)$ converges also in this case, $f^{(n)} \to 0$ and $h_n \to 1$, [**4**]. Hence, $\Phi_n(w_n, 0) = \mathcal{O}(\delta_n/f^{(n)}) = \mathcal{O}(\delta_n/a_{n+1})$ if $w_n \to 0$.

Type 3. $a = -\frac{1}{4}$. This is a trickier case. $K(a_n/1)$ may diverge, but if $a_n \in P_{\alpha,n}$ (for fixed α) from some n on, then $K(a_n/1)$ converges to a value f, $f^{(n)} \to -\frac{1}{2}$, $h_n \to \frac{1}{2}$, [**14**], and $S_n(w_n) \to f$ for every sequence $w_n \in W_{\alpha,n}$ by Thron's parabola sequence theorem. It is clear that then $g_n \to \frac{1}{2}$, so $\operatorname{dist}(-h_n, \partial W_{\alpha,n}) \geq d_n g_n \cos\alpha \sim \frac{1}{2} d_n \cos\alpha$. Hence $h_n/(h_n + f^{(n)}) = \mathcal{O}(d_n^{-1})$ and $\Phi_n(w_n, 0) = \mathcal{O}(R_n/d_n)$ if $\{R_n\}$ is chosen such that also $a_n \in E_n$ from some n on. This means in particular that $R_n \to 0$ and $w_n \to -\frac{1}{2}$ are necessary to get $\Phi_n(w_n, 0) \to 0$, and thus $\Delta_n \to \Delta = 0$.

Type 4. $a = \infty$. Again $K(a_n/1)$ may converge or diverge, but if $a_n \in P_{\alpha,n}$ from some n on and $T_n \to 0$, then $K(a_n/1)$ converges to some $f \in \widehat{\mathbf{C}}$ and $S_n(w_n) \to f$ for every $w_n \in W_{\alpha,n}$. Also here $\operatorname{dist}(-h_n, \partial W_{\alpha,n}) \geq d_n g_n \cos\alpha$, but g_n does not have to approach $\frac{1}{2}$ as $n \to \infty$. Hence $\Phi_n(w_n, 0) = \mathcal{O}(R_n/d_n g_n f^{(n)})$. If R_n stays bounded as $n \to \infty$, then $\Phi_n(w_n, 0) = \mathcal{O}(R_n/d_n g_n w_n)$.

We shall suggest a number of choices for w_n, and see how these affect the approximants of $K(a_n/1)$ of our four types. For more information on these methods for choosing $\{w_n\}$, we refer to [**13**].

6.1. The fixed point method. This method is very easy and efficient for continued fractions of type 1.

Type 1. The idea is that, since $a_n \to a$, we may replace the nth tail by the value of the periodic continued fraction $K(a/1)$, that is,

(6.1)
$$w_n := w := \frac{a}{1} + \frac{a}{1} + \frac{a}{1} + \dots = \frac{q-1}{2}; \quad q := \sqrt{1+4a}, \quad \operatorname{Re} q > 0.$$

Then $\hat{a}_n = w(1 + w) = a$, and thus this simple device gives an improvement of the order $\Phi_n(w, 0) = \mathcal{O}(\delta_n)$ as $n \to \infty$, where $\delta_n := \sup_{m \geq n} |a_m - a|$.

Type 2. If $a = 0$, then (6.1) gives $w = 0$, and we are back to the classical approximants. This actually means that the classical approximants $S_n(0)$ are rather good for this type of continued fractions.

Type 3. If $a = -\frac{1}{4}$, then (6.1) gives $w = -\frac{1}{2}$, and it seems plausible that approximants $S_n(-\frac{1}{2})$ should do well. However, since $S_n(-h_n) = \infty$, we do not want w to be too close to $-h_n \to -\frac{1}{2}$. A second problem is that $h_n/(h_n + w_n) \to \infty$ in this case. Still, it follows from results by Thron and Waadeland [**20**] that

(6.2)
$$\frac{f - S_n(-\frac{1}{2})}{f - S_n(0)} = \begin{cases} \mathcal{O}(n^{2-\mu}) & \text{if } |a_n + \frac{1}{4}| \le cn^{-\mu} \text{ for all } n \text{ for } \mu > 2 \\ \mathcal{O}(nr^n) & \text{if } |a_n + \frac{1}{4}| \le cr^n \text{ for all } n \text{ for } 0 < r < 1, \end{cases}$$

where c is a positive constant.

Type 4. In this case (6.1) does not make sense. The choice $S_n(\infty)$ brings back the classical approximants $S_n(\infty) = S_{n-1}(0)$.

6.2. The square root modification. The idea here is strongly related to (6.1), only this time we choose

(6.3)
$$w_n := \frac{a_{n+1}}{1} + \frac{a_{n+1}}{1} + \dots = \frac{q_n - 1}{2}; \quad q_n := \sqrt{1 + 4a_{n+1}}, \ \operatorname{Re} q_n \ge 0$$

if a_{n+1} is not negative and $< -\frac{1}{4}$. This was suggested by Gill [**2**] for the case $a_n \to 0$. However,

$$\hat{a}_{n+1} := w_n(1 + w_{n+1}) = a_{n+1} + w_n(w_{n+1} - w_n), \tag{6.4}$$

so $(a_{n+1} - \hat{a}_{n+1}) \to 0$ if $w_n(w_{n+1} - w_n) \to 0$. Hence we expect this modification to do well in several cases, in particular if $a_n \to a$ in a monotone fashion. Indeed, (6.3) was applied successfully to a continued fraction expansion $K(a_n/1)$ of the incomplete gamma function where $a_n \to \infty$, [**6**], [**7**]. For continued fractions of our type 1, (6.3) gives that $\Phi_n(w_n, 0) = \mathcal{O}(\sup_{m \ge n} |q_{m+1} - q_m|)$, and for type 2 we get $\Phi_n(w_n, 0) = \mathcal{O}(\sup_{m \ge n} |(q_{m+1} - q_m)(q_m - 1)/a_{n+1}|)$.

An alternative version of this modification is

$$w_n := \frac{a_{n+1}}{1} + \frac{a_{n+2}}{1} + \dots + \frac{a_{n+k}}{1} + \frac{a_{n+1}}{1} + \frac{a_{n+2}}{1} + \dots + \frac{a_{n+k}}{1} + \\ + \frac{a_{n+1}}{1} + \dots +$$

for some $k \in \mathbf{N}$, but we shall not pursue this idea in this paper.

6.3. The improvement machine. The idea here is based on the following lemma:

Lemma 6.1 [**9**]. *Let* $(f^{(n)} - w_n) \to 0$. *Then*

$$\frac{f^{(n+1)} - w_{n+1}}{f^{(n)} - w_n} \to t \quad \textit{if and only if} \quad \frac{a_{n+1} - \hat{a}_{n+1}}{a_n - \hat{a}_n} \to t.$$

This was used in [**8**] in the following way. Let $(f^{(n)} - w_n) \to 0$ and $(a_{n+1} - \hat{a}_{n+1})/(a_n - \hat{a}_n) \to t$. Then

$$\frac{a_{n+1} - \hat{a}_{n+1}}{f^{(n)} - w_n} \sim 1 + w_{n+1} + tw_n,$$

and thus

$$f^{(n)} - w_n \sim \frac{a_{n+1} - \hat{a}_{n+1}}{1 + w_{n+1} + tw_n}.$$

This means that if we have chosen $\{w_n\}$ such that $(f^{(n)} - w_n) \to 0$, then

$$w_n^{(1)} := w_n + \frac{a_{n+1} - \hat{a}_{n+1}}{1 + w_{n+1} + tw_n} \tag{6.5}$$

is an even better choice when the limit t exists. Indeed, if $(h_n + w_n)/(h_n + w_n^{(1)}) \to 1$, which is normally the case, then

$$\Phi_n(w_n^{(1)}, w_n) = \frac{f - S_n(w_n^{(1)})}{f - S_n(w_n)} = \frac{h_n + w_n}{h_n + w_n^{(1)}} \cdot \frac{f^{(n)} - w_n^{(1)}}{f^{(n)} - w_n} \to 0. \tag{6.6}$$

This idea was applied successfully to a number of continued fraction expansions of ratios of hypergeometric functions, [**12**].

6.4. Asymptotic expansion. Here the idea is to choose a sequence $\{\mu_j(n)\}_{j=\nu}^{\infty}$ of functions of n such that $\mu_{j+1}(n)/\mu_j(n) \to 0$ as $n \to \infty$ for every j, and to match coefficients c_j such that

$$w_n := \hat{w}_n^{(N)} := \sum_{j=\nu}^{N} c_j \mu_j(n) \tag{6.7}$$

gives $a_n - \hat{a}_n^{(N)} = \mathcal{O}(\mu_M(n))$ for $\hat{a}_n^{(N)} := \hat{w}_{n-1}^{(N)}(1+\hat{w}_n^{(N)})$ and M as large as practical or possible. Evidently this will only work if $\{a_n\}$ is of a suitable form. The square root modification (6.3) suggests the choice $\mu_j(n) := q_n^j$ if $q_n \to 0$, Type 3, and $\mu_j(n) = q_n^{-j}$ if $q_n \to \infty$, Type 4, since (6.3) then gives the first two terms of the expansion in these cases.

We can also choose simpler expressions for $\mu_j(n)$, inspired by q_n. For instance, if $q_n = \mathcal{O}(n^{-1/2})$ or $q_n = \mathcal{O}(n^{1/2})$ as $n \to \infty$, then $\mu_j(n) := n^{-j/2}$ is a possible choice. Then (6.7) takes the form

$$\hat{w}_n^{(N)} w_n := \sum_{j=\nu}^{N} c_j n^{-j/2}. \tag{6.8}$$

The fact that

$$(n-1)^{-j/2} = n^{-j/2}\left\{1 + \frac{j}{2n} + \frac{j(j+2)}{8n^2} + \cdots + \frac{j(j+2)\cdots(j+2k)}{(k+1)!2^{k+1}n^{k+1}} + \dots\right\} \tag{6.9}$$

for $j \geq 0$, $n \in \mathbf{N}$, $k \in \mathbf{N}$ helps to determine the coefficients c_j.

6.5. Linear approximation. This is a totally different idea. In short it relies on Waadeland's idea from 1986 [**21**]. He regarded $K(a_n/1)$ as a function of its elements a_n. For $K(a_n/1)$ of Type 1 or 3 he then got the linear approximation

$$F(a,a,a,\dots) + \sum_{n=1}^{\infty} \frac{\partial F(a,a,a,\dots)}{\partial a_n}\,(a_n - a) \tag{6.10}$$

to its value f, where $F(a,a,a,\dots)$ is the value $w = (q-1)/2$ of the periodic continued fraction $K(a_n/1)$ and

$$\frac{\partial F}{\partial a_n} = \frac{f}{a_n}\prod_{j=1}^{n-1}\frac{-f^{(j)}}{1+f^{(j)}}$$

which evaluated at $(a,a,a,\dots)$ is just

$$\frac{\partial F(a,a,a,\dots)}{\partial a_n} = \frac{w}{a}\left(\frac{-w}{1+w}\right)^{n-1};$$
$$w := \frac{q-1}{2}, \quad q := \sqrt{1+4a},\ \operatorname{Re} q \geq 0.$$

Hence

$$w_n := w + \sum_{k=n+1}^{n+N} \left(\frac{-w}{1+w}\right)^{k-n-1} \frac{a_k - a}{1+w} \tag{6.11}$$

is a possible choice. This choice was investigated for Gauss continued fractions in [**22**].

Remarks 6.6. We shall see that the various modifications work very well, in particular for slowly converging continued fractions. The problem is to derive a priori truncation error bounds which reflect the improvement obtained. To illustrate how this may be done, we shall concentrate on a few of these modifications.

7. Example 1: The arctangent function. The continued fraction $K(a_n/1)$ in (1.1) converges to $\arctan z$ for $z \in Z_1 \cup \{\pm i\}$. The cases $z = 0$ and $z = \pm i$ are trivial, since $\arctan z = z$ for these values of z. So in this section we shall see how different choices for w_n perform for $z \in Z_1 \setminus \{0\}$.

7.1. Choice of w_n. We observe that

$$a_{n+1} = n^2 z^2/(4n^2 - 1) \to a = z^2/4 \quad \text{as } n \to \infty,$$

so the fixed point modification is

$$w_n = w = (q-1)/2 \quad \text{where } q = \sqrt{1+z^2} \text{ with } \operatorname{Re} q > 0, \tag{7.1}$$

and the square root modification is

$$\begin{gathered} w_n = w = (q_n - 1)/2 \\ \text{where} \quad q_n = \sqrt{1 + z^2 + z^2/(4n^2-1)} \quad \text{with } \operatorname{Re} q_n > 0. \end{gathered} \tag{7.2}$$

If we apply the improvement machine to $w_n = w$ given by (7.1), we get $t = 1$, and thus

$$w_n^{(1)} := w + \frac{a_{n+1} - a}{1 + 2w} = \frac{q-1}{2} + \frac{z^2/4q}{4n^2 - 1}. \tag{7.3}$$

We can repeat this trick and apply the improvement machine on (7.3). The limit t is still 1, and so

$$
\begin{aligned}
w_n^{(2)} &:= w_n^{(1)} + \frac{a_{n+1} - \hat{a}_{n+1}^{(1)}}{1 + w_n^{(1)} + w_{n+1}^{(1)}} \\
&= \frac{q-1}{2} + \frac{z^2(q-1)(2n+1-(q+1)/8q)}{(2n+1)^2[2q^2(4n^2+4n-3)+z^2]},
\end{aligned} \tag{7.4}
$$

where $\hat{a}_{n+1}^{(1)} := w_n^{(1)}(1 + w_{n+1}^{(1)})$, is a possible choice. We can do this repeatedly to improve the approximation $f \approx S_n(w_n^{(k)})$.

The improvement machine applied to (7.2) also gives $t = 1$, and thus the modification

$$
\tilde{w}_n^{(2)} := w_n + \frac{a_{n+1} - \hat{a}_{n+1}}{1 + w_n + w_{n+1}} = \frac{(q_n - 1)q_n}{q_n + q_{n+1}} \tag{7.5}
$$

which really has a very nice closed form.

A different approach is to expand a_n in a series $\sum c_j \mu_n^{-j}$. In view of (7.3) it seems reasonable to try $\mu_n := q(2n-1)$. This gives for instance

$$
\begin{aligned}
\hat{w}_n^{(4)} &:= \sum_{j=0}^{4} c_j \mu_n^{-j} = \frac{q-1}{2} + \frac{qz^2}{4\mu_n^2} - \frac{qz^2}{2\mu_n^3} - \frac{9q^4 - 34q^2 + 25}{16\mu_n^4}\, q \\
&= \frac{q-1}{2} + qz^2\, \frac{4(\mu_n - 1)^2 - 9z^2 + 12}{16\mu_n^4}.
\end{aligned} \tag{7.6}
$$

Finally, the linear approximation in (6.11) gives for instance

$$
\begin{aligned}
w_n &:= w + \frac{a_{n+1} - a}{1+w} - \frac{w(a_{n+2} - a)}{(1+w)^2} + \frac{w^2(a_{n+3} - a)}{(1+w)^3} \\
&= \frac{q-1}{2} + \frac{z^2/2}{(q+1)(4n^2-1)} - \frac{q-1}{(q+1)^2}\, \frac{z^2/2}{4(n+1)^2 - 1} \\
&\quad + \frac{(q-1)^2}{(q+1)^3}\, \frac{z^2/2}{4(n+2)^2 - 1}.
\end{aligned} \tag{7.7}
$$

7.2. The improvement $\Phi_n(w_n, 0)$. Since $a_n \to a$ monotonically, we find that

(7.8)
$$\Phi_n(w_n,\,0) = \begin{cases} \mathcal{O}(a_{n+1}-a) = \mathcal{O}(n^{-2}) & \text{if } w_n := w = (q-1)/2, \\ \mathcal{O}(a_{n+1}-a_n) = \mathcal{O}(n^{-3}) & \text{if } w_n := (q_n-1)/2. \end{cases}$$

Since $a_n - \hat{a}_n^{(1)} \sim z^2(1-1/q)/16n^3$ for $\hat{a}_n^{(1)} := w_{n-1}^{(1)}(1+w_n^{(1)})$, the choices (7.3) and (7.4) give

$$\Phi_n(w_n^{(1)},\,0) = \mathcal{O}(n^{-3}), \qquad \Phi_n(w_n^{(2)},\,0) = \mathcal{O}(n^{-4}), \tag{7.9}$$

respectively. The improvement using (7.5) is similarly of the order $\Phi_n(\tilde{w}_n^{(2)}, 0) = \mathcal{O}(n^{-4})$.

To estimate the effect of the choice (7.6), we observe that $a_{n+1} - \hat{a}_{n+1}^{(4)} = \mathcal{O}(n^{-5})$ and decreases monotonically in absolute value, at least from some n on, and thus $\Phi_n(\hat{w}_n^{(4)},\,0) = \mathcal{O}(n^{-5})$. Finally, the linear approximation (7.7) gives $\Phi_n(w_n, 0) = \mathcal{O}(n^{-2})$ as $n \to \infty$. Increasing N will still give $\Phi_n(w_n, 0) = \mathcal{O}(n^{-2})$, but with $\limsup n^2|\Phi_n(w_n, 0)|$ smaller.

7.3. Tables of approximants. In the two tables below we have computed $S_n(w_n)$ for $z = 1$, i.e., $\arctan z = \pi/4$, for the various modifications w_n. The quantity $m(k)$ is the smallest natural number for which $S_n(w_n)$ is correct, after rounding, with k decimals for all $n \geq m(k)$.

$z = 1. \quad \arctan z = 0.785398163397448309615660845819875 72\ldots$

n	$w_n = 0$	$w_n = (q-1)/2$	$w_n = (q_n - 1)/2$	(7.3)
1	1.0000...	0.828427...	0.79128784...	0.78986923...
2	0.7500...	0.783611...	0.78524116...	0.78525453...
3	0.7916...	0.785533...	0.78540726...	0.78540681...
4	0.7843...	0.785385...	0.78539745...	0.78539747...
5	0.7855...	0.785399...	0.78539822...	0.78539822...
$m(6)$	9	6	5	5
$m(35)$	46	40	38	38

n	(7.5)	(7.4)	(7.6)	(7.7)
1	0.7863101667...	0.7860773121...	0.7760582401...	0.78496161...
2	0.7853818831...	0.7853835353...	0.7853246655...	0.78540709...
3	0.7853989151...	0.7853988690...	0.7854001013...	0.78539768...
4	0.7853981141...	0.7853981162...	0.7853980804...	0.78539820...
5	0.7853981673...	0.7853981671...	0.7853981681...	0.78539815...
$m(6)$	4	4	4	3
$m(35)$	35	35	34	37

The effect of the modifications is not so good here, since $K(a_n/1)$ converges quite fast for $z = 1$, also in the classical sense. We get a different picture for $z := 0.01 + 2i$, which lies closer to the boundary of Z_1.

$z = 0.01 + 2i$. $\quad \arctan z = 1.567463153946 \cdots + 0.5492839233 \ldots i$

n	$w_n = 0$	$w_n = (q-1)/2$	$w_n = (q_n - 1)/2$
1	$0.01 + 2.00 \ldots i$	$1.727 \cdots + 0.997 \ldots i$	$1.5575 \cdots + 0.7481 \ldots i$
2	$2.20 \cdots - 5.99 \ldots i$	$1.595 \cdots + 0.462 \ldots i$	$1.5911 \cdots + 0.5297 \ldots i$
3	$0.01 \cdots + 0.09 \ldots i$	$1.532 \cdots + 0.559 \ldots i$	$1.1558 \cdots + 0.5465 \ldots i$
4	$0.03 \cdots + 2.07 \ldots i$	$1.582 \cdots + 0.562 \ldots i$	$1.5687 \cdots + 0.5533 \ldots i$
5	$0.40 \cdots - 5.60 \ldots i$	$1.569 \cdots + 0.537 \ldots i$	$1.5689 \cdots + 0.5475 \ldots i$
$m(6)$	$>> 1000$	320	72

n	(7.3)	(7.5)	(7.4)
1	$1.5412 \cdots + 0.7279 \ldots i$	$1.5257 \cdots + 0.6365 \ldots i$	$1.5215 \cdots + 0.6247 \ldots i$
2	$1.5910 \cdots + 0.5311 \ldots i$	$1.5792 \cdots + 0.5474 \ldots i$	$1.5789 \cdots + 0.5479 \ldots i$
3	$1.5583 \cdots + 0.5464 \ldots i$	$1.5653 \cdots + 0.5469 \ldots i$	$1.5654 \cdots + 0.5469 \ldots i$
4	$1.5686 \cdots + 0.5533 \ldots i$	$1.5671 \cdots + 0.5504 \ldots i$	$1.5671 \cdots + 0.5503 \ldots i$
5	$1.5689 \cdots + 0.5476 \ldots i$	$1.5679 \cdots + 0.5491 \ldots i$	$1.5679 \cdots + 0.5491 \ldots i$
$m(6)$	72	30	30

n	(7.6)	(7.7)
1	$-1.84953\cdots - 0.25433\ldots i$	$1.5816\cdots + 0.5410\ldots i$
2	$1.54943\cdots + 0.89921\ldots i$	$1.5684\cdots + 0.5389\ldots i$
3	$1.56647\cdots + 0.55273\ldots i$	$1.5603\cdots + 0.5522\ldots i$
4	$1.56815\cdots + 0.54898\ldots i$	$1.5720\cdots + 0.5527\ldots i$
5	$1.56726\cdots + 0.54916\ldots i$	$1.5679\cdots + 0.5449\ldots i$
$m(6)$	15	317

The effect is now quite spectacular! Indeed, $S_n(0)$ approximates $\arctan(0.01 + 2i)$ rather poorly, even for quite large n. We have for instance

$$\begin{aligned}
S_{995}(0) &= 1.57598778\cdots + 0.54395517\ldots i\\
S_{996}(0) &= 1.55868637\cdots + 0.54461720\ldots i\\
S_{997}(0) &= 1.56776338\cdots + 0.55919115\ldots i\\
S_{998}(0) &= 1.57584063\cdots + 0.54404631\ldots i\\
S_{999}(0) &= 1.55883632\cdots + 0.54469776\ldots i\\
S_{1000}(0) &= 1.56775974\cdots + 0.55902097\ldots i
\end{aligned}$$

which means that even $m(1) > 1000$. The moral is: do not use $S_n(0)$ to approximate the value of a continued fraction of type 1 if a is close to the boundary of $\mathcal{A}_1$. Indeed, since it does not cost any extra to compute $S_n(w)$ instead of $S_n(0)$, one should never use $S_n(0)$ to approximate the value of $K(a_n/1)$ of type 1.

7.4. The Gragg Warner bound and Thron's parabola sequence bound. $K(a_n/1)$ is a Stieltjes fraction. For classical approximants we may choose between the Gragg Warner bound (4.3) or Thron's parabola sequence bound $2T_n$ given by (4.1). In both cases we choose $\alpha = \arg z$ if $|\arg z| < \pi/2$. If $z^2 < 0$ we choose $\alpha = 0$. We demonstrate the effect for the case $|\arg z| < \pi/2$. The Gragg Warner bound gives

$$|f - S_n(0)| \le 2Z \prod_{\nu=1}^{n-1} K_\nu \quad \text{where } Z := \frac{|z|}{\cos\alpha} = \frac{|z|^2}{\operatorname{Re} z} \tag{7.10}$$

and

$$K_\nu := 1 - \frac{2}{1+\sqrt{1+\frac{4\nu^2 Z^2}{4\nu^2-1}}} \to K \ := 1 - \frac{2}{1+\sqrt{1+Z^2}} \quad \text{as } \nu \to \infty.$$

For $z = 1$ we get $Z = 1$, $K_1 = 0.20871$, $K_2 = 0.17952$ and $K = 0.171573$. Hence, if we replace K_ν by K_2 for $\nu > 2$, we get

$$|f - S_n(0)| \le 0.41742 \cdot 0.17952^{n-2} \quad \text{for } z = 1,\ n \ge 2. \tag{7.11}$$

For $z = 0.01 + 2i$ we get $Z = 400.01$, $K_1 = 0.995679$, $K_2 = 0.995170$, $K_3 = 0.995082$ and $K = 0.99501$ which does not look very promising. If we replace K_ν by K_3 for $\nu > 3$, we get

$$|f - S_n(0)| \le 792.716 \cdot 0.995082^{n-3} \quad \text{for } z = 0.01 + 2i,\ n \ge 3. \tag{7.12}$$

Thron's parabola sequence bound (4.1) is a bound for $|f - S_n(w_n)|$ for every w_n with $\operatorname{Re}(w_n e^{-i\alpha}) \ge 0$, and thus also for our modifying factors w_n and for $w_n = 0$. It gives

$$|f - S_n(w_n)| \le 2T_n = Z \Big/ \prod_{\nu=1}^{n-1} L_\nu \quad \text{where } Z := \frac{|z|^2}{\operatorname{Re} z} \tag{7.13}$$

and

$$L_\nu := 1 + \frac{4\nu^2 - 1}{\nu^2 Z^2} \longrightarrow L := 1 + \frac{4}{Z^2} \quad \text{as } \nu \to \infty.$$

For $z = 1$ we get $Z = 1$, $L_1 = 4$, $L_2 = 19/4$ and $L = 5$, and thus for instance

$$|f - S_n(w_n)| \le \frac{1}{4}\left(\frac{4}{19}\right)^{n-2} \quad \text{for } z = 1,\ n \ge 2,\ \operatorname{Re} w_n \ge 0. \tag{7.14}$$

For $z = 0.01 + 2i$ we get $L = 1.000025$, $L_1 = 1.000019$, $L_2 = 1.000023$, so we can for instance use

$$\begin{gathered} |f - S_n(w_n)| \le 400.00/1.000023^{n-2} \\ \text{for } z = 0.01 + 2i,\ n \ge 2,\ \operatorname{Re}\left(w_n \frac{\overline{z}}{|z|}\right) \ge 0. \end{gathered} \tag{7.15}$$

The tables below show how this compares to the actual truncation errors.

$z = 1$.

n	$\lvert f - S_n(0)\rvert$	(7.11)	(7.14)
5	$3.17 \cdot 10^{-7}$	$1.34 \cdot 10^{-2}$	$2.33 \cdot 10^{-3}$
10	$3.63 \cdot 10^{-19}$	$2.50 \cdot 10^{-4}$	$9.65 \cdot 10^{-7}$
20	$< 10^{-40}$	$8.72 \cdot 10^{-14}$	$1.65 \cdot 10^{-13}$
40	$<< 10^{-40}$	$1.05 \cdot 10^{-28}$	$4.83 \cdot 10^{-27}$

$z = 0.01 + 2i$.

n	$\lvert f - S_n(0)\rvert$	(7.12)	(7.15)
10	$2.139\ \ldots$	765.8	400
50	$4.574\ \ldots$	628.8	400
100	$1.535\ \ldots$	491.4	399
1000	$0.0974\ldots$	5.8	391

7.5. Error bounds for the fixed point modification. The fixed point modification uses $w_n := w := (q-1)/2$ where $q = \sqrt{1+z^2}$ with $\operatorname{Re} q > 0$.

7.5.1. *Bounds based on Q_n.* The choice (5.1) for R_n gives

$$R_{n+1} := \frac{2|z^2/4|}{(4n^2-1)\Delta} = \frac{C}{4n^2-1} \tag{7.16}$$
$$\text{where} \quad C := \frac{|z^2|}{|q+1| - |q-1|} \quad \text{for } n \geq 1$$

where $z^2 = q^2 - 1$. We have to check that $a_n \in E_n$ in (2.13) with this choice of R_n in order to use the bound Q_n in OST.

Case 1. $z \in \mathbf{R} \setminus \{0\}$. Then $a_n > 0$ for $n \geq 2$, $q > 1$ and $C = z^2/2$. Hence it follows from Remark 2.2.2 that $a_{n+1} \in E_{n+1}$ if

$$(w - R_n)(1 + w + R_{n+1}) \leq a_{n+1} \leq (w + R_n)(1 + w - R_{n+1}),$$

which is equivalent to

$$-12n^2 - 8(q-2)n - (q-2)^2 \le 0 \le 4n^2 + 8qn - q(q+4).$$

The left inequality holds trivially for $n \ge 2$, and the right inequality holds if also $n \ge q(\sqrt{2+4/q} - 1)$. Clearly $n \ge q(\sqrt{2+4/q} - 1)$ for all $n \ge 2$ if $q \le 2$; i.e., if $|z| \le \sqrt{3}$, and thus $a_n \in E_n$ for all $n \ge 3$ for these values of z. Hence, in view of Remarks 2.2.3 and 3.1.2, we have by (3.5) that

(7.17)

$$\begin{aligned}|f - S_{n+1}(w)| &\le \left(\frac{1+w+R_2}{1+w+a_2+R_2}\right)^2 \frac{a_2|z|}{1+w-R_2} \\ &\quad \cdot \frac{R_{n+1}}{1+w} \prod_{k=2}^{n} \frac{w+R_k}{1+w+R_k} \\ &= \frac{2|z|^3}{4-q}\left(\frac{q+2}{3q}\right)^2 \frac{(\frac{q-1}{q+1})^n}{4n^2-1} \prod_{k=1}^{n-1}\left(1+\frac{2}{4k^2+q-2}\right)\end{aligned}$$

for $n \ge 2$ and $0 < z^2 \le 3$. For $z = 1$ we get in particular that

$$|f - S_{n+1}(w)| \le \frac{0.00453231}{4n^2-1} \cdot 0.181262^{n-2} \quad \text{for } n \ge 2. \tag{7.18}$$

Case 2. Otherwise. According to Lemma 5.1 we have that $a_n \in E_n$ for $n > N$ if R_{n+1} is given by (7.16) for all $n > N$ and

$$\frac{C}{4N^2-1} \le \frac{\Delta}{2}; \quad \text{i.e.,} \ \frac{|q^2-1|}{(|q+1|-|q-1|)^2} \le N^2 - \frac{1}{4}.$$

This condition is probably more restrictive than necessary, but it is easy to check. For instance, it works for $N := 1$ if $5|z^2| \le 3(|z^2+1|+1)$; that is, if $z^2 \in U_1$, where U_1 is a closed, bounded, simply connected domain with $0 \in U_1$, whose boundary ∂U_1 intersects the real axis at the two points $-3/4$ and 3, and where $\{z^2 : \ |z^2| \le 3/4\} \subset U_1 \subset \{z^2 : |z^2| \le 3\}$. (The condition is sharp for $z^2 > 0$.) By (3.5) we thus have

$$|f - S_{n+1}(w)| \le \frac{|a_1|H_2^2|a_2|}{|1+w|(|1+w|-R_2)} \cdot \frac{|z^2|/(2\Delta)}{4n^2-1} \prod_{k=1}^{n-1} M_{k+1} \tag{7.19}$$

for $n \geq 2$ and $z^2 \in U_1$, where

$$H_2 = \frac{|\overline{a_2}(1+w) + |1+w|^2 - R_2^2| + |a_2|R_2}{|1+a_2+w|^2 - R_2^2}$$

by Remark 3.1.2. Similarly, $N = 2$ works if $C/15 \leq \Delta/2$; that is, $34|z^2| \leq 30(|z^2+1|+1)$, which means that z^2 belongs to a larger closed domain U_2 of similar shape, with the points $-15/16$ and 15 on the boundary. By (3.7) this gives

(7.20)

$$|f - S_{n+1}(w)| \leq \frac{|a_1a_2a_3|H_3^2}{|1+a_2|^2|1+w|(|1+w|-R_3)} \cdot \frac{|z^2|/(2\Delta)}{4n^2-1} \prod_{k=2}^{n-1} M_{k+1}$$

for $n \geq 1$ and $z^2 \in U_2$ where

$$H_3 = \frac{||1+w|^2 + \frac{a_3}{1+a_2}(1+\overline{w}) - R_3|^2 + |\frac{a_3}{1+a_2}|R_3}{|1+w+\frac{a_3}{1+a_2}|^2 - R_3^2}.$$

The efficiency of these bounds also depends on

$$M_k = \frac{|w + |w^2| - R_k^2| + R_k}{|1+w|^2 - R_k^2} \tag{7.21}$$

which converges to $M := |w|/|1+w| < 1$. If M is close to 1, we may well have $M_k > 1$ even for quite large k.

For $z = 0.01+2i$ we get $z^2 = -3.9999+0.04i$, $|1+w| = 1.00290335354$ and $|w| = 0.997129979$. Hence $|z^2|/\Delta^2 \leq 4N^2 - 1$ only for $N \geq 174$. It therefore makes sense to be more careful. By Remark 5.1.1 we have $a_{n+1} \in E_{n+1}$ if

$$(2R_n - R_{n+1})|1+w| - R_{n+1}|w| - 2R_nR_{n+1} \geq 0,$$

which holds with our choice (7.16) for R_n if

$$(4n^2 + 8n - 5)|1+w| - (4n^2 - 8n + 3)|w| \geq 2C = |z^2|/\Delta.$$

If this holds for $n = N$, then it holds for all $n \geq N$. For $z = 0.01 + 2i$ this is all right for $N = 42$, which possibly also is on the large side.

It takes some computation to find that $|h_{43} + w| = 1.005787273$ and $|f_{41} - f_{42}| = 4.91352$, so that by (3.4) and Remark 3.1.2

(7.22)
$$|f - S_{n+1}(w)| \le \frac{|f_{41} - f_{42}|(|1+w| + R_{43})^2}{|h_{42}|(|h_{43} + w| - R_{43})^2} \, |f^{(42)} - S^{(42)}_{n-41}(w)|$$
$$\le 1.81480416 \cdot R_{n+1} \prod_{k=43}^{n} M_k \quad \text{for } n \ge 42,\ z = 0.01 + 2i,$$

where M_k is given by (7.21) with $R_{k+1} = 344.8/(4k^2 - 1)$ as given by (7.16). (We may get better bounds by increasing N.)

A natural choice for a slower converging sequence $\{R_n\}$ is $R_n := \widehat{C}/(2n-1)^\lambda$ for some constants $\widehat{C} > 0$ and $1 \le \lambda < 2$ to be determined. Then $|a_{n+1} - a| \le r_{n+1}$, as given in Remark 2.2.2, if and only if

$$\frac{|z^2|/4}{4n^2 - 1} \le \frac{\widehat{C}|1+w|}{(2n-1)^\lambda} - \frac{\widehat{C}|w|}{(2n+1)^\lambda} - \frac{\widehat{C}^2}{(4n^2-1)^\lambda}. \tag{7.23}$$

With the simplest choice $\lambda := 1$, (7.23) holds for all $n \ge 1$ and all $z \in Z_1$ when $\widehat{C} := |z^2/2|/(|1+w| + |w|)$. Hence the bound Q_n in OST based on the choice

$$R_k := \widehat{C}/(2k-1) \tag{7.24}$$

works for all $z \in Z_1$. With $N = 1$ we get from OST that

$$|f - S_n(w)| \le \frac{|z|}{|1+w| - \widehat{C}} \cdot \frac{\widehat{C}}{|1+w|(2n-1)} \prod_{k=1}^{n-1} M_k \quad \text{for } n \ge 1,\ z \in Z_1, \tag{7.25}$$

where M_k still is given by (7.21), but this time with R_k given by (7.24). For $z = 0.01 + 2i$ it gives

$$|f - S_n(w)| \le \frac{688.856}{2n-1} \prod_{k=1}^{n-1} M_k \quad \text{for } z = 0.01 + 2i,\ n \ge 1. \tag{7.26}$$

However, as seen from the table below comparing the various error bounds, this bound is almost useless, although it gives a better bound than Thron's parabola sequence bound (7.15) in the previous subsection. But it helps to increase N. From (3.7) and Remark 3.1.2 we get for instance

$$(7.27)\qquad |f-S_n(w)| \leq \frac{1.825418}{2n-1}\prod_{k=3}^{n-1} M_k \quad \text{for } z=0.01+2i,\ n\geq 3.$$

Larger N in (3.4) will give even better bounds.

7.5.2. *Bounds based on* (3.13). We assume that $|\arg z| < \pi/2$. Then $\arg a_n = 2\alpha$ for $n \geq 2$ when $\alpha := \arg z$. By (4.2) it follows therefore that

$$|f-S_n(w)| \leq \frac{2R_nT_n}{1+\mathrm{Re}\,(we^{-i\alpha})}$$

where $2T_n$ is given by (4.1) as in (7.13). The advantage of this bound is that we only need $a_k \in E_k$ for all $k > n$ to conclude that $|f^{(n)} - w| \leq R_n$, and we do not have to compute h_N or f_N for large Ns. It improves the bound from Thron's parabola sequence theorem if $R_n/(1+\mathrm{Re}\,(\sqrt{1+z^2}e^{-i\alpha})) < 1$ which at least happens from some n on, since this positive expression tends to 0. So let $N \in \mathbf{N}$ be such that $a_{n+1} \in E_{n+1}$ for $n > N$ with R_{n+1} given by (7.16) for $n \geq N$. Then

(7.28)

$$|f-S_{n+1}(w)| \leq \frac{|z^2|/(2\Delta)}{4n^2-1} \cdot \frac{Z}{1+\mathrm{Re}(\sqrt{1+z^2}\,e^{-i\alpha})}\prod_{\nu=1}^{n}\left(1+\frac{4\nu^2-1}{\nu^2Z^2}\right)^{-1}$$

for $n \geq N$ where $Z = |z|/\cos\alpha$ as in (7.13). For $z^2 \in U_1$ this holds for $n \geq N = 1$, whereas $z = 0.01+2i$ requires $n \geq N = 42$ as seen above.

The tables below show the values of these bounds compared to the actual error for $z = 1$ and $z = 0.01 + 2i$ for various values of n.

$z = 1$.

n	$f - S_n(w)$	(7.18)	(7.28)
3	$-1.4 \cdot 10^{-4}$	$7.6 \cdot 10^{-4}$	$7.3 \cdot 10^{-4}$
4	$1.3 \cdot 10^{-5}$	$6.4 \cdot 10^{-5}$	$6.4 \cdot 10^{-5}$
5	$-1.0 \cdot 10^{-6}$	$6.4 \cdot 10^{-6}$	$7.2 \cdot 10^{-6}$
20	$2.9 \cdot 10^{-19}$	$5.0 \cdot 10^{-18}$	$1.0 \cdot 10^{-17}$

$z = 0.01 + 2i$.

n	$\lvert f - S_n(w)\rvert$	(7.22)	(7.26)	(7.27)	(7.28)
43	$1.13 \cdot 10^{-4}$	$4.6 \cdot 10^{-2}$	$1.5 \cdot 10^{4}$	$1.51 \cdot 10^{-1}$	3.0
44	$1.08 \cdot 10^{-4}$	$4.4 \cdot 10^{-2}$	$1.4 \cdot 10^{4}$	$1.48 \cdot 10^{-1}$	3.4
45	$1.02 \cdot 10^{-4}$	$4.4 \cdot 10^{-2}$	$1.4 \cdot 10^{4}$	$1.46 \cdot 10^{-1}$	3.2
100	$2.17 \cdot 10^{-5}$	$1.2 \cdot 10^{-2}$	$6.9 \cdot 10^{3}$	$7.11 \cdot 10^{-2}$	$6.5 \cdot 10^{-1}$
500	$5.9 \cdot 10^{-8}$	$1.5 \cdot 10^{-4}$	$3.1 \cdot 10^{2}$	$3.16 \cdot 10^{-3}$	$5.2 \cdot 10^{-2}$
1000	$3.9 \cdot 10^{-11}$	$2.3 \cdot 10^{-6}$	$1.2 \cdot 10^{1}$	$1.25 \cdot 10^{-4}$	$1.3 \cdot 10^{-2}$

7.6. Error bounds for the modification (7.6). To illustrate what happens for sharper (and more complicated) choices for w_n, we choose to consider $\hat{w}_n^{(4)}$ given by (7.6). It gives

$$\begin{aligned}\varepsilon_{n+1}^{(4)} := a_{n+1} - \hat{a}_{n+1}^{(4)} = \frac{z^2/(z^2+1)}{(4n^2-1)^4}\\ \cdot \left\{ \frac{36q^2-52}{q} n^3 + \frac{153q^4-378q^2+33}{8q^2} n^2 \right.\\ \left. - \frac{9q^4+62q^2+25}{8q^3} n - \frac{9q^6+189q^4+939q^2-625}{256q^4} \right\}\end{aligned}$$

since $q^2 - 1 = z^2$. Thron's parabola sequence bound $2T_n$ in (4.1) is independent of w_n, so there is no need to repeat the analysis of this bound. We shall concentrate on the OST-bound Q_n in combination with (3.4), and the bound (3.13) which combines $2T_n$ with estimates from OST.

7.6.1. *Bounds based on Q_n.* Lemma 5.1 leads us to expect that $R_n = \mathcal{O}(\varepsilon_n^{(4)})$ works. So, inspired by (7.16) we first try

$$R_n := 2|\varepsilon_n^{(4)}|/\Delta_n^{(4)}. \tag{7.29}$$

Then it follows from Remark 5.1.1 that $a_{n+1} \in E_{n+1}$ for $n \geq N$ if

(7.30)
$$\frac{2|\varepsilon_n^{(4)}|}{\Delta_n^{(4)}}\,|1+\widehat{w}_{n+1}^{(4)}| - \left(1+\frac{2|\widehat{w}_n^{(4)}|}{\Delta_{n+1}^{(4)}}\right)|\varepsilon_{n+1}^{(4)}| - \frac{4|\varepsilon_n^{(4)}\varepsilon_{n+1}^{(4)}|}{\Delta_n^{(4)}\Delta_{n+1}^{(4)}} \geq 0 \quad \text{for } n \geq N,$$

where $\Delta_n^{(4)} := |1+\widehat{w}_n^{(4)}| - |\widehat{w}_{n-1}^{(4)}|$. If (7.30) holds for $N := 2$, then $a_n \in E_n$ for all $n \geq 3$, and by (3.5) we have

$$\begin{aligned} |f - S_n(\widehat{w}_n^{(4)})| \leq{} & \frac{|z|(|1+\widehat{w}_2^{(4)}| + R_2)^2}{(|1+a_2+\widehat{w}_2^{(4)}| - R_2)^2} \cdot \frac{|a_2|}{|1+\widehat{w}_2^{(4)}| - R_2} \\ & \cdot \frac{R_n}{|1+\widehat{w}_n^{(4)}|} \prod_{k=2}^{n-1} M_k^{(4)} \end{aligned} \tag{7.31}$$

for $n \geq 2$, where $M_k^{(4)}$ has the obvious meaning. Straightforward computation shows that (7.30) holds with $N := 2$ for $z = 1$. Hence (7.31) gives an error bound for $z = 1$ and $n \geq 2$. A slight improvement is obtained by increasing N. For $N = 3$ we get, for instance, by (3.7) that

(7.32)
$$\begin{aligned} |f - S_n(\widehat{w}_n^{(4)})| \leq{} & \frac{|z^3/3|(|1+\widehat{w}_3^{(4)}| + R_3)^2}{(|1+\frac{3z^2}{5} + (1+\frac{z^2}{3})\widehat{w}_3^{(4)}| - |1+\frac{z^2}{3}|R_3)^2} \\ & \cdot \frac{|4z^2/15|}{|1+\widehat{w}_3^{(4)}| - R_3} \cdot \frac{R_n}{|1+\widehat{w}_n^{(4)}|} \prod_{k=3}^{n-1} M_k^{(4)} \quad \text{for } n \geq 3 \end{aligned}$$

when (7.30) holds for $N = 3$, which of course also is the case for $z = 1$.

For $z = 0.01 + 2i$ we find that (7.30) holds for $n \geq 5$. Hence $a_n \in E_n$ for $n \geq 6$, and

$$\begin{aligned} |f - S_n(\widehat{w}_n^{(4)})| \leq{} & \frac{|f_4 - f_5|(|1+\widehat{w}_6^{(4)}| + R_6)^2}{|h_5|(|h_6+\widehat{w}_6^{(4)}| - R_6)^2} \\ & \cdot \frac{|a_6|}{|1+\widehat{w}_6^{(4)}| - R_6} \cdot \frac{R_n}{|1+\widehat{w}_n^{(4)}|} \prod_{k=6}^{n-1} M_k^{(4)} \end{aligned} \tag{7.33}$$

for $n \geq 6$, where $R_k = 2|\varepsilon_k^{(4)}|/\Delta_k^{(4)}$, M_k is given by (7.21) and

$$\left|\frac{f_4 - f_5}{h_5}\right| = \frac{|a_1 a_2 a_3 a_4 a_5|}{|1 + a_2 + a_3 + a_4 + a_5 + a_2 a_4 + a_2 a_5 + a_3 a_5|^2} = \frac{64|z|^9}{|105 + 350z^2/3 + 43z^4|^2}.$$

That is,

$$|f - S_n(\widehat{w}_n^{(4)})| \leq 0.2427 \cdot \frac{R_n}{|1 + \widehat{w}_n^{(4)}|} \prod_{k=6}^{n-1} M_k^{(4)} \quad \text{for } z = 0.01 + 2i,\ n \geq 6. \tag{7.34}$$

These error bounds work very well. (See the tables below.) But they require a technique to find an N which guarantees that $a_{n+1} \in E_{n+1}$ for all $n > N$ for a given $z \in Z_1$. Here we have found such an N by testing (7.30).

A less accurate bound which makes it easier to find such an N can be found if we accept that $R_n \to 0$ slower than $\mathcal{O}(\varepsilon_{n+1}^{(4)})$. Indeed, if we can find $\{R_n\}$ and $N \in \mathbf{N}$ such that

$$|\varepsilon_{n+1}^{(4)}| \leq R_n|1 + \widehat{w}_{n+1}^{(4)}| - R_{n+1}|\widehat{w}_n^{(4)}| - R_n R_{n+1} \quad \text{for } n > N \tag{7.35}$$

for all $z \in Z_1$, then there is no need for further checking.

7.6.2. *Bounds based on* (3.13). For $|\arg z| < \pi/2$ we get as in (7.28) that

$$|f - S_{n+1}(\widehat{w}_{n+1}^{(4)})| \leq \frac{R_{n+1} \cdot 2T_{n+1}}{1 + \operatorname{Re}(\sqrt{1+z^2}\, e^{-i\alpha})} \tag{7.36}$$

if $|f^{(n+1)} - \widehat{w}_{n+1}^{(4)}| \leq R_{n+1}$ and $\operatorname{Re}(\widehat{w}_{n+1}^{(4)} e^{-i\alpha}) \geq 0$.

Let E_n be given by (2.13) with R_n as given by (7.29) and $\widehat{w}_n^{(4)}$ by (7.6). Then we have already found that $a_{n+1} \in E_{n+1}$ for all $n \geq 2$ if $z = 1$, and that $a_{n+1} \in E_{n+1}$ for all $n \geq 5$ if $z = 0.01 + 2i$. It is also straightforward to see that $\mathrm{Re}\,(\widehat{w}_n^{(4)}e^{-i\alpha}) \geq 0$ for these values of n. We therefore find that

(7.37)

$$|f-S_{n+1}(\widehat{w}_{n+1}^{(4)})| \leq \frac{2|\varepsilon_{n+1}^{(4)}|}{\Delta_{n+1}^{(4)}}\cdot\frac{|z|/\cos\alpha}{1+\mathrm{Re}\,(\sqrt{1+z^2}\,e^{-i\alpha})}\prod_{k=2}^{n+1}\left(1+\frac{\cos^2\alpha}{|a_k|}\right)^{-1}$$

for $n \geq 2$ and $n \geq 5$, respectively.

The tables below show the actual error $|f - S_n(\widehat{w}_n^{(4)})|$ and our error bounds for some values of n at the two points $z = 1$ and $z = 0.01 + 2i$.

$z = 1$.

n	$\lvert f - S_n(\widehat{w}_n^{(4)})\rvert$	(7.31)	(7.32)	(7.33)	(7.37)
3	$1.9 \cdot 10^{-6}$	$5.9 \cdot 10^{-5}$	$4.9 \cdot 10^{-5}$	———	$1.3 \cdot 10^{-4}$
4	$8.2 \cdot 10^{-8}$	$1.5 \cdot 10^{-6}$	$1.2 \cdot 10^{-6}$	———	$2.3 \cdot 10^{-6}$
5	$4.8 \cdot 10^{-9}$	$6.4 \cdot 10^{-8}$	$5.3 \cdot 10^{-8}$	———	$9.0 \cdot 10^{-8}$
20	$1.6 \cdot 10^{-23}$	$9.5 \cdot 10^{-23}$	$7.9 \cdot 10^{-23}$	$7.5 \cdot 10^{-23}$	$8.0 \cdot 10^{-22}$

$z = 0.01 + 2i$.

n	$\lvert f - S_n(\widehat{w}_n^{(4)})\rvert$	(7.34)	(7.37)
6	$8.69 \cdot 10^{-5}$	$2.48 \cdot 10^{-2}$	25.9
7	$3.85 \cdot 10^{-5}$	$8.10 \cdot 10^{-3}$	7.0
8	$1.91 \cdot 10^{-5}$	$3.36 \cdot 10^{-3}$	2.7
100	$2.91 \cdot 10^{-11}$	$2.89 \cdot 10^{-9}$	$1.7 \cdot 10^{-6}$
500	$9.07 \cdot 10^{-16}$	$8.82 \cdot 10^{-14}$	$6.8 \cdot 10^{-11}$

8. Example 2: The tangent function. The continued fraction $K(a_n/1)$ in (1.2) converges to $\tan z$ for all $z \in \mathbf{C}$. It is of type 2, indeed $a_{n+1} = -z^2/(4n^2-1)$ which approaches 0 rather fast. Hence $S_n(0)$ is really a fixed point modification, and we expect $S_n(0)$ to converge reasonably fast unless $|z|$ is large.

8.1. Truncation error bounds for $S_n(0)$.

8.1.1. *The Craviotto-Jones-Thron bound.* By a value set method similar to OST, Craviotto, Jones and Thron [**1**] proved that

$$|f - S_n(0)| \leq \frac{\rho_n^3/|\widehat{B}_{n-1}|^2}{|h_n|(|h_n| - \rho_n^2)} \quad \text{for } n \geq k+1 \tag{8.1}$$

where h_n is given by (2.11), $\rho_n := |z|/(2n-1)$, $\widehat{B}_k$ is given recursively by $\widehat{B}_k = (-1)^{k+1}(2k-1)\widehat{B}_{k-1}/z + \widehat{B}_{k-2}$ for $k = 1, 2, 3, \ldots$ with $\widehat{B}_0 := 1$ and $\widehat{B}_{-1} := 0$, and $k \geq 0$ is chosen such that $|z^2| \leq 4k - 2$ and $\rho_n + 1/\rho_{n-1} \geq 2$ for $n \geq k+1$. This bound is quite sharp, as they showed for an example with $z = 2e^{i\pi/4}$, but it is awkward to compute.

8.1.2. *The bound based on* Q_n. If we use OST with $w_n = 0$ for all n and $R_n := 2|a_n|$ for $n \geq 1$, then we get that if $|a_{n+1}| \leq R_n(1 - R_{n+1})$ for $n > N$, then by (3.4)

$$\begin{aligned} |f - S_n(0)| \leq \frac{|f_{N-1} - f_N|}{|h_N|} \cdot H_{N+1}^2 \\ \cdot \frac{|a_{N+1}| \cdot 2|a_n|}{1 - 2|a_{N+1}|} \prod_{k=N+1}^{n-1} M_k \quad \text{for } n \geq N+1 \end{aligned} \tag{8.2}$$

where $M_k = R_k/(1 - R_k)$ by Remark 2.2.3 and

$$H_{N+1} = \frac{|h_{N+1} - 4|a_{N+1}|^2| + 2|h_{N+1} - 1| \cdot |a_{N+1}|}{|h_{N+1}|^2 - 4|a_{N+1}|^2} \tag{8.3}$$

by Remark 3.1.2. Here $|a_{n+1}| \leq R_n(1 - R_{n+1})$ if and only if $|z^2| \leq n^2 + 2n - 5/4$; that is, we can use any $N \in \mathbf{N}$ with $N \geq \sqrt{|z|^2 + 9/4} - 2$.

To compare with the bounds in [**1**], we set $z = 2e^{i\pi/4}$ which means that $z^2 = 4i$ and $N \geq \frac{1}{2}$ works. If we use $N = 2$ in (8.2), we get (in view of Remark 3.1.4) that

(8.4)

$$|f-S_n(0)| \le |a_1a_2|\left\{\frac{|B_3-4B_2|a_3|^2|+2|a_3|^2}{|B_3|^2-4|B_2a_3|^2}\right\}^2\frac{2|a_3a_n|}{1-2|a_3|}$$
$$\cdot\prod_{k=3}^{n-1}\frac{R_k}{1-R_k}$$
$$\le 0.963256\,|a_n|\prod_{k=3}^{n-1}2|a_k|/(1-2|a_k|) \quad \text{for } n\ge 3,\ z=2e^{i\pi/4},$$

where $B_2 := 1+a_2$ and $B_3 := 1+a_2+a_3$. Indeed, the first bound in (8.4) holds for all $|z^2| \le 55/4$, i.e., $|z| \le \sqrt{55}/2$. If we instead increase N to $N := 3$, we get

$$|f-S_n(0)| \le |a_1a_2a_3|\left\{\frac{|B_4-4B_3|a_4|^2|+2|1+a_2|\cdot|a_4|^2}{|B_4|^2-4|a_4B_3|^2}\right\}^2 \tag{8.5}$$
$$\cdot\frac{2|a_4a_n|}{1-2|a_4|}\prod_{k=4}^{n-1}M_k$$

where $B_3 := 1+a_2+a_3$ and $B_4 := 1+a_2+a_3+a_4+a_2a_4$ which holds for $|z^2| \le 91/4$. In particular

$$|f-S_n(0)| \le 0.05867966\,|a_n|\prod_{k=4}^{n-1}2|a_k|/(1-2|a_k|) \tag{8.6}$$
$$\text{for } n\ge 4,\ z=2e^{i\pi/4},$$

and so on.

The table below shows how these bounds compare to the bound in [**1**]:

$z = 2e^{i\pi/4}$.

n	$\lvert f - S_n(0)\rvert$	(8.1)	(8.4)	(8.6)
3	$2.23 \cdot 10^{-2}$	$3.20 \cdot 10^{-2}$	$2.57 \cdot 10^{-1}$	———
6	$1.55 \cdot 10^{-6}$	$1.84 \cdot 10^{-6}$	$1.92 \cdot 10^{-3}$	$1.02 \cdot 10^{-4}$
9	$6.17 \cdot 10^{-12}$	$6.90 \cdot 10^{-12}$	$1.66 \cdot 10^{-7}$	$8.84 \cdot 10^{-9}$
12	$2.56 \cdot 10^{-18}$	$3.87 \cdot 10^{-18}$	$1.47 \cdot 10^{-12}$	$7.86 \cdot 10^{-14}$
15	$4.80 \cdot 10^{-25}$	$5.13 \cdot 10^{-24}$	$12.59 \cdot 10^{-18}$	$1.38 \cdot 10^{-19}$

This is not at all impressive, but one should take into consideration the kind of work involved in computing the different bounds. The computation of (8.1) requires the recursive computation of $\widehat{B}_n$. (The value of h_n is then given by $h_n = \rho_n \widehat{B}_n / \widehat{B}_{n-1}$.)

Another matter is that, for $z = x + iy$, we have $\tan z = (\tan x + \tan(iy))/(1 - \tan x \tan(iy))$. Hence it is probably just as easy to compute $\tan z$ for $z \in \mathbf{R}$ and $z \in i\mathbf{R}$. We may therefore concentrate on $z \in \mathbf{R}$ and $z \in i\mathbf{R}$, which means that $a_n \in \mathbf{R}$ for all $n \geq 2$. In particular $a_n > 0$ if $z \in i\mathbf{R}$, and then the a posteriori bound

$$|f - S_n(0)| \leq |S_{n+1}(0) - S_n(0)|$$

which follows from [**11**, p. 87], [**16**, p. 97] is much easier to compute than (8.1), even if it is used as an a priori bound.

In the rest of this example we shall look at the two values $z = 1$ and $z = 15i$. For $z = 1$ the bounds (8.4) and (8.5) take the forms

$$(8.7) \qquad |f - S_n(0)| \leq \frac{0.122258}{4n^2 - 8n + 3} \cdot \prod_{k=2}^{n-2} \frac{2}{4k^2 - 3} \quad \text{for } z = 1,\ n \geq 3$$

and

$$(8.8) \qquad |f - S_n(0)| \leq \frac{4.4515 \cdot 10^{-4}}{4n^2 - 8n + 3} \cdot \prod_{k=3}^{n-2} \frac{2}{4k^2 - 3} \quad \text{for } z = 1,\ n \geq 4,$$

respectively.

For $z = 15i$ we have $R_k = 2|a_k| > 1$ for $k < 12$, and $|a_{n+1}| \leq 2|a_n|(1 - 2|a_{n+1}|)$ only for $n \geq 15$. This means that (8.2) only holds with $N \geq 14$. It requires some computation to find

$$h_{15} = 1 + \frac{a_{15}}{1} + \frac{a_{14}}{1} + \cdots + \frac{a_2}{1},$$

so we rather turn to the bounds (4.1) or (4.3) with $\alpha = 0$. That is,

$$|f - S_n(0)| \leq 2T_n = 15 \prod_{\nu=1}^{n-1} \frac{225}{224 + 4\nu^2} \quad \text{for } z = 15i,\ n \geq 1 \tag{8.9}$$

and

$$|f - S_n(0)| \leq 30 \prod_{\nu=1}^{n-1} \frac{\sqrt{1 + 900/(4\nu^2 - 1)} - 1}{\sqrt{1 + 900/(4\nu^2 - 1)} + 1} \quad \text{for } z = 15i,\ n \geq 1. \tag{8.10}$$

8.1.3. *The bound based on* (3.13). Since $a_{n+1} \in E_{n+1}$ for $n \geq 14$ when $z = 15i$ and $R_n = 2|a_n|$, we get from (4.2) that

$$\begin{aligned} |f - S_{n+1}(0)| &\leq 2R_{n+1}T_{n+1} \\ &= 15 \cdot \frac{450}{4n^2 - 1} \prod_{\nu=1}^{n} \frac{225}{224 + 4\nu^2} \quad \text{for } z = 15i,\ n \geq 14. \end{aligned} \tag{8.11}$$

Compared to (8.9) the term R_{n+1} gives a positive effect since $R_{n+1} < 1$ for $n \geq 11$.

Combining Q_n with the Gragg-Warner bound (8.10) gives similarly

$$\begin{gathered} |f - S_{n+1}(0)| \leq 30 \frac{450}{4n^2 - 1} \prod_{\nu=1}^{n} \frac{\sqrt{1 + 900/(4\nu^2 - 1)} - 1}{\sqrt{1 + 900/(4\nu^2 - 1)} + 1} \\ \text{for } z = 15i,\ n \geq 14 \end{gathered} \tag{8.12}$$

which improves (8.10) for these values of n.

The tables below show how these bounds compare to the actual truncation error.

$z = 1$.

n	$\lvert - S_n(0)\rvert$	(8.7)	(8.8)
3	$1.85 \cdot 10^{-3}$	$8.15 \cdot 10^{-3}$	———
6	$2.25 \cdot 10^{-9}$	$3.78 \cdot 10^{-7}$	$8.93 \cdot 10^{-9}$
9	$1.44 \cdot 10^{-16}$	$4.44 \cdot 10^{-13}$	$1.05 \cdot 10^{-14}$
12	$1.32 \cdot 10^{-24}$	$5.82 \cdot 10^{-20}$	$1.38 \cdot 10^{-21}$
15	$2.79 \cdot 10^{-33}$	$1.55 \cdot 10^{-27}$	$3.66 \cdot 10^{-29}$

$z = 15i$.

n	$\lvert f - S_n(0)\rvert$	(8.9)	(8.10)	(8.11)	(8.12)
3	1.64	14.48	20.66	———	
6	0.12	9.72	4.31	———	
9	$5.90 \cdot 10^{-3}$	3.30	0.29	———	———
12	$9.97 \cdot 10^{-5}$	0.49	$6.88 \cdot 10^{-3}$	———	
15	$6.72 \cdot 10^{-7}$	$7.39 \cdot 10^{-4}$	$6.28 \cdot 10^{-5}$	$4.24 \cdot 10^{-4}$	$3.61 \cdot 10^{-5}$
30	$1.43 \cdot 10^{-22}$	$2.16 \cdot 10^{-17}$	$4.16 \cdot 10^{-20}$	$2.89 \cdot 10^{-18}$	$5.57 \cdot 10^{-21}$

8.2. Choice of w_n. The improvement $\Phi_n(w_n, 0)$. The square root modification takes the form

$$w_n := (q_n - 1)/2 \quad \text{where } q_n = \sqrt{1 - z^2/(n^2 - 1/4)}\,. \tag{8.13}$$

Gill [**2**] proved that if $\max_{m\geq n} |w_m - w_{m+1}| \leq \varepsilon_n |w_{n+1}|$ for all $n \geq 1$, where $0 \leq \varepsilon_n \leq 1$ and $0 < |w_m| < \sigma_n$ for $m \geq n \geq 1$, then $|\Phi_n(w_n, 0)| \leq \sigma_n \varepsilon_n/(1 - 5\sigma_n)^2$. For our continued fraction we can use $\sigma_n = |w_n| = |q_n - 1|/2 = \mathcal{O}(a_{n+1})$ and $\varepsilon_n = |q_n - q_{n+1}|/|q_{n+1} - 1| = \mathcal{O}((a_{n+1} - a_{n+2})/a_{n+2})$, and thus $\Phi_n(w_n, 0) = \mathcal{O}(a_n - a_{n+1}) = \mathcal{O}(n^{-3})$. This is also what we get from (2.14) and (6.4):

$$\begin{aligned} \Phi_n(w_n, 0) &= \mathcal{O}((a_n - \hat{a}_n)/w_n) = \mathcal{O}(w_{n-1} - w_n) \\ &= \mathcal{O}(a_n - a_{n+1}) = \mathcal{O}(n^{-3}). \end{aligned} \tag{8.14}$$

This holds true for every $z \in \mathbf{C}$, and in particular for $z \in \mathbf{R}$ and $z \in i\mathbf{R}$.

The improvement machine applied to this w_n gives $t=1$, and

$$\tilde{w}_n^{(2)} = (q_n - 1)q_n/(q_n + q_{n+1}). \tag{8.15}$$

The improvement is now of the order $\Phi_n(\tilde{w}_n^{(2)}, 0) = \mathcal{O}(n^{-4})$.

Finally, the asymptotic expansion should probably be done in powers of $(q_n - 1)$. However, to keep the computation of w_n simpler, we shall rather use a polynomial in $1/n$. This gives for instance

$$\begin{aligned}\widehat{w}_n^{(7)} &:= \sum_{j=2}^{7} c_j n^{-j}\\ &= -\frac{z^2}{4n^2} - \frac{z^2(1+z^2)}{16n^4} + \frac{z^4}{8n^5}\\ &\quad - \frac{z^2(1+14z^2+2z^4)}{64n^6} + \frac{z^4(11+5z^2)}{32n^7}\end{aligned} \tag{8.16}$$

which gives an improvement $\Phi_n(\widehat{w}_n^{(7)}, 0) = \mathcal{O}((a_n - \widehat{a}_n^{(7)})/a_n) = \mathcal{O}(n^{-8})$.

The tables below show $S_n(w_n)$ for $z=1$ and $z=15i$ for the various choices of w_n. The values for $S_n(w_n)$ are just truncated, with no rounding.

$z=1$. $\tan z = 1.5574077246549022305069748074583601730 87$

n	$w_n = 0$	$w_n = (q_n - 1)/2$	(8.15)	(8.16)
1	1.0000000	———	———	1.01587301587
2	1.5000000	1.560373755	1.5572005678	1.55554048555
3	1.5555555	1.557434164	1.5574071043	1.55740148064
4	1.5573770	1.557407913	1.5574077225	1.55740770406
5	1.5574074	1.557407725	1.5574077246	1.55740772459
$m(40)$	18	16	15	16

$z = 15i$. $\tan z = 0.999999999999812847540623214020923208746 9i$

n	$w_n = 0$	$w_n = (q_n - 1)/2$	(8.15)	(8.16)
3	$2.63736263i$	$1.087640589i$	$1.0155773319i$	$0.18872854i$
6	$0.88135751i$	$0.994032636i$	$0.9994064402i$	$1.01188070i$
9	$1.00590032i$	$1.000174372i$	$1.0000110822i$	$1.0000110822i$
12	$0.99990034i$	$0.999998035i$	$0.9999999152i$	$0.9999999152i$
15	$1.00000067i$	$1.000000009i$	$1.0000000002i$	$0.99999998i$
$m(35)$	40	38	37	38

8.3. Error bounds for the square root modification. We clearly see that there is not much to gain by changing to modified approximants for our $K(a_n/1)$. The classical approximants converge almost as fast as the modified ones, and the truncation error bounds (8.4) and (8.5) are reasonably good and easy to compute, in particular if we replace the product of M_k for $k \geq 6$ by powers of M_6 for instance. For completeness we shall still derive some bounds for $|f - S_n(w_n)|$, where w_n is given by (8.13), since the techniques can easily be adapted to other continued fractions of type 2.

8.3.1. *Bounds based on Q_n.* In view of Lemma 5.1, we try the choice

(8.17)
$$R_{n+1} := \frac{2|\varepsilon_{n+1}|}{\Delta_{n+1}} = \frac{2|w_n - w_{n+1}| \cdot |w_n|}{|1 + w_{n+1}| - |w_n|} = \frac{|q_n - q_{n+1}| \cdot |q_n - 1|}{|q_{n+1} + 1| - |q_n - 1|}$$

where $q_n = \sqrt{1 - z^2/(n^2 - 1/4)}$. Let first $z^2 > 0$ and $n \geq \sqrt{z^2 + 1/4}$, which for instance holds for $n \geq 2$ and $|z| \leq \sqrt{15}/2$ or for $n \geq 3$ and $|z| \leq \sqrt{35}/2$. Then $a_n < 0$, $0 \leq q_n < 1$, $\{q_n\}$ increases monotonically to 1 and

$$R_{n+1} = (q_{n+1} - q_n)(1 - q_n)/(q_{n+1} + q_n). \tag{8.18}$$

Clearly $(1 - q_n)$ decreases monotonically, $(q_{n+1} + q_n)$ increases monotonically and

$$q_{n+1} - q_n = \frac{q_{n+1}^2 - q_n^2}{q_{n+1} + q_n} = \frac{16z^2}{(2n-1)(2n+1)(2n+3)(q_{n+1} + q_n)}$$

decreases monotonically, so R_{n+1} decreases monotonically. Hence, by Lemma 5.1, $a_{n+1} \in E_{n+1}$ if $R_{n+1} \leq \Delta_{n+1}/2 = (q_{n+1}+q_n)/4$; that is, if

$$4(q_{n+1}-q_n)(1-q_n) \leq (q_{n+1}+q_n)^2$$

which always holds for $z \in \mathbf{R}$ with $|z| \leq \sqrt{15}/2$ and $n \geq 2$ or for $|z| \leq \sqrt{35}/2$ and $n \geq 3$. Hence we get from (3.7) that

(8.19)

$$|f-S_n(w_n)| \leq \frac{|z^3/3|}{(1-z^2/3)^2} H_3^2 \frac{|2z^2/15|}{q_3+1-2R_3} \cdot \frac{2R_n}{q_n+1} \prod_{k=3}^{n-1} M_k \quad \text{for } n \geq 3$$

for $-\sqrt{35}/2 \leq z \leq \sqrt{35}/2$, where R_k is given by (8.18), $q_k = \sqrt{1-4z^2/(4k^2-1)}$, $w_n = (q_n-1)/2$, and by Remarks 2.2.3 and 3.1.2

(8.20)
$$\begin{aligned} M_k &= \frac{|w_k|+R_k}{1-|w_k|-R_k} = \frac{1-q_k+2R_k}{1+q_k-2R_k} \to 0, \\ H_3 &= \frac{|(\frac{\overline{a_3}}{1+\overline{a_2}} + \frac{\overline{q_3}+1}{2})\frac{q_3+1}{2} - R_3^2| + |\frac{a_3}{1+a_2}|R_3}{|\frac{a_3}{1+a_2} + \frac{q_3+1}{2}|^2 - R_3^2}. \end{aligned}$$

For $z=1$ this gives in particular $H_3 < 0.907179$ and

(8.21)

$$|f-S_n(w_n)| \leq 0.0853895 \cdot \frac{R_n}{1+q_n} \prod_{k=3}^{n-1} M_k \leq 0.0853895 \cdot \frac{R_n}{1+q_n} \cdot 0.03757^{n-3}$$

for $n \geq 3$. Since

$$\frac{R_n}{1+q_n} = \frac{(q_n-q_{n-1})(1-q_{n-1})}{(q_n+q_{n-1})(1+q_n)} = \frac{(q_n^2-q_{n-1}^2)(1-q_{n-1}^2)}{(q_n+q_{n-1})^2(1+q_n)(1+q_{n-1})}$$

where $q_n = \sqrt{1-4z^2/(4n^2-1)} > 1-4z^2/(4n^2-1)$, we have

(8.22)
$$\begin{aligned} \frac{R_n}{1+q_n} &\leq \frac{1}{\left(2-\frac{8z^2}{(2n-3)(2n+1)}\right)^2 \left(2-\frac{4z^2}{(2n-3)(2n-1)}\right)^2} \\ &\quad \cdot \frac{64z^4}{(2n-3)^2(2n-1)^2(2n+1)} \end{aligned}$$

which for $z = 1$ gives the simpler bound

(8.23)

$$|f - S_n(w_n)| \leq 0.34156 \cdot \frac{(2n-3)^2(2n+1)}{(4n^2-4n-7)^2(4n^2-8n+1)^2} \cdot 0.03757^{n-3}.$$

For $z = iy$ with $y \in \mathbf{R}$, we have $a_n > 0$ for $n \geq 2$, and thus $q_n > 1$, $\{q_n\}$ decreases monotonically to 1, and

$$R_{n+1} = \frac{(q_n - q_{n+1})(q_n - 1)}{2 - (q_n - q_{n+1})} \tag{8.24}$$

when $q_n - q_{n+1} < 2$. Since $(q_n - q_{n+1}) \to 0$, $(q_n - q_{n+1})$ is certainly < 2 from some n on. Let n satisfy $q_n - q_{n+1} < 2$. Then $\{R_{n+1}\}$ decreases if $(q_n - q_{n+1})$ decreases, which absolutely seems to be the case, but it has to be checked. Let $f(n) := 4y^2/(4n^2-1)$ so that $q_n = \sqrt{1+f(n)}$. Then $(q_n - q_{n+1})$ decreases if $\frac{d}{dn}(\sqrt{1+f(n)} - \sqrt{1+f(n+1)}) < 0$; that is, if

$$f'(n)\sqrt{1+f(n+1)} - f'(n+1)\sqrt{1+f(n)} < 0$$

where $f'(n) = -32y^2n/(4n^2-1)^2$. Hence $(q_n - q_{n+1})$ decreases if

$$\frac{n^2}{(4n^2-1)^4}\left(1+\frac{4y^2}{4n^2+8n+3}\right) > \frac{(n+1)^2}{(4n^2+8n+3)^4}\left(1+\frac{4y^2}{4n^2-1}\right)$$
$$n^2(2n+3)^3(4n^2+8n+3+4y^2) > (n+1)^2(2n-1)^3(4n^2-1+4y^2)$$

which holds for all $y \in \mathbf{R}$ if $n^2(2n+3)^3 > (n+1)(2n-1)^3$ which is easy to verify for all $n \geq 0$. Hence $a_{n+1} \in E_{n+1}$ if $q_n - q_{n+1} < 2$ and $R_{n+1} \leq \Delta_{n+1}/2$. Since $(q_n - q_{n+1})$ is decreasing, we have $(q_n - q_{n+1}) < 2$ for $n \geq N$ if $q_N - q_{N+1} < 2$. Straightforward computation shows that this holds if and only if

$$4y^2 < (4N^2-1)(4N^2+8N+3).$$

Hence we have for instance that $q_n - q_{n+1} < 2$ for $n \geq 2$ for $|y| < 5\sqrt{21}/2$ and for $n \geq 3$ for $|y| < 21\sqrt{5}/2$. Moreover, $R_{n+1} \leq \Delta_{n+1}/2$ if

$$\begin{aligned} (q_n - q_{n+1})q_n &\leq 1 + (q_n - q_{n+1})^2/4, \\ \text{i.e.,}\quad q_n^2 &\leq 1 + (q_n + q_{n+1})^2/4, \\ \text{i.e.,}\quad y^2 &\leq \frac{(2n-1)(2n+1)(2n+3)^2}{4(4n+7)}. \end{aligned}$$

Hence (8.19) still holds for $n \geq 3$ when $z = iy$ with

$$|y| \leq \min\left\{\frac{21\sqrt{5}}{2},\ \sqrt{\frac{5\cdot 7\cdot 9^2}{4\cdot 19}}\right\} = \frac{3}{2}\sqrt{\frac{35}{19}} \approx 6.1,$$

where R_k is given by (8.17) and q_k, w_k and M_k are as before.

For $z = 15i$ we need $n \geq 6$ to ensure that $a_{n+1} \in E_{n+1}$, and thus by (3.4)

(8.25)

$$|f - S_n(w_n)| \leq \frac{|f_{N-1} - f_N|}{|h_N|}\,\frac{H_{N+1}^2|a_{N+1}|}{|1+w_{N+1}| - R_{N+1}}\,\frac{R_n}{|1+w_n|}\prod_{k=N+1}^{n-1} M_k \quad \text{for } n \geq N+1$$

where the bound improves slightly if we increase $N \geq 5$. For instance, $N = 6$ gives

$$(8.26)\quad |f - S_n(w_n)| \leq 0.724648 \cdot \frac{R_n}{1+w_n}\prod_{k=7}^{n-1} M_k \quad \text{for} \quad z = 15i,\ n \geq 7$$

where $M_k = (w_k + R_k)/(1 + w_k + R_k)$. This is not too bad. (See the table below.)

8.3.2. *Bounds based on* (3.13). For $0 < z^2 \leq 15/4$ we already know that $a_n \in E_n$ for $n \geq 3$ when R_n is given by (8.17). Hence by (4.2)

$$(8.27)\qquad |f - S_n(w_n)| \leq \frac{2R_nT_n}{1+w_n} = \frac{4|z|^{2n-1}}{\prod_{\nu=1}^{n-1}(|z|^2 + 4\nu^2 - 1)} \cdot \frac{R_n}{1+q_n}$$

for $0 < z^2 < 15/4$ and $n \geq 3$, where $R_n/(1+q_n)$ satisfies (8.22). For $z = 1$ this gives in particular

(8.28)

$$|f - S_n(w_n)| \leq \frac{(2n-3)^2(2n+1)}{(4n^2 - 4n - 7)^2(4n^2 - 8n + 1)^2 4^{n-3}((n-1)!)^2} \quad \text{for } n \geq 3.$$

$z = 1$.

n	$\vert f - S_n(w_n)\vert$	(8.7) for $\vert f - S_n(0)\vert$	(8.23)	(8.28)
3	$2.64 \cdot 10^{-5}$	$8.15 \cdot 10^{-3}$	$4.44 \cdot 10^{-4}$	$3.23 \cdot 10^{-4}$
4	$1.88 \cdot 10^{-7}$	$5.37 \cdot 10^{-4}$	$1.59 \cdot 10^{-6}$	$3.42 \cdot 10^{-6}$
5	$4 \cdot 10^{-10}$	$1.81 \cdot 10^{-5}$	$1.32 \cdot 10^{-8}$	$4.72 \cdot 10^{-8}$

For $z^2 < 0$ we have $a_n > 0$ for all $n \geq 2$. The choice (8.17) for R_n reduces to (8.24), and we have already found that $|f^{(n)} - w_n| \leq R_n$ for $n \geq 6$ when $z = 15i$. Hence by (4.2) we have that

(8.29)

$$|f - S_n(w_n)| \leq \frac{2R_nT_n}{1 + w_n} = \frac{4|z|^{2n-1}}{\prod_{\nu=1}^{n-1}(|z|^2 + 4\nu^2 - 1)} \cdot \frac{R_n}{1 + q_n} \quad \text{for } n \geq 6,$$

that is,

(8.30)

$$|f - S_n(w_n)| \leq \frac{15^{2n+3}}{4^{n-3}} \cdot \frac{(2n-3)^2(2n+1)}{(4n^2 - 4n - 903)(4n^2 - 8n - 447)^2 \prod_{\nu=1}^{n-1}(56 + \nu^2)}$$

for $n \geq 6$.

$z = 15i$.

n	$\vert f - S_n(w_n)\vert$	(8.9)	(8.26)	(8.30)
3	$8.76 \cdot 10^{-2}$	14.48	———	———
6	$5.97 \cdot 10^{-3}$	9.72	———	1.01
9	$1.74 \cdot 10^{-4}$	3.30	$1.21 \cdot 10^{-2}$	$4.96 \cdot 10^{-3}$
12	$1.97 \cdot 10^{-6}$	0.49	$1.33 \cdot 10^{-4}$	$7.66 \cdot 10^{-5}$
15	$9.00 \cdot 10^{-9}$	$7.39 \cdot 10^{-4}$	$6.01 \cdot 10^{-7}$	$4.86 \cdot 10^{-7}$

9. Example 3: The incomplete gamma function. The continued fraction $K(a_n/1)$ in (1.3) converges to the incomplete gamma function $\Gamma(a, z)$ for $z \in Z_3$, that is, for $|\arg z| < \pi$, as long as $(a - z)$ is not a positive, odd integer. (Otherwise the continued fraction is not defined.) Hence we assume that $\frac{a-z-1}{2} \notin \mathbf{N}$ in this section. Since

$$\tag{9.1} \begin{gathered} a_{n+1} = \frac{-n(n-a)}{(2n-1+z-a)(2n+1+z-a)} = -\frac{1}{4} + \frac{\varepsilon_{n+1}}{4} \\ \text{where} \quad \varepsilon_{n+1} := \frac{4nz+(z-a)^2-1}{4n^2+4(z-a)n+(z-a)^2-1} = \frac{z}{n} + \mathcal{O}(n^{-2}) \\ \text{as } n \to \infty, \end{gathered}$$

we expect that the continued fraction converges slowly except if $|\arg z|$ is small or $a \in \mathbf{N}$. In the latter case $a_{n+1} = 0$ for $n = a$, and $\Gamma(a, z)$ reduces to a rational function in z with a finite continued fraction expansion. Hence we also assume that $a \notin \mathbf{N}$ in this section.

9.1. Choice of w_n. The fixed point modification $w_n = -\frac{1}{2}$ is not a good idea here, since $S_n^{-1}(\infty) \to -\frac{1}{2}$. Since $a_n \to -\frac{1}{4}$ in a monotone way, the square root modification

$$\tag{9.2} w_n = \frac{q_n - 1}{2} \quad \text{where } q_n = \sqrt{1+4a_{n+1}} = \sqrt{\varepsilon_{n+1}},$$

where ε_{n+1} is given by (9.1) and $\operatorname{Re} q_n > 0$, may be a better idea. The improvement machine in Section 6.3 applied to (9.2) gives $t = 1$ and

$$\tag{9.3} \begin{aligned} w_n^{(1)} &= w_n + \frac{a_{n+1} - w_n(1+w_{n+1})}{1+w_{n+1}+w_n} \\ &= \frac{q_n - 1}{2} + \frac{4a_{n+1} - (q_n-1)(q_{n+1}+1)}{2(q_n+q_{n+1})} \\ &= \frac{q_n(q_n-1)}{q_n+q_{n+1}}. \end{aligned}$$

For the asymptotic expansion (6.7), the choice $\mu_j(n) := q_n^{-j}$ is probably a good choice. We shall rather choose the simpler expression $\mu_j(n) = n^{-j/2}$, and

(9.4)

$$w_{n+1} = \widehat{w}_{n+1}^{(N)} := \sum_{j=0}^{N} c_j n^{-j/2} \quad \text{for } n \geq 1, \quad \widehat{w}_1^{(N)} := a_2/(1+\widehat{w}_2^{(N)}).$$

By means of a computer algebra package (here Maple), it is simple to derive as many coefficients c_n as one may wish. The first 11 ones are

$$c_0 = \frac{-1}{2}, \quad c_1 = \frac{\sqrt{z}}{2}, \quad c_2 = \frac{-1}{8}, \quad c_3 = \frac{4(a+z)^2 - (4z+1)^2}{64\sqrt{z}},$$

$$c_4 = -\frac{4(a+z)^2 - (4z+1)^2}{128z},$$

$$c_5 = \frac{-1}{4096z^{3/2}}(25 - 48z - 264z^2 + 48az - 104a^2 - 576z^3 + 384az^2 + 192a^2z - 192a^3z - 368z^4 - 32a^2z^2 + 16a^4),$$

$$c_6 = \frac{1}{2048z^2}(13 - 16z - 72z^2 + 16az - 56a^2 - 192z^3 + 128az^2 + 64a^2z - 112z^4 + 192az^3 - 32a^2z^2 - 64a^3z + 16a^4),$$

$$c_7 = \frac{1}{131072z^{5/2}}(-1073 + 1000z - 484z^2 - 1000az + 4748a^2 - 5440z^3 + 1920az^2 - 4160a^2z - 13744z^4 + 10560az^3 + 864a^2z^2 + 4160a^3z - 1840a^4 - 14720z^5 + 23040az^4 - 1280a^2z^3 - 7680a^3z^2 + 640a^4z - 5824z^6 + 14720az^5 - 8768a^2z^4 - 3840a^3z^3 + 4288a^4z^2 - 640a^5z + 64a^6),$$

$$c_8 = \frac{-1}{8192z^3}(-103 + 78z - 15z^2 - 78az + 465a^2 - 176z^3 + 96az^2 - 336a^2z - 552z^4 + 432az^3 + 336a^3z - 216a^4 - 672z^5 + 1152az^4 - 192a^2z^3 - 384a^3z^2 + 96a^4z - 240z^6 + 672az^5 - 528a^2z^4 - 64a^3z^3 + 240a^4z^2 - 96a^5z + 16a^6),$$

$$c_9 = \frac{-1}{16777216z^{7/2}}(375733 - 240352z + 19760z^2 + 240352az - 1721744a^2 + 34944z^3 - 224000az^2 + 1063552a^2z - 409120z^4 + 108416az^3 + 58944a^2z^2 - 1063552a^3z + 899808a^4 - 1358336z^5 + 1218560az^4 - 379904a^2z^3 + 931840a^3z^2 - 412160a^4z - 1858816z^6 + 3078656az^5 - 624384a^2z^4 - 336896a^3z^3 - 572160a^4z^2 + 412160a^5z - 98560a^6 - 1304576z^7 + 3297280az^6 - 1964032a^2z^5 - 860160a^3z^4 + 960512a^4z^3 - 143360a^5z^2 + 14336a^6z - 371456z^8$$

$$
\begin{aligned}
&+1304576az^7-1467392a^2z^6+243712a^3z^5+609792a^4z^4\\
&-387072a^5z^3+80896a^6z^2-14336a^7z+1280a^8),\\
c_{10}=&\frac{1}{524288z^4}(23797-13184z+320z^2+13184az-110272a^2\\
&+2176z^3-9984az^2+59520a^2z-6112z^4+1920az^3+4800a^2z^2\\
&-59520a^3z+62496a^4-21504z^5+22528az^4-16384a^2z^3\\
&+43008a^3z^2-27648a^4z-39424z^6+70656az^5-19968a^2z^4\\
&-4096a^3z^3-26112a^4z^2+27648a^5z-8704a^6-30720z^7\\
&+86016az^6-67584a^2z^5-8192a^3z^4+30720a^4z^3-12288a^5z^2\\
&+2048a^6z-7936z^8+30720az^7-41984a^2z^6+18432a^3z^5\\
&+9728a^4z^4-14336a^5z^3+7168a^6z^2-2048a^7z+256a^8).
\end{aligned}
$$

If we let $a := \frac{1}{2}$, as we do in our numerical experiments, we get the first 13 coefficients

$$
\begin{aligned}
&c_0=-\frac{1}{2},\quad c_1=\frac{\sqrt{z}}{2},\quad c_2=-\frac{1}{8},\quad c_3=-\frac{1+3z}{16}\sqrt{z},\quad c_4=\frac{1+3z}{32},\\
&c_5=\frac{5+16z+23z^2}{256}\sqrt{z},\quad c_6=-\frac{1+6z+7z^2}{128},\\
&c_7=-\frac{15+69z+115z^2+91z^3}{2048}\sqrt{z},\quad c_8=\frac{4+27z+84z^2+60z^3}{2048},\\
&c_9=\frac{99+1092z+2254z^2+2548z^3+1451z^4}{65536}\sqrt{z},\\
&c_{10}=-\frac{1+3z+27z^2+60z^3+31z^4}{2048},\\
&c_{11}=\frac{201-1365z-10626z^2-5797z^3-13059z^4-14882z^5}{524288}\sqrt{z},\\
&c_{12}=\frac{8+93z-40z^2+504z^3+1240z^4+732z^5}{65536}.
\end{aligned}
$$

The tables below show how the approximants $S_n(w_n)$ perform for $z=1$, where we expect them to converge relatively fast, and for $z=-2+0.1i$ which is close to the boundary of Z_3. The values in the tables are just truncated, with no rounding.

$z = 1$. $\Gamma(\frac{1}{2}, 1) = 0.27880558528066197649923261 1$

n	$S_n(0)$	$S_n(-\frac{1}{2})$	$S_n((q_n-1)/2)$	(9.3)
3	0.2764	0.2846	0.27862	0.278810
6	0.27865	0.27908	0.278797	0.27880598
9	0.278788	0.278834	0.27880479	0.27880562
12	0.2788027	0.2788099	0.27880547	0.278805591
15	0.278805027	0.2788064	0.278805565	0.2788055863
30	0.2788055843	0.2788055865	0.278805585257	0.2788055852817
$m(6)$	18	15	13	6
$m(35)$	422	432	373	344

n	$S_n(-\frac{1}{2}(1-\sqrt{\frac{z}{n}}))$	$S_n(-\frac{1}{2}(1-\sqrt{\frac{z}{n}+\frac{1}{4n}}))$	(9.4) with $N = 3$
3	0.27828	0.27880584	0.27880584
6	0.278788	0.278805566	0.278805566
9	0.2788042	0.2788055840	0.27880558403
12	0.27880541	0.27880558515	0.27880558515
15	0.278805555	0.278805585263	0.278805585263
30	0.278805585249	0.278805585280654	0.278805585280654
$m(6)$	14	5	4
$m(35)$	373	273	273

n	(9.4) with $N = 12$
3	0.2788103
6	0.2788055863
9	0.2788055852870
12	0.27880558528080
15	0.2788055852806684
30	0.27880558528066197662 5
$m(6)$	4
$m(35)$	157

$z=-2+0.1i$. $\Gamma(\frac{1}{2},-2+0.1i)=1.25056710-6.66810491i$

n	$S_n(0)$	$S_n(-\frac{1}{2})$	$S_n((q_n-1)/2)$
3	$2.049065-12.1200i$	$-0.192079-8.3924i$	$1.423033-6.980900i$
10	$-0.056675-7.4533i$	$0.015170-5.5030i$	$1.173930-6.660657i$
100	$0.556692-6.6339i$	$2.321004-6.2896i$	$1.249460-6.660389i$
500	$1.153556-6.5586i$	$1.362853-6.7757i$	$1.250984-6.667677i$
997	$1.288672-6.6522i$	$1.214746-6.6870i$	$1.250619-6.668206i$
998	$1.289878-6.6557i$	$1.212170-6.6803i$	$1.250609-6.668210i$
999	$1.290753-6.6593i$	$1.211335-6.6768i$	$1.250599-6.668214i$

n	(9.3)	$S_n(-\frac{1}{2}(1-\sqrt{\frac{z}{n}+\frac{1}{4n}}))$	(9.4) with $N=3$
3	$1.3361990-6.72598506i$	$0.982045-7.477938i$	$1.22044278-7.129332i$
10	$1.2337356-6.67403715i$	$1.200983-6.531474i$	$1.23099422-6.649664i$
100	$1.2503563-6.66809019i$	$1.255394-6.667001i$	$1.25060741-6.667921i$
500	$1.2505624-6.66809956i$	$1.250689-6.668223i$	$1.25056932-6.667677i$
997	$1.2505679-6.66681044i$	$1.250546-6.668115i$	$1.25056718-6.668206i$
998	$1.2505680-6.66810462i$	$1.250545-6.668113i$	$1.25056716-6.668210i$
999	$1.2505680-6.66810465i$	$1.250545-6.668111i$	$1.25056714-6.668214i$
$m(4)$	179	479	145

n	(9.4) with $N=12$
3	$1.248929501067419799885850-6.671402555508192973093313i$
10	$1.250567125006118886935756-6.668104602095381387437780i$
500	$1.250567104272837837836407-6.668104914779757974760128i$
997	$1.250567104272837836131794-6.668104914779757974123697i$
998	$1.250567104272837836131815-6.668104914779757974123431i$
999	$1.250567104272837836131859-6.668104914779757974123171i$
$m(4)$	5

These tables show that $K(a_n/1)$ converges reasonably fast for $z=1$, but painfully slowly for $z=-2+0.1i$. As expected we do not gain much by replacing $S_n(0)$ by $S_n(-\frac{1}{2})$. But the choices (9.3) and (9.4) seem to be doing relatively well, in particular (9.4) with $N = 12$, as was to be expected. The simple form of (9.3) is also an important point. Hence it really makes sense to use modifying factors to make $S_n(w_n)$ converge faster to $\Gamma(a, z)$.

9.2. Truncation error bounds for $S_n(0)$. This time $K(a_n/1)$ is not a Stieltjes fraction, and the Gragg Warner bound does not apply. To be sure, $K(a_n/1)$ is the even part of the Stieltjes fraction, if $a < 1$,

$$\frac{z^{a-1}e^{-z}}{1} + \frac{(1-a)/z}{1} + \frac{1/z}{1} + \frac{(2-a)/z}{1} + \frac{2/z}{1} + \frac{(3-a)/z}{1} + \dots,$$

which is limit periodic with $a_n \to \infty$. However, the point in this section is to demonstrate how to treat $K(a_n/1)$ where $a_n \to -\frac{1}{4}$, and to see what we can expect to gain for such continued fractions. Hence, we shall forget about this connection to Stieltjes fractions.

The choice for α in Thron's parabola sequence theorem is not so obvious in this case. However, (9.1) indicates that $\alpha := \frac{1}{2}\arg z$ is a possible choice. Then $a_{n+1} \in P_{\alpha,n+1}$ if

$$|a_{n+1}| - \operatorname{Re}(a_{n+1}|z|/z) \le g_n(1 - g_{n+1})(1 + \cos\arg z). \tag{9.5}$$

Let first $a \in \mathbf{R}$ and $z > 0$. Then $a_{n+1} \ge -\frac{1}{4}$ for all $n \in \mathbf{N}$ with

$$n \ge n_0(a, z) := \max\left\{\frac{1+a-z}{2}, \ \frac{1-(z-a)^2}{4z}\right\}.$$

Hence $a_{n+1} \in P_{0,n+1}$ with $g_n = g_{n+1} = \frac{1}{2}$ for $n \ge n_0(a, z)$. Note that $d_n = 1/n$ and $k_n = 2(|a_n| - a_n) \le 4|a_n|$ in this case. If $n_0(a, z) \le 1$, we thus have by (2.8) that

$$\begin{aligned} |f - S_n(w_n)| \le 2T_n &= |a_1| \Big/ \prod_{\nu=1}^{n-1}\left(1 + \frac{1 - 4|a_{\nu+1}| + 1/\nu}{4|a_{\nu+1}|}\right) \\ &= \frac{e^{-z}z^a}{|1+z-a|} \Big/ \prod_{\nu=1}^{n-1} \frac{(1+1/\nu)|(2\nu+z-a)^2 - 1|}{4\nu|\nu-a|} \end{aligned} \tag{9.6}$$

for $n \ge 1$ whenever $\operatorname{Re} w_n \ge -\frac{1}{2}$. Now $n_0(a, z) \le 0$ if $(z - a) \ge 1$. Moreover, if $(z - a) < 1$, then

$$n_0(a, z) = \begin{cases} (1 + a - z)/2 & \text{if } a + z \ge 1, \\ (1 - (z-a)^2)/4z & \text{if } a + z \le 1. \end{cases}$$

Hence $n_0(a,z) \le 1$ if either

$$\begin{aligned} &z-a \ge 1, \quad \text{or} \\ &a+z \ge 1 \quad \text{and} \quad |a-z| \le 1, \quad \text{or} \\ &a+z \le 1 \quad \text{and} \quad 5-4a \le 0, \quad \text{or} \\ &a+z \le 1 \quad \text{and} \quad (z-a+2)^2 > 5-4a > 0. \end{aligned}$$

This holds in particular for $a = \frac{1}{2}$ and $z = 1$. Hence

$$(9.7) \qquad \begin{gathered} |f - S_n(w_n)| \le \frac{2/e}{3} \Big/ \prod_{\nu=1}^{n-1} \frac{(\nu+1)((2\nu+\frac{1}{2})^2-1)}{4\nu^2(\nu-\frac{1}{2})} \\ \text{for } n \ge 1,\ z=1,\ a=\frac{1}{2} \end{gathered}$$

when $\operatorname{Re} w_n \ge -\frac{1}{2}$. A slightly smaller, but more complicated, bound can be obtained by choosing g_n more carefully.

Let us now turn to complex values of z. Then (9.5) holds with $g_n = g_{n+1} = \frac{1}{2}$ and $a < n$ if $n \in \mathbf{N}$ with

$$\frac{n(n-a)|z|}{|(2n+z-a)^2-1|} + \operatorname{Re} \frac{n(n-a)\overline{z}}{(2n+z-a)^2-1} \le \frac{1}{4}\,(|z| + \operatorname{Re} z)$$

which holds with $a = \frac{1}{2}$, $z = -2 + 0.1i$ and $n \ge 3$. That is, $a_n \in P_{\alpha,n}$ with $\alpha := (\arg z)/2$ for $n \ge 4$. By combining (3.4) with (3.11), as done in (3.12), but using the bounds from Thron's parabola sequence theorem for $|f^{(2)} - S^{(2)}_{n-2}(w_n)|$, we then get

$$(9.8) \quad |f - S_n(w_n)| \le \frac{(1+|a_3|/d_3 g_3 \cos\alpha)|f_1 - f_2| \cdot |a_3|}{(d_2 g_2 \cos\alpha)^2 (1-g_2)\cos\alpha} \Big/ \prod_{\nu=4}^{n} \tilde{M}_\nu,$$

where $g_2 = g_3 = \frac{1}{2}$, $d_\nu = 1/\nu$, $|f_1 - f_2| = |a_1 a_2|/|1+a_2|$, $\alpha = \frac{1}{2}\arg z = \frac{1}{2}\tan^{-1}(-0.05) \in (0, \pi/2)$ and

$$(9.9) \qquad \tilde{M}_{\nu+1} = 1 + \frac{1 - k_{\nu+1} + 1/\nu}{4|a_{\nu+1}|}\cos^2\alpha \quad \text{where } k_{\nu+1} \le 1.$$

Inserting this into (9.8) and replacing $k_{\nu+1}$ by 1 then gives

$$(9.10) \quad |f - S_n(w_n)| \le 7.51948 \cdot 10^9 \Big/ \prod_{\nu=3}^{n-1} \left(1 + \frac{6.2383056 \cdot 10^{-4}}{4\nu|a_{\nu+1}|}\right)$$

for $z = -2 + 0.1i$, $a = \frac{1}{2}$, $n \geq 4$.

A second expression can be obtained from (3.8), using Remark 3.1.3 to estimate $\tilde{H}_3$.

9.3. Error bounds for the modification (9.4). We want to involve the oval sequence theorem, so we want to find radii R_n such that $a_n \in E_n$, at least from some n on. If $a_n \in P_{\alpha,n}$ with $g_{n-1} = g_n = \frac{1}{2}$ for $n \geq N+2$, it seems reasonable to try $R_n := |1/2 + w_n|$. Then $-\frac{1}{2}$ is a boundary point of V_n, and thus $M_k = 1$ in the oval sequence theorem. The bound (3.12) then takes the form

$$\begin{aligned}|f - S_n(w_n)| \leq \frac{4N^2}{\cos^3 \alpha} &\Big\{ \cos \alpha + 2(N+1)|a_{N+1}| \Big\} \\ &\cdot \frac{|a_{N+1}| \cdot |f_{N-1} - f_N| R_n}{(|1 + w_{N+1}| - R_{N+1})|1 + w_n|}\end{aligned}$$

for $n \geq N+1$ if this R_n works for $n \geq N+1$. However, we can do much better.

In the following we let $a = \frac{1}{2}$. Then the modification (9.4) has coefficients $c_0, c_1, \ldots, c_{12}$ as given in part 9.1. Moreover, $a_n - \widehat{a}_n^{(N)} = \mathcal{O}(n^{-(N+1)/2})$ for $\widehat{a}_n^{(N)} := \widehat{w}_{n-1}^{(N)}(1 + \widehat{w}_n^{(N)})$, whereas

$$\Delta_n^{(N)} := |1 + \widehat{w}_n^{(N)}| - |\widehat{w}_{n-1}^{(N)}| = \mathcal{O}(n^{-1/2}).$$

Hence, by Lemma 5.1 we expect that some $R_n = \mathcal{O}(n^{-N/2})$ will work if $N \geq 2$.

First let $z = 1$. For $N = 3$ we have

$$\widehat{w}_{n+1}^{(3)} = -\frac{1}{2} + \frac{1}{2\sqrt{n}} - \frac{1}{8n} - \frac{1}{4n^{3/2}}. \tag{9.11}$$

It is straightforward to prove that $0 > \widehat{w}_n^{(3)} > \widehat{w}_{n+1}^{(3)} > -\frac{1}{2}$, $0 < a_{n+1} - \widehat{a}_{n+1}^{(3)} < a_n - \widehat{a}_n^{(3)}$ and

$$R_n := \frac{2|a_n - \widehat{a}_n^{(3)}|}{\Delta_n^{(3)}} \leq \frac{1}{2}\, \Delta_n^{(3)}$$

for $n \geq 4$. Hence it follows from Lemma 5.1 that $a_n \in E_n$ for $n \geq 5$ with this radius R_n. Hence it follows from (3.12) with $N = 3$ that

$$\begin{aligned} |f - S_n(\widehat{w}_n^{(3)})| &\leq 16(1+6|a_3|)|f_1 - f_2| \frac{|a_3|}{|1+\widehat{w}_3^{(3)}| - R_3} \\ &\cdot \frac{R_n}{|1+\widehat{w}_n^{(3)}|} \prod_{k=3}^{n-1} M_k^{(3)} \end{aligned} \tag{9.12}$$

where we have used $R_3 := R_4$, $d_\nu = 1/\nu$ and $g_\nu = \frac{1}{2}$. Now, $|1 + \widehat{w}_n^{(3)}| > \frac{1}{2}$, $|f_1 - f_2| = |a_1 a_2|/|1 + a_2|$, and by Remark 2.2.3 we have

$$M_k^{(3)} = \frac{|\widehat{w}_k^{(3)}| + R_k}{1 + \widehat{w}_k^{(3)} - R_k}$$

since $-\frac{1}{2} < \widehat{w}_k^{(3)} < 0$. Hence

$$|f - S_n(\widehat{w}_n^{(3)})| \leq 0.438219\, R_n \prod_{k=3}^{n-1} \frac{-\widehat{w}_k^{(3)} + R_k}{1 + \widehat{w}_k^{(3)} - R_k} \tag{9.13}$$

for $z = 1$, $a = \frac{1}{2}$ and $\widehat{w}_n^{(3)}$ given by (9.11). Here $M_k^{(3)} < 1$, but $M_k^{(3)} \to 1$. Hence we also have the simpler bound

$$|f - S_n(\widehat{w}_n^{(3)})| \leq 0.438219\, R_n = 0.856438\, \frac{|a_n - \widehat{a}_n^{(3)}|}{\Delta_n^{(3)}} \tag{9.14}$$

for $n \geq 4$ for this situation. The table below shows the effect of the various bounds for $z = 1$.

n	$\lvert f - S_n(\widehat{w}_n^{(3)})\rvert$	(9.7)	(9.13)	(9.14)
3	$2.5 \cdot 10^{-7}$	$1.9 \cdot 10^{-2}$	$1.1 \cdot 10^{-1}$	$1.1 \cdot 10^{-1}$
6	$1.9 \cdot 10^{-8}$	$4.6 \cdot 10^{-3}$	$1.7 \cdot 10^{-2}$	$2.5 \cdot 10^{-2}$
9	$1.3 \cdot 10^{-9}$	$2.0 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$	$1.2 \cdot 10^{-2}$
12	$1.3 \cdot 10^{-10}$	$1.1 \cdot 10^{-3}$	$3.0 \cdot 10^{-4}$	$7.1 \cdot 10^{-3}$
15	$1.8 \cdot 10^{-11}$	$7.1 \cdot 10^{-4}$	$5.6 \cdot 10^{-5}$	$4.9 \cdot 10^{-3}$
30	$7.8 \cdot 10^{-15}$	$1.7 \cdot 10^{-4}$	$6.1 \cdot 10^{-7}$	$1.6 \cdot 10^{-3}$

The situation gets more complicated when z is close to the boundary of Z_3 for this continued fraction. For $z = -2 + 0.1i$ the quantity $\Delta_n^{(N)}$ is negative for $n \leq 52$, even for large values of N.

10. Example 4: The error function. The continued fraction in (1.4) converges to the complementary error function $\operatorname{erfc}(z)$ for $z \in Z_4$, where Z_4 is the open right half plane where $\operatorname{Re} z > 0$. Since $a_n = \mathcal{O}(n)$ as $n \to \infty$, we expect that the continued fraction converges slowly, in particular for z close to the boundary of Z_4.

10.1. Choice of w_n. The fixed point modification does not make sense in this case, but the square root modification still works. It gives

$$w_n = \frac{q_n - 1}{2} \quad \text{where } q_n := \sqrt{1 + 2n/z^2}\,,\ \operatorname{Re} q_n > 0. \tag{10.1}$$

The improvement machine gives $t = 1$, so also this time

$$w_n^{(1)} = \frac{q_n(q_n - 1)}{q_n + q_{n+1}} \tag{10.2}$$

is a useful choice which can be improved by applying the machine repeatedly.

As in the previous example, we shall use the asymptotic expansion in $\sqrt{n}$ instead of q_n. This leads to modifications of the form

$$\widehat{w}_n^{(N)} := \frac{\sqrt{n}}{\sqrt{2}\,z} + \sum_{j=0}^{N} c_j n^{-j/2} \quad \text{for } n \geq 1. \tag{10.3}$$

Since $a_{n+1} = n/2z^2$ for $n \geq 1$, the first coefficients c_j in (10.3) are given by

$$c_{-1} = \frac{1}{\sqrt{2}\,z}, \quad c_0 = -\frac{1}{2}, \quad c_1 = \frac{-1+z^2}{4\sqrt{2}\,z}, \quad c_2 = \frac{1}{8}, \quad c_3 = \frac{1+2z^2-z^4}{32\sqrt{2}\,z},$$

$$c_4 = \frac{1-z^2}{16}, \quad c_5 = \frac{5 - 13z^2 - 3z^4 + z^6}{128\sqrt{2}\,z}, \quad c_6 = \frac{-5 - 8z^2 + 4z^4}{128},$$

$$c_7 = \frac{-21-300z^2+230z^4+20z^6-5z^8}{2048\sqrt{2}\,z}, \quad c_8 = \frac{-23+30z^2+12z^4-4z^6}{256},$$

$$c_9 = \frac{-399 + 1215z^2 + 1750z^4 - 770z^6 - 35z^8 + 7z^{10}}{8192\sqrt{2}\,z},$$
$$c_{10} = \frac{53 + 304z^2 - 180z^4 - 32z^6 + 8z^8}{1024},$$
$$c_{11} = \frac{869 + 34806z^2 - 29387z^4 - 14700z^6 + 4515z^8 + 126z^{10} - 21z^{12}}{65536\sqrt{2}\,z},$$
$$c_{12} = \frac{1186 - 1625z^2 - 2120z^4 + 800z^6 + 80z^8 - 16z^{10}}{4096},$$
$$c_{13} = \frac{39325 - 122101z^2 - 440605z^4 + 207823z^6 + 51975z^8}{262144\sqrt{2}\,z}$$
$$\frac{-12243z^{10} - 231z^{12} + 33z^{14}}{262144\sqrt{2}\,z},$$
$$c_{14} = \frac{-5165 - 64344z^2 39420z^4 + 21760z^6 - 6000z^8 - 384z^{10} + 64z^{12}}{32768},$$
$$c_{15} = \frac{-334477 - 26968760z^2 + 23047356z^4 + 27283256z^6 - 8847982z^8}{8388608\sqrt{2}\,z}$$
$$\frac{-1321320z^{10} + 252252z^{12} + 3432z^{14} - 429z^{16}}{8388608\sqrt{2}\,z}.$$

The tables below illustrate the effect of the various modifications. We have chosen the two values $z = 1$ which is well inside Z_4, and $z = 0.1+2i$ which is closer to the boundary of Z_4. Obviously there is a lot to be gained by these modifications.

$z = 1$. $\operatorname{erfc}(1) = 0.1394027926403309882496163$

n	$S_n(0)$	(10.1)
4	0.135534	0.13954
5	0.141492	0.13934
24	0.139401389	0.139402800
25	0.139403851	0.139402786
50	0.139402789	0.139402792630
51	0.139402795	0.139402792649
$m(25)$	434	238

n	(10.2)	(10.3) with $N = 15$
4	0.1394066	0.13940266
5	0.1394011	0.139402803
24	0.13940279273	0.1394027926403309769
25	0.13940279257	0.1394027926403309942
50	0.13940279264038	26 true decimals
51	0.13940279264028	26 true decimals
$m(25)$	250	72

z=0.1+2i.

$\mathrm{erfc}(z) = -4.4118706347832286456999 40-15.380492381244562690780755 49i$

n	$S_n(0)$	(10.1)
3	$-5.13593 - 15.30575i$	$-4.2140653562 - 15.3224376370i$
10	$-4.84716 - 15.81604i$	$-4.4109453127 - 15.3667487640i$
100	$-4.51276 - 15.38294i$	$-4.4117336325 - 15.3804692715i$
500	$-4.41408 - 15.37818i$	$-4.4118700388 - 15.3804929285i$
1000	$-4.41164 - 15.38044i$	$-4.4118706622 - 15.3804923874i$
$m(5)$	2210	369

n	(10.2)
3	$-4.4084646709 - 15.4441032288i$
10	$-4.4113209516 - 15.3800344500i$
100	$-4.4118701012 - 15.3804924209i$
997	$-4.4118706337 - 15.3804923818i$
998	$-4.4118706343 - 15.3804923817i$
$m(5)$	58

n	(10.3) with $N = 12$
3	$-4.3629068370677 \quad - 15.38226877i$
10	$-4.4118705713853718 - 15.3804917778090479i$
100	$-4.41187063478322852 - 15.380492381244562617i$
997	> 26 true decimals
998	> 26 true decimals
$m(5)$	9

10.2. Error bounds for $S_n(0)$. The continued fraction (1.4) for $\operatorname{erfc}(z)$ is a Stieltjes fraction. Since $a_{n+1} = n/2z^2$, we choose $\alpha = -\arg z \in (-\pi/2, \pi/2)$ in Thron's parabola sequence theorem. The bound (4.1) then takes the form

$$\begin{aligned} |f - S_n(w_n)| &\le \frac{|e^{-z^2}/z|}{2\cos\alpha} \Big/ \prod_{\nu=1}^{n-1} \left(1 + \frac{\cos^2\alpha}{|a_{\nu+1}|}\right) \\ &= \frac{|e^{-z^2}|}{2\operatorname{Re} z} \Big/ \prod_{\nu=1}^{n-1} \left(1 + \frac{2(\operatorname{Re} z)^2}{\nu}\right) \end{aligned} \tag{10.4}$$

for $\operatorname{Re}(w_n z/|z|) \ge 0$. In particular this gives

(10.5)

$$|f - S_n(w_n)| \le \left\{ 2e \prod_{\nu=1}^{n-1} \left(1 + \frac{2}{\nu}\right) \right\}^{-1} \quad \text{for } z = 1,\ \operatorname{Re} w_n \ge 0,\ n \ge 2,$$

and for $z = 0.1 + 2i$ and $\operatorname{Re}\Big(w_n \frac{z}{|z|}\Big) \ge 0$ we get

$$|f - S_n(w_n)| \le 54.055 \Big/ \prod_{\nu=1}^{n-1} \left(1 + \frac{0.02}{\nu}\right) \quad \text{for } n \ge 2. \tag{10.6}$$

As a comparison, the Gragg-Warner bound (4.3) gives

$$|f - S_n(0)| \le \frac{|e^{-z^2}|}{\operatorname{Re} z} \prod_{\nu=1}^{n-1} \frac{\sqrt{1 + 2\nu/(\operatorname{Re} z)^2} - 1}{\sqrt{1 + 2\nu/(\operatorname{Re} z)^2} + 1} \quad \text{for } n \ge 2, \tag{10.7}$$

which means that

$$|f - S_n(0)| \le \frac{1}{e} \prod_{\nu=1}^{n-1} \frac{\sqrt{1+2\nu}-1}{\sqrt{1+2\nu}+1} \quad \text{for } z=1,\ n \ge 2, \tag{10.8}$$

and

(10.9)

$$|f - S_n(0)| \le 108.11 \prod_{\nu=1}^{n-1} \frac{\sqrt{1+200\nu}-1}{\sqrt{1+200\nu}+1} \quad \text{for } z=0.1+2i,\ n \ge 2.$$

In the table below we compare these bounds to the actual truncation errors.

n	$z=1$ $\lvert f-S_n(0)\rvert$	(10.5)	(10.8)	$z=0.1+2i$ $\lvert f-S_n(0)\rvert$	(10.6)	(10.9)
10	$1.7\cdot10^{-4}$	$3.3\cdot10^{-3}$	$5.8\cdot10^{-4}$	$6.2\cdot10^{-1}$	51.1	55.6
100	$8.4\cdot10^{-13}$	$3.6\cdot10^{-5}$	$2.1\cdot10^{-11}$	$1.0\cdot10^{-1}$	48.8	7.9
500	$5.8\cdot10^{-27}$	$1.5\cdot10^{-6}$	$1.3\cdot10^{-26}$	$3.2\cdot10^{-3}$	47.2	2.4
1000	$< 10^{-35}$	$3.7\cdot10^{-7}$	$5.6\cdot10^{-38}$	$2.3\cdot10^{-4}$	46.6	$1.7\cdot10^{-2}$

10.3. Error bounds for (10.3). The first question is how many terms do we want to use in the expression (10.3) for w_n. The more terms we choose, the better effect we have from the modification, and the easier it normally is to find suitable radii R_n.

Let first $z=1$. Then we expect everything to go through smoothly, so we start with $N=3$. Then

$$\widehat{w}_n^{(3)} = \sqrt{\frac{n}{2} - \frac{1}{2} + \frac{1}{8n} + \frac{\sqrt{2}}{32n^{3/2}}} < \frac{\sqrt{n}}{2} - \frac{7}{16} \quad \text{for } n \ge 1,$$

$$\Delta_n^{(3)} = 1 + \frac{\sqrt{n}-\sqrt{n-1}}{2} - \frac{1}{8n(n-1)} - \frac{\sqrt{2}}{32} \cdot \left(\frac{1}{(n-1)^{3/2}} - \frac{1}{n^{3/2}} \right) > 1 \quad \text{for } n \ge 2.$$

Moreover, $|a_n - \widehat{a}_n^{(3)}|/\Delta_n^{(3)} \to 0$ monotonically and

$$\frac{2|a_n - \widehat{a}_n^{(3)}|}{\Delta_n^{(3)}} \le 2|a_n - \widehat{a}_n^{(3)}| < \frac{1}{2}\Delta_n^{(3)} \quad \text{for } n \ge 2.$$

Hence, by Lemma 5.1 we may use $R_n := 2|a_n - \widehat{a}_n^{(3)}|$ for $n \geq 2$ in (3.5), where

$$H_2 = \frac{1 + \widehat{w}_2^{(3)} + R_2}{1 + a_2 + \widehat{w}_2^{(3)} + R_2}$$

by Remark 3.1.2. This gives

(10.10)

$$\begin{aligned}|f - S_n(\widehat{w}_n^{(3)})| &\leq \frac{(1 + \widehat{w}_2^{(3)} + R_2)^2}{(\frac{3}{2} + \widehat{w}_2^{(3)} + R_2)^2} \cdot \frac{1/(2e)}{|1 + \widehat{w}_2^{(3)}| - 2|a_2 - \widehat{a}_2^{(3)}|} \\ &\quad \cdot \frac{|a_n - \widehat{a}_n^{(3)}|}{|1 + \widehat{w}_n^{(3)}|} \prod_{\nu=2}^{n-1} \frac{\widehat{w}_\nu^{(3)} + 2|a_\nu - \widehat{a}_\nu^{(3)}|}{1 + \widehat{w}_\nu^{(3)} + 2|a_\nu - \widehat{a}_\nu^{(3)}|} \\ &\leq 0.5937 \cdot \frac{|a_n - \widehat{a}_n^{(3)}|}{\sqrt{n/2} + 1/2} \prod_{\nu=2}^{n-1} \left(1 - \frac{1}{\sqrt{\frac{\nu}{2}} + \frac{9}{16} + \frac{2}{10}}\right)\end{aligned}$$

for $n \geq 2$ and $\operatorname{Re} w_n \geq 0$.

11. Concluding remarks. Which approximants to use and which error bounds to use, depends to a large degree on the situation.

1. The better bounds one requires, the more computation it usually takes to find them. To make a catalogue over truncation error bounds for a given function, one may put in a considerable amount of work to find good bounds. On the other hand, if the idea is to program a computer to compute the function values to any accuracy required by the user, it is a good idea to have bounds that are easy and fast to compute by the machine.

2. It is important to find bounds that are valid for all, or almost all, z of interest for a given function.

3. We want the computation to be fast. Evidently it is hard to accelerate a continued fraction which already converges fast. But it is easy, and much to be gained, by modifying the approximants of slowly converging continued fractions. In view of point 2 above, we should therefore use modifications also where we have fast convergence, if it can be done with little extra work.

4. The computation of the truncation error bounds does not need high precision. Some of the bounds contain for instance square roots.

They can be simplified by the simple observation that $1-x < \sqrt{1-x} < 1-x/2$ and $1 < \sqrt{1+x} < 1+x/2$ for $0 < x < 1$.

5. The computation of the modification w_n can generally be done with less accuracy than wanted for $S_n(w_n)$. The value w_n is just an approximation to $f^{(n)}$ anyway.

6. The fixed point modification can be seen as a way to make the approximants $S_n(w)$ of $K(a_n/1)$ of type 1 to behave more like the classical approximants of continued fractions of type 2.

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