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SYMM'S LOG KERNEL INTEGRAL OPERATORS

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ABSTRACT. The Bochner integral is applied to prove the compactness of Symm's log kernel integral operators on $\mathcal{L}^1, \mathcal{L}^\infty$ and weighted $\mathcal{L}^p$ spaces when $1 < p < \infty$. Moreover, the ranges of these operators on weighted $\mathcal{L}^p$ spaces are determined, and this is applied to solve singular integral equations.

1. Introduction. The integral equation

$$\pi^{-1} \int_{-1}^{1} f(s) \ln |t-s| \, ds = g(t), \quad t \in]-1,1[, \tag{1.1}$$

for a given function g is called *Symm's integral equation* by I.H. Sloan and E.P. Stephan [**15**], who named it after G.T. Symm [**16**]. This integral equation arises in many areas in analysis and has applications to potential theory and scattering theory (see [**4, 16**], for example, and the references therein). In [**4**, Section 3], the integral equation (1.1) is solved when g is suitably smooth. In the present paper the equation (1.1) is considered from the viewpoint of operator theory, as in [**7**, Section 13].

To be more precise, let λ denote the Lebesgue measure in the open interval $]-1,1[$. Symm's log kernel integral operator L_p on the Banach space $\mathcal{L}^p(\lambda)$ is defined by

$$L_p(f)(t) = \pi^{-1} \int_{-1}^{1} f(s) \ln |t-s| \, ds, \quad t \in]-1,1[, \tag{1.2}$$

for each $f \in \mathcal{L}^p(\lambda)$ whenever $1 \le p \le \infty$.

According to [**7**, Section 13], the operator L_p is compact, whenever $1 < p < \infty$, the proof of which is based upon the Hille-Tamarkin theorem. Furthermore, the range of L_p in the case in which $1 < p < 2$ has been described there, but note that the description in [**7**, Section

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13] of the range of L_p for $2 < p < \infty$ is not correct. The main aim of this paper is to extend these results to weighted $\mathcal{L}^p$-spaces; the correct description of the range of L_p, $2 < p < \infty$, will be given also (see Proposition 5.2). Symm's integral equations in special weighted $\mathcal{L}^p$-spaces have been solved in terms of Chebyshev-Fourier series; see [**1**] and [**15**], for example.

In Section 3 operators $L_{p,\rho}$ are introduced which have the same form as L_p (see (1.2)) and are defined on the space $\mathcal{L}^p(\lambda)$ with weight ρ, when $1 < p < \infty$. In the case where $\rho = (1-\mathbf{x})^\alpha(1+\mathbf{x})^\beta$ with $\alpha, \beta \in]-1, p-1[$, it is shown that $L_{p,\rho}$ is compact (see Proposition 3.4). Since the Hille-Tamarkin theorem is not applicable to weighted $\mathcal{L}^p$-spaces in an obvious way, we shall instead use Bochner integrals as in [**3**, Chapter III]. Our method applies even to the case when $p = 1$ (see Proposition 3.5). The operator L_∞ will be shown to be the transpose operator of L_1 so that it also is compact (see Proposition 5.4).

Section 4 collects results on finite Hilbert transforms for use in Section 5.

In Section 5 we shall determine the range of $L_{p,\rho}$ (see Proposition 5.2). As an application, a singular integral equation more general than (1.1) will be solved in a simpler manner than that done originally by M. Schleiff [**13**]; see (5.12).

2. Preliminaries.

Let Ω denote the open interval $]-1,1[$. The Lebesgue measure in Ω is denoted by λ; the domain of λ is the σ-algebra $\mathcal{S}$ of Lebesgue measurable subsets of Ω. The identity function on Ω is denoted by $\mathbf{x}$; that is, $\mathbf{x}(t) = t$ for each $t \in \Omega$. We denote the constant function one by $\mathbf{1}$.

Let $1 \le p < \infty$. Let α and β be real numbers. Define $\rho = (1-\mathbf{x})^\alpha(1+\mathbf{x})^\beta$. Let $\mathcal{L}^p(\rho\lambda)$ denote the space of all Lebesgue measurable functions f on Ω with values in the complex plane $\mathbf{C}$ such that $|f|^p\rho$ is λ-integrable, and equip $\mathcal{L}^p(\rho\lambda)$ with the seminorm $|\cdot|_{p,\rho}$ given by

$$|f|_{p,\rho} = \left(\int_\Omega |f|^p \rho \, d\lambda\right)^{1/p}, \quad f \in \mathcal{L}^p(\rho\lambda).$$

In other words, $\mathcal{L}^p(\rho\lambda)$ is the usual $\mathcal{L}^p$-space with respect to the measure $\rho\lambda$ given by $(\rho\lambda)(E) = \int_E \rho \, d\lambda$ for each $E \in \mathcal{S}$. The seminormed space $\mathcal{L}^p(\rho\lambda)$ is complete. To make the presentation simpler, we shall identify $\mathcal{L}^p(\rho\lambda)$ with its quotient space with respect to the closed subspace of λ-null functions. So $\mathcal{L}^p(\rho\lambda)$ is then regarded as a Banach space with norm $|\cdot|_{p,\rho}$. When the weight function ρ is $\mathbf{1}$, we write $\mathcal{L}^p(\mathbf{1}\lambda)$ simply as $\mathcal{L}^p(\lambda)$ and $|\cdot|_{p,\mathbf{1}}$ as $|\cdot|_p$.

Let $\mathcal{L}^\infty(\lambda)$ denote the space of $\mathbf{C}$-valued, λ-measurable, λ-essentially bounded functions on Ω. As above, $\mathcal{L}^\infty(\lambda)$ will be regarded as a Banach space with λ-essential supremum norm.

Lemma 2.1. *Let $1 < p < \infty$, and let $\rho = (1-\mathbf{x})^\alpha(1+\mathbf{x})^\beta$ with real numbers α and β.*

(i) *$\mathcal{L}^\infty(\lambda) \subset \mathcal{L}^p(\rho\lambda)$ if and only if $\alpha > -1$ and $\beta > -1$.*

(ii) *$\mathcal{L}^p(\rho\lambda) \subset \mathcal{L}^r(\lambda)$ for some $r \in]1,p]$ if and only if $\alpha < p-1$ and $\beta < p-1$.*

Proof. Statement (i) is straightforward. Statement (ii) follows from the generalized Hölder inequality. Indeed, if either $\alpha > 0$ or $\beta > 0$, then $r < p(1+\max\{\alpha,\beta\})^{-1}$, and if not then we can take $r = p$. □

From now on, we always assume that $\alpha, \beta \in]-1, p-1[$ when $1 < p < \infty$. This condition guarantees also that the finite Hilbert transform on $\mathcal{L}^p(\rho\lambda)$ is continuous (see Lemma 4.2).

The dual space of the Banach space $\mathcal{L}^p(\rho\lambda)$ is identified with $\mathcal{L}^q((1/\rho^{q/p})\lambda)$ with $q = p/(p-1)$ so that the duality is given by

$$\int_\Omega gf\,d\lambda = \int_\Omega (g/\rho^{1/p})(f\rho^{1/p})\,d\lambda$$

for each $g \in \mathcal{L}^q((1/\rho^{q/p})\lambda)$ and $f \in \mathcal{L}^p(\rho\lambda)$.

Let X be a complex Banach space with norm $|\cdot|$ and X' its dual space. The bilinear form associated with the duality between X and X' is denoted by $\langle\cdot,\cdot\rangle$, that is, $\langle x', x\rangle = x'(x)$ for each $x' \in X'$ and $x \in X$. Let $S : X \to X$ be a continuous linear operator and define its transpose ${}^tS : X' \to X'$ by

$$\langle {}^tSx', x\rangle = \langle x', Sx\rangle, \quad x' \in X',\ x \in X$$

(cf. [**18**, Chapter 18]). The null space and range of S are defined by

$$\mathcal{N}(S) = S^{-1}(\{0\}) \quad\text{and}\quad \mathcal{R}(S) = S(X),$$

respectively. The operator S is called a *Noether* or *Fredholm* operator if it has closed range and if both $\mathcal{N}(S)$ and $\mathcal{N}({}^tS)$ are finite-dimensional. The index $\kappa(S)$ of such an operator S is defined by

$$\kappa(S) = \dim \mathcal{N}(S) - \dim \mathcal{N}({}^tS).$$

In this case $\mathcal{N}({}^tS) = \dim X/\mathcal{R}(S)$ (cf. [**9**, p. 1]).

A function from Ω into X is called *strongly λ-measurable* if it is the limit of a sequence of X-valued $\mathcal{S}$-simple functions on Ω. A strongly λ-measurable function $\Phi : \Omega \to X$ is said to be *Bochner λ-integrable* if the scalar function $|\Phi| : t \mapsto |\Phi(t)|$, $t \in \Omega$, is λ-integrable.

Suppose that $\Phi : \Omega \to X$ is Bochner λ-integrable. Then, given a set $E \in \mathcal{S}$, there is a unique vector $x_E \in X$ such that

$$\langle x', x_E \rangle = \int_E \langle x', \Phi(t) \rangle \, dt, \quad x' \in X'.$$

The vector

$$\int_E \Phi \, d\lambda = x_E$$

is called the Bochner λ-integral of Φ over E. Further properties of Bochner integrals can be found in [**3**, Chapter II], for example.

3. Compactness. Let λ be the Lebesgue measure on the σ-algebra of Lebesgue measurable subsets of the open interval $\Omega = \,]-1, 1[$.

Let $f \in \mathcal{L}^1(\lambda)$. It follows from the Fubini theorem that the integral

$$(L_1 f)(t) = \pi^{-1} \int_{-1}^{1} f(s) \ln |t - s| \, ds$$

exists λ-almost every $t \in \Omega$, and the resulting function $L_1 f$ belongs to $\mathcal{L}^1(\lambda)$. Moreover, the so-defined linear operator $L_1 : f \mapsto L_1 f$, $f \in \mathcal{L}^1(\lambda)$, is a continuous linear operator on the Banach space $\mathcal{L}^1(\lambda)$.

The following example shows that there are functions $f \in \mathcal{L}^1(\lambda)$, which are defined everywhere in Ω such that $(L_1 f)(t)$ does not exist for some point $t \in \Omega$.

Example 3.1. Let $f(t) = t^{-1}(\ln t)^{-2}$ for each $t \in\,]0, 1/2[$ and $f(t) = 0$ for each $t \in \Omega\backslash\,]0, 1/2[$. The so-defined function f belongs to $\mathcal{L}^1(\lambda)$ but $L_1(f)(0) = -\infty$ and

$$\lim_{n\to\infty}(L_1 f)(-n^{-1}) = -\infty. \tag{3.1}$$

In fact, (3.1) is a consequence of the Fatou lemma (cf. [**6**, (12.23)]) as follows:

$$\begin{aligned}\limsup_{n\to\infty} L_1(f)(-n^{-1}) &= \liminf_{n\to\infty}(-L_1 f)(-n^{-1})\\ &\le -\pi^{-1}\int_{-1}^{1}\liminf_{n\to\infty}(-f(s)\ln|-n^{-1}-s|)\,ds\\ &= \pi^{-1}\int_0^{1/2} f(s)\ln s\,ds\\ &= \int_0^{1/2}(s\ln s)^{-1}\,ds = -\infty.\end{aligned}$$

Note that

$$-(L_1 f)(-n^{-1}) = O(\ln\ln n) \tag{3.2}$$

for a large $n \in \mathbf{N}$, which has been proved by D. Elliott (personal communication).

According to K. Jörgens [**7**, Example 11.2], we have $L_1(\mathcal{L}^p(\lambda)) \subset \mathcal{L}^p(\lambda)$ and the restriction L_p of L_1 to $\mathcal{L}^p(\lambda)$ is a compact operator whenever $1 < p < \infty$. However, his proof does not seem to apply to the case when $p = 1$ or that of weighted $\mathcal{L}^p$-spaces, $1 < p < \infty$. So we shall present a totally different proof which also covers those cases, by using Bochner integrals.

Given $t \in \Omega$, define a function $G(t) : \Omega\backslash\{t\} \to \mathbf{C}$ by

$$G(t)(s) = \pi^{-1}\ln|t-s|, \quad s \in \Omega\backslash\{t\}.$$

Lemma 3.2. *Let* $1 < p < \infty$. *Let* $\rho = (1-\mathbf{x})^\alpha(1+\mathbf{x})^\beta$ *with* $\alpha, \beta \in\,]-1, p-1[$.

(i) *Given* $t \in \Omega$, *the function* $G(t)$ *belongs to the space* $\mathcal{L}^p(\rho\lambda)$.

(ii) *The so-defined vector-valued function* $G : \Omega \to \mathcal{L}^p(\rho\lambda)$ *is continuous and has relatively compact range.*

(iii) *Let* $f \in \mathcal{L}^p(\rho\lambda)$. *Then the function* $fG : t \mapsto f(t)G \in \mathcal{L}^p(\rho\lambda)$, $t \in \Omega$, *is Bochner* λ*-integrable and*

$$\int_\Omega fG\,d\lambda = L_1 f \quad (\textit{as elements of } \mathcal{L}^p(\rho\lambda)). \tag{3.3}$$

Proof. Let $r \in \,]1,\infty[$ be a number such that $\alpha, \beta \in \,]-r^{-1},\infty[$, in which case $\rho \in \mathcal{L}^r(\lambda)$. Let $r' = r/(r-1)$. Let $\bar{\Omega}$ denote the closed interval $[-1,1]$. Given $t \in \bar{\Omega}$, by the Hölder inequality

(3.4)
$$\begin{aligned}\left(\int_{-1}^{1} |\ln|t-s||^p \rho(s)\,ds\right)^{1/p} &\le \left[\left(\int_{-1}^{1} |\ln|t-s||^{pr'}\,ds\right)^{1/r'} |\rho|_r\right]^{1/p} \\ &\le \left(\int_{-2}^{2} |\ln|s||^{pr'}\,ds\right)^{1/(pr')} |\rho|_r^{1/p} < \infty.\end{aligned}$$

(i) Apply (3.4) to each $t \in \Omega$.

(ii) By (3.4) we can extend G to $\bar{\Omega}$ and shall denote the extension by $\bar{G}$. Statement (ii) will follow at once if we establish that $\bar{\Omega} \to \bar{G} : \mathcal{L}^p(\rho\lambda)$ is continuous because $\bar{\Omega}$ is compact. To this end, fix a point $t_0 \in \Omega$ and a positive number ε. Choose a $\delta > 0$ such that

$$\left(\int_{-2\delta}^{2\delta} |\ln|s||^{pr'}\,ds\right)^{1/(pr')} < \varepsilon.$$

Let χ_A denote the characteristic function of the set $A = \{s \in \Omega : |s - t_0| < \delta\}$. It then follows from the above inequity that

$$|\chi_A \bar{G}(t)|_{pr'} < \varepsilon \tag{3.5}$$

for every $t \in \bar{\Omega}$ satisfying $|t - t_0| < \delta/2$. In particular,

$$|\chi_A \bar{G}(t_0)|_{pr'} < \varepsilon. \tag{3.6}$$

Moreover,

$$|\ln|s-t|-\ln|s-t_0|| \leq 2\max\{\ln 2, |\ln\delta|, |\ln(\delta/2)|\}$$

whenever $|t-t_0| < \delta/2$ and $s \in \Omega\backslash A$. The Lebesgue dominated convergence theorem implies that there exists a $\delta_0 \in]0,\delta/2[$ such that

$$\left(\int_{\Omega\backslash A} |\ln|s-t|-\ln|s-t_0||^{pr'}\,ds\right)^{1/(pr')} < \varepsilon \tag{3.7}$$

for every $t \in \bar{\Omega}$ satisfying $|t-t_0| < \delta_0$. By the Hölder inequality together with (3.5), (3.6) and (3.7), we have

$$\begin{aligned}|\bar{G}(t)-\bar{G}(t_0)|_{p,\rho} &\leq |\rho|_r^{1/p}|\bar{G}(t)-\bar{G}(t_0)|_{pr'}\\ &\leq |\rho|_r^{1/p}(|\chi_A\bar{G}(t)|_{pr'}+|\chi_A\bar{G}(t_0)|_{pr'}\\ &\qquad + |\chi_{\Omega\backslash A}(\bar{G}(t)-\bar{G}(t_0)|_{pr'})\\ &< 3|\rho|_r^{1/p}\varepsilon,\end{aligned}$$

which implies that $\bar{G}$ is continuous at t_0.

(iii) Since $\mathcal{L}^p(\rho\lambda)$ is separable the continuous function $\bar{G}$ is strongly λ-measurable (see [**17**, pp. 67–68], for example). Let $K = \sup\{|G(t)|_{p,\rho} : t \in \Omega\}$ which is finite by (ii). If $q = p/(p-1)$, then

$$\begin{aligned}\int_\Omega |f(t)G(t)|_{p,\rho}\,dt &\leq K\int_\Omega |f(t)|\,dt\\ &\leq K|f|_{p,\rho}\cdot|\rho^{-1/p}|_q < \infty\end{aligned} \tag{3.8}$$

and hence fG is Bochner λ-integrable. Furthermore,

$$\left|\int_\Omega fG\,d\lambda\right|_{p,\rho} \leq \int_\Omega |fG|_{p,\rho}\,d\lambda \tag{3.9}$$

(cf. [**3**, Theorem II.2.4]).

In view of Lemma 2.1 (ii), let $J : \mathcal{L}^p(\rho\lambda) \to \mathcal{L}^1(\lambda)$ denote the natural injection. Since the measure λ is finite, we have

$$J\left(\int_\Omega fG\,d\lambda\right) = \int_\Omega J\circ(fG)\,d\lambda$$

(cf. [**3**, Theorem II.2.6]). To show (3.3), let $h \in \mathcal{L}^\infty(\lambda)$. Then

$$\Big\langle h, J\Big(\int_\Omega fG\,d\lambda\Big)\Big\rangle = \langle h, L_1 f\rangle,$$

which follows from the Fubini theorem. Thus (3.3) holds because J is injective. □

By Lemma 3.2 (iii), we have $L_1(\mathcal{L}^p(\rho\lambda)) \subset \mathcal{L}^p(\rho\lambda)$. So let

$$L_{p,\rho} : \mathcal{L}^p(\rho\lambda) \longrightarrow \mathcal{L}^p(\rho\lambda) \tag{3.10}$$

denote the restriction of L_1 to $\mathcal{L}^p(\rho\lambda)$. If $\rho = \mathbf{1}$, then we write $L_{p,\mathbf{1}}$ as L_p.

Corollary 3.3. *The linear operator $L_{p,\rho}$ is continuous.*

Proof. This follows from (3.8) and (3.9) applied to each $f \in \mathcal{L}^p(\rho\lambda)$. □

Proposition 3.4. *Let p, ρ and G be as in Lemma* 3.2*. Then the linear operator $L_{p,\rho} : \mathcal{L}^p(\rho\lambda) \to \mathcal{L}^p(\rho\lambda)$ is compact.*

Proof. Since the range of the function G is relatively compact in $\mathcal{L}^p(\rho\lambda)$, there exist Borel simple functions $G_n : \Omega \to \mathcal{L}^p(\rho\lambda)$, $n \in \mathbf{N}$, such that

$$\lim_{n\to\infty} \sup_{t\in\Omega} |G(t) - G_n(t)|_{p,\rho} = 0.$$

Let $n = 1, 2, \ldots$. The linear operator $K_n : \mathcal{L}^p(\rho\lambda) \to \mathcal{L}^p(\rho\lambda)$ defined by

$$K_n f = \int_\Omega fG_n\,d\lambda, \quad f \in \mathcal{L}^p(\rho\lambda),$$

is compact because its range is finite-dimensional. Let $q = p/(p-1)$. The Hölder inequality implies that

$$|(L_{p,\rho} - K_n)f|_{p,\rho} \le \sup_{t\in\Omega} |G(t) - G_n(t)|_{p,\rho} |f|_{p,\rho} |\rho^{-1/p}|_q.$$

Consequently, the operator $L_{p,\rho}$, which is the limit of compact operators K_n, $n \in \mathbf{N}$, in the operator norm, is compact. $\square$

Proposition 3.5. *The operator* $L_1 : \mathcal{L}^1(G) \to \mathcal{L}^1(G)$ *is compact.*

Proof. Let p, ρ and G be as in Lemma 3.2 and $J : \mathcal{L}^p(\rho\lambda) \to \mathcal{L}^1(\lambda)$ the natural injection. Then $J \circ G : \Omega \to \mathcal{L}^1(\lambda)$ is strongly λ-measurable, and a routine computation shows that, given $f \in \mathcal{L}^1(\lambda)$, the function $f(J \circ G) : \Omega \to \mathcal{L}^1(\lambda)$ is Bochner λ-integrable such that $L_1 f = \int_\Omega f(J \circ G)\, d\lambda$. In the terminology of [**3**, Chapter III], the operator L_1 is Bochner representable. Since $J \circ G$ has relatively compact range, it follows from [**3**, Theorem III.2.2] that L_1 is compact. $\square$

We can prove the compactness of L_1 also by taking the Borel simple functions G_n, $n \in \mathbf{N}$, as in the proof of Proposition 3.4 because $J(G_n(t)) \to J(G(t))$ in $\mathcal{L}^1(\lambda)$ as $n \to \infty$ uniformly with respect to $t \in \Omega$.

We shall show in Proposition 5.4 that Symm's log kernel integral operator on $\mathcal{L}^\infty(\lambda)$ is compact.

4. Finite Hilbert transforms. Throughout this section let λ denote the Lebesgue measure in the open interval $\Omega =]-1, 1[$.

Let $f \in \mathcal{L}^1(\lambda)$. The Cauchy principal value

$$(Tf)(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \left[\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1} \right] \frac{f(s)}{s-t}\, ds, \tag{4.1}$$

exists for λ-almost every $t \in \Omega$, and the resulting function Tf defined λ-almost everywhere in Ω is λ-measurable; see [**2**, Theorem 8.1.6], for example. We call Tf the *finite Hilbert transform* of f.

If f is the function defined in Example 3.1, then Tf does not belong to $\mathcal{L}^1(\lambda)$; see [**8**, Proof of Theorem 1(b)].

Let w be the function on Ω defined by

$$w = (1 - \mathbf{x}^2)^{1/2}. \tag{4.2}$$

Lemma 4.1. *Let $1 < p < \infty$. Then the following statements hold on a function $f \in \mathcal{L}^p(\lambda)$.*

(i) *$Tf = 0$ if and only if $f = C/w$ for some $C \in \mathbf{C}$ (with $C = 0$ when $p \geq 2$).*

(ii) *$(1/w)T(wf) \in \mathcal{L}^r(\lambda)$ whenever $1 < r < \min\{2, p\}$ and*

$$\int_{-1}^{1} (1/w)T(wf)\, d\lambda = 0.$$

(iii) *$T((1/w)T(wf)) = -f$.*

Proof. Statement (i) can be found in [**19**, p. 176] and (ii) in [**9**, Theorem II.4.4], for example. For (iii), see [**10**, Proof of Proposition 2.4]. □

It would be interesting to know whether or not Lemma 4.1 can be extended to the case when $p = 1$.

Let $1 < p < \infty$. It follows from the M. Riesz theorem (cf. [**2**, Proposition 8.1.9]) that $T(\mathcal{L}^p(\lambda)) \subset \mathcal{L}^p(\lambda)$.

Let $\rho = (1 - \mathbf{x})^\alpha (1 + \mathbf{x})^\beta$ with $\alpha, \beta \in \,]-1, p-1[$. Then

$$T(\mathcal{L}^p(\rho\lambda)) \subset \mathcal{L}^p(\rho\lambda) \tag{4.3}$$

and the restriction

$$T_{p,\rho} : \mathcal{L}^p(\rho\lambda) \longrightarrow \mathcal{L}^p(\rho\lambda) \tag{4.4}$$

of T to the weighted space $\mathcal{L}^p(\rho\lambda)$ is continuous. This is a result by B.V. Khevedelidze and can be found, for example, in [**9**, Theorem II.3.7]. Further results are given in the following lemma. The linear span of a vector a is denoted by span $\{a\}$.

Lemma 4.2. *Let $1 < p < \infty$. Let $\rho = (1 - \mathbf{x})^\alpha (1 + \mathbf{x})^\beta$ with $\alpha, \beta \in \,]-1, p-1[$. Then the following statements hold.*

(i) *Let $q = p/(p-1)$. Then the Parseval identity*

$$\int_\Omega (gTf + fTg)\, d\lambda = 0$$

holds for each $f \in \mathcal{L}^p(\rho\lambda)$ *and* $g \in \mathcal{L}^q((1/\rho^{q/p})\lambda)$. *In other words, if* $\rho^* = 1/\rho^{q/p}$, *then the identification* $(\mathcal{L}^p(\rho\lambda))' = \mathcal{L}^q(\rho^*\lambda)$, *cf. Section* 2, *gives*

$$ {}^tT_{p,\rho} = -T_{q,\rho^*}. \tag{4.5} $$

(ii) $\mathcal{N}(T_{p,\rho}) = \mathcal{L}^p(\rho\lambda) \cap \operatorname{span}\{1/w\}$.

(iii) $\dim \mathcal{N}(T_{p,\rho}) = 1$ *if and only if both* α *and* β *belong to* $]p/2-1, p-1[$.

(iv) *The operator* $T_{p,\rho}$ *is injective if and only if either* α *or* β *belongs to* $]-1, p/2-1]$.

(v) $\dim \mathcal{N}({}^tT_{p,\rho}) = 1$ *if and only if both* α *and* β *belong to* $]-1, p/2-1[$.

(vi) *The transpose* ${}^tT_{p,\rho}$ *is injective if and only if either* α *or* β *belongs to* $[p/2-1, p-1[$.

Proof. Statement (i) has been given in [**9**, Theorem I.4.2], for instance. Statement (ii) follows from Lemma 2.1(ii) and Lemma 4.1(i). Statements (iii) and (iv) are immediate consequences of (ii) which also implies (v) and (vi) because of (4.5). □

Let T_p denote $T_{p,\mathbf{1}}$ whenever $1 < p < \infty$.

When $1 < p < 2$, the operator T_p is a Noether operator with $\kappa(T_p) = 1$ (see [**7**, Section 13] or [**10**, Proposition 2.4], for example). The following proposition generalizes this.

Proposition 4.3. *Let* $1 < p < \infty$. *Let* $\rho = (1-\mathbf{x})^\alpha(1+\mathbf{x})^\beta$ *with* $\alpha, \beta \in]p/2-1, p-1[$. *Then*

(i) $\mathcal{N}(T_{p,\rho}) = \operatorname{span}\{1/w\}$ *and* $\mathcal{N}({}^tT_{p,\rho}) = \{0\}$; *and*

(ii) $T_{p,\rho}$ *is a surjective Noether operator, with* $\kappa(T_{p,\rho}) = 1$, *satisfying that*

$$ T_{p,\rho}^{-1}(\{f\}) = -(1/w)T(wf) + \operatorname{span}\{1/w\}, \quad f \in \mathcal{L}^p(\rho\lambda). $$

Proof. (i) See Lemma 4.2(i), (ii).

(ii) Let $f \in \mathcal{L}^p(\rho\lambda)$. Since $wf \in \mathcal{L}^p((\rho/w^p)\lambda)$, it follows from (4.3) that $T(wf) \in \mathcal{L}^p((\rho/w^p)\lambda)$, and hence $(1/w)T(wf) \in \mathcal{L}^p(\rho\lambda)$. Now (ii) is a consequence of (i) and Lemma 4.1(iii). □

The following result has been given in [**19**, p. 179].

Lemma 4.4. *If $f : \Omega \to \mathbf{C}$ is a λ-measurable function such that $f/w \in \mathcal{L}^1(\lambda)$, then*

$$wT(f/w) = (1/w)\left[T(wf) + \pi^{-1}\mathbf{x}\int_\Omega \frac{f}{w}\,d\lambda + \pi^{-1}\int_\Omega \frac{\mathbf{x}f}{w}\,d\lambda\right]$$

holds λ-almost everywhere.

Proposition 4.5. *Let $1 < p < \infty$. Let $\rho = (1-\mathbf{x})^\alpha(1+\mathbf{x})^\beta$ with $\alpha, \beta \in \,]-1, p/2-1[$. Then the following statements hold.*

(i) $\mathcal{N}(T_{p,\rho}) = \{0\}$ *and* $\mathcal{N}({}^tT_{p,\rho}) = \operatorname{span}\{1/w\}$.

(ii) *The range of the linear operator $T_{p,\rho} : \mathcal{L}^p(\rho\lambda) \to \mathcal{L}^p(\rho\lambda)$ is given by*

$$\mathcal{R}(T_{p,\rho}) = \left\{f \in \mathcal{L}^p(\rho\lambda) : \int_\Omega \frac{f}{w}\,d\lambda = 0\right\}. \tag{4.6}$$

Moreover, given $f \in \mathcal{R}(L_{p,\rho})$,

$$T_{p,\rho}^{-1}(f) = -wT(f/w) = -(1/w)\left[T(wf) + \pi^{-1}\int_\Omega \frac{\mathbf{x}f}{w}\,d\lambda\right]. \tag{4.7}$$

Hence, $T_{p,\rho}$ is a Noether operator with $\kappa(T_{p,\rho}) = -1$.

Proof. (i) See Lemma 4.2(i), (ii).

(ii) Let $f \in \mathcal{R}(T_{p,\rho})$ and $g = T_{p,\rho}^{-1}(f)$. It then follows from Lemmas 4.1(i) and 4.2(i) that

$$\int_\Omega \frac{f}{w}\,d\lambda = \int_\Omega \frac{Tg}{w}\,d\lambda = -\int_\Omega gT(1/w)\,d\lambda = 0.$$

Conversely, let f be an arbitrary element of the righthand side of (4.6). Then $wT(f/w) \in \mathcal{L}^p(\rho\lambda)$, which follows from (4.3). Lemma 4.4

implies the second identity in (4.7) so that $f = T(-wT(f/w))$ by Lemma 4.1(iii). Therefore, $f \in \mathcal{R}(T_{p,\rho})$ and $T_{p,\rho}^{-1}(f) = -wT(f/w)$. $\square$

A special case of the above proposition is when $2 < p < \infty$ and $\rho = \mathbf{1}$, which has been presented, for example, in [**7**, Section 13] and [**10**, Proposition 2.6].

Let $\mu = (1-\mathbf{x})^{1/2}(1+\mathbf{x})^{-1/2}$. We shall generalize M. Schleiff's result [**14**, p. 149] which asserts that $T_{2,\mu} : \mathcal{L}^2(\mu\lambda) \to \mathcal{L}^2(\mu\lambda)$ is bijective.

Proposition 4.6. *Let* $1 < p < \infty$. *Let* $\rho = (1-\mathbf{x})^\alpha(1+\mathbf{x})^\beta$ *with* α, β *satisfying* $\alpha \in \,]p/2-1, p-1[$ *and* $\beta \in \,]-1, p/2-1[$. *The operator* $T_{p,\rho} : \mathcal{L}^p(\rho\lambda) \to \mathcal{L}^p(\rho\lambda)$ *is bijective and*

$$\begin{aligned} T_{p,\rho}^{-1}(f) &= -(1/\mu)T(\mu f) \\ &= -(1/w)\left[T(wf) - \pi^{-1}\int_\Omega f\mu\, d\lambda\right], \quad f \in \mathcal{L}^p(\rho\lambda). \end{aligned} \tag{4.8}$$

Proof. By Lemma 4.2(iv), the operator $T_{p,\rho}$ is injective. To show the surjectivity of $T_{p,\rho}$, let $f \in \mathcal{L}^p(\rho\lambda)$. The second identity in (4.8) is straightforward. That $(1/\mu)T(\mu f) \in \mathcal{L}^p(\rho\lambda)$ follows from (4.3). Therefore, $T(-(1/\mu)T(\mu f)) = f$ by Lemma 4.1(iii), which establishes (4.8). Hence, $T_{p,\rho}$ is surjective. $\square$

The proof of the following proposition, which is similar to that of Proposition 4.6, will be omitted.

Proposition 4.7. *Let* $1 < p < \infty$. *Let* $\rho = (1-\mathbf{x})^\alpha(1+\mathbf{x})^\beta$ *with* α, β *satisfying* $\alpha \in \,]-1, p/2-1[$ *and* $\beta \in \,]p/2-1, p-1[$. *The operator* $T_{p,\rho} : \mathcal{L}^p(\rho\lambda) \to \mathcal{L}^p(\rho\lambda)$ *is a bijection and*

$$T_{p,\rho}^{-1}(f) = -\mu T(f/\mu) = -(1/w)\left[T(fw) + \pi^{-1}\int_\Omega \frac{f}{\mu}\, d\lambda\right], \quad f \in \mathcal{L}^p(\rho\lambda).$$

If either $\alpha = p/2-1$ or $\beta = p/2-1$, then it is known that $T_{p,\rho}$ is not a Noether operator (cf. [**5**, Theorem 9.5.3]), and accordingly its range is not so well characterized as in the cases discussed in Propositions 4.3, 4.5, 4.6 and 4.7. However, we have the following result.

Proposition 4.8. *Let* $1 < p < \infty$. *Let* $\rho = (1-\mathbf{x})^{\alpha}(1+\mathbf{x})^{\beta}$ *such that either* $\alpha = p/2 - 1$ *or* $\beta = p/2 - 1$.

(i) *The linear operator* $T_{p,\rho} : \mathcal{L}^p(\rho\lambda) \to \mathcal{L}^p(\rho\lambda)$ *is injective and its range is a proper dense subspace of the Banach space* $\mathcal{L}^p(\rho\lambda)$ *so that*

$$\mathcal{N}(T_{p,\rho}) = \{0\} \text{ and } \mathcal{N}({}^tT_{p,\rho}) = \{0\}. \tag{4.9}$$

(ii) *There is a linear functional* $\Lambda : \mathcal{R}(T_{p,\rho}) \to \mathbf{C}$ *such that*

$$T_{p,\rho}^{-1}(f) = -(1/w)T(wf) + \Lambda(f)/w, \quad f \in \mathcal{R}(L_{p,\rho}).$$

(iii) *If either* $\alpha = p/2 - 1$ *and* $\beta \in]-1, p/2-1]$ *or* $\alpha \in]-1, p/2-1]$ *and* $\beta = p/2 - 1$, *then the constant function* $\mathbf{1}$ *does not belong to* $\mathcal{R}(T_{p,\rho})$.

(iv) *If either* $\alpha = p/2 - 1$ *and* $\beta \in]p/2-1, p-1[$ *or* $\alpha \in]p/2-1, p-1[$ *and* $\beta = p/2 - 1$, *then* $\mathcal{R}(T_{p,\rho})$ *contains all polynomial functions on* Ω.

Proof. (i) We have (4.9) by Lemma 4.2(iv), (vi). Consequently, $\mathcal{R}(T_{p,\rho})$ is dense in $\mathcal{L}^p(\rho\lambda)$. Since $T_{p,\rho}$ is not a Noether operator (cf. [**5**, Theorem 9.5.3]), it follows that $\mathcal{R}(T_{p,\rho}) \neq \mathcal{L}^p(\rho\lambda)$.

(ii) By Lemma 2.1(ii) we have $\mathcal{L}^p(\rho\lambda) \subset \mathcal{L}^r(\lambda)$ for some $r \in]1, 2[$. Since $T_{p,\rho}$ is injective, statement (ii) follows from Proposition 4.3(ii) applied to T_r.

(iii) The function $\mathbf{1}$ does not belong to $\mathcal{R}(T_{p,\rho})$ because $T(\mathbf{x}/w) = \mathbf{1}$ and $(\mathbf{x} - C)/w \notin \mathcal{L}^p(\rho\lambda)$ for any constant C.

(iv) Suppose that $\alpha = p/2 - 1$ and $\beta \in]p/2-1, p-1[$. Let $n = 0, 1, 2, \ldots$. Define Chebyshev polynomial functions of first and second kind by

$$\mathbf{T}_n(\cos\xi) = \cos n\xi \quad \text{and} \quad \mathbf{U}_n(\cos\xi) = \frac{\sin(n\xi + \xi)}{\sin\xi}$$

for each $\xi \in]0, \pi[$, respectively. There is a constant C such that the factor $1 - t$ divides $\mathbf{T}_n(t) - C$ so that

$$\frac{\mathbf{T}_n(t) - C}{w(t)} = \frac{\mathbf{T}_n(t) - C}{1 - t}\left(\frac{1-t}{1+t}\right)^{1/2}, \quad t \in \Omega.$$

Then $(\mathbf{T}_n - C)/w \in \mathcal{L}^p(\rho\lambda)$ because $(\mathbf{T}_n - C)/(1 - \mathbf{x}) \in \mathcal{L}^\infty(\lambda)$ and $(1 - \mathbf{x})^{1/2}(1 + \mathbf{x})^{-1/2} \in \mathcal{L}^p(\rho\lambda)$. Hence $\mathbf{U}_n \in \mathcal{R}(T_{p,\rho})$, which follows from Lemma 4.1(i) and the fact that $T(\mathbf{T}_{n+1}/w) = \mathbf{U}_n$ (cf. [**19**, p. 180]).

The remaining case when $\alpha \in]p/2 - 1, p - 1[$ and $\beta = p/2 - 1$ can be handled similarly. □

Remark 4.9. If $p = 2$ and $\rho = \mathbf{1}$, then the linear functional Λ in Proposition 4.8 has been obtained explicitly in [**10**, Theorem 3.2 and Corollary 3.3].

5. Range of Symm's log kernel integral operator. We now come to the main aim of this paper which is to determine the range of Symm's log kernel integral operator $L_{p,\rho}$. To be precise, let λ be the Lebesgue measure in the open interval $\Omega =]-1, 1[$. Let $1 < p < \infty$ and ρ the weight function $(1-\mathbf{x})^\alpha(1+\mathbf{x})^\beta$ with $\alpha, \beta \in]-1, p-1[$. Let $L_{p,\rho} : \mathcal{L}^p(\rho\lambda) \to \mathcal{L}^p(\rho\lambda)$ be the linear operator defined by

$$(5.1) \qquad (L_{p,\rho} f)(t) = \pi^{-1} \int_\Omega f(s) \ln|s - t|\, ds, \quad t \in \Omega,$$

for each $f \in \mathcal{L}^p(\rho\lambda)$. Let $W^{(1)}_{p,\rho}(\Omega)$ denote the domain of the differentiation operator $D_{p,\rho} : f \mapsto f'$ in the Banach space $\mathcal{L}^p(\rho\lambda)$. Recall that $T_{p,\rho} : \mathcal{L}^p(\rho\lambda) \to \mathcal{L}^p(\rho\lambda)$ is the operator defined by finite Hilbert transforms (see (4.4)). We then have

$$(5.2) \qquad \mathcal{R}(L_{p,\rho}) \subset W^{(1)}_{p,\rho}(\Omega)$$

and

$$(5.3) \qquad D_{p,\rho} \circ L_{p,\rho} = -T_{p,\rho}.$$

These results when $\rho = \mathbf{1}$ have been given in [**7**, Section 13]. The general case can be proved similarly by using the fact that the operator $D_{p,\rho}$ is closed.

When we do not need to emphasize the fact that $D_{p,\rho}(f)$ belongs to $\mathcal{L}^p(\rho\lambda)$ for a function $f \in W^{(1)}_{p,\rho}(\Omega)$, we may write $D_{p,\rho}(f)$ simply as Df.

The domain of a linear operator S is denoted by $\mathcal{D}(S)$.

Lemma 5.1. *Let $1 < p < \infty$. Let $\rho = (1-\mathbf{x})^\alpha(1+\mathbf{x})^\beta$ such that $\alpha, \beta \in \,]-1, p-1[$. Then the operator $L_{p,\rho} : \mathcal{L}^p(\rho\lambda) \to \mathcal{L}^p(\rho\lambda)$ is a continuous linear injection satisfying* (5.2) *and* (5.3) *such that*

$$L^{-1}_{p,\rho}(g) = \frac{1}{w}\left[T(wDg) - \pi^{-1}(\ln 2)^{-1}\int_\Omega \frac{g}{w}\,d\lambda\right], \qquad g \in \mathcal{R}(L_{p,\rho}). \tag{5.4}$$

Moreover, if $T_{p,\rho}$ is injective, i.e., either α or β belongs to $]-1, p/2-1]$ by Lemma 4.2, then the operator $-T^{-1}_{p,\rho} \circ D_{p,\rho}$ is a proper extension of $L^{-1}_{p,\rho}$; in other words,

$$L^{-1}_{p,\rho} \subsetneqq -T^{-1}_{p,\rho} \circ D_{p,\rho}. \tag{5.5}$$

Proof. By (5.3) and Lemma 4.2(ii), we have

$$\mathcal{N}(L_{p,\rho}) \subset \mathcal{N}(T_{p,\rho}) = \mathcal{L}^p(\rho\lambda) \cap \operatorname{span}\{1/w\}.$$

Therefore, $L_{p,\rho}$ is injective because

$$-(\ln 2)^{-1} L_1(1/w) = \mathbf{1}, \tag{5.6}$$

cf. [**11**, Corollary, p. 138]. The inverse formula (5.4) for $L_{p,\rho}$ is due to [**4**, Section 3].

Finally, suppose that $T_{p,\rho}$ is injective. It then follows from (5.3) that

$$(-T^{-1}_{p,\rho} \circ D_{p,\rho}) \circ L_{p,\rho} = T^{-1}_{p,\rho} \circ T_{p,\rho}.$$

Since $1/w \notin \mathcal{L}^p(\rho\lambda)$, it follows from (5.6) that

$$\mathbf{1} \in \mathcal{D}(T_{p,\rho}^{-1} \circ D_{p,\rho}) \backslash \mathcal{D}(L_{p,\rho}^{-1});$$

that is, (5.5) holds. □

Consider the special case in which $2 < p < \infty$ and $\rho = \mathbf{1}$. By the above lemma, the range $\mathcal{R}(L_{p,\mathbf{1}})$ of the operator $L_{p,\mathbf{1}} : \mathcal{L}^p(\lambda) \to \mathcal{L}^p(\lambda)$ is strictly smaller than the domain $\mathcal{D}(T_{p,\mathbf{1}}^{-1} \circ D_{p,\mathbf{1}})$ of $T_{p,\mathbf{1}}^{-1} \circ D_{p,\mathbf{1}}$. This corrects the claim in [**7**, Section 13] that $\mathcal{R}(L_{p,\mathbf{1}}) = \mathcal{D}(T_{p,\mathbf{1}}^{-1} \circ D_{p,\mathbf{1}})$.

The following proposition determines the range of the operator $L_{p,\rho} : \mathcal{L}^p(\rho\lambda) \to \mathcal{L}^p(\rho\lambda)$.

Proposition 5.2. *Let $1 < p < \infty$, and let $\rho = (1-\mathbf{x})^\alpha(1+\mathbf{x})^\beta$ with $\alpha, \beta \in]-1, p-1[$. Then the following statements hold.*

(i) *If $\alpha, \beta \in]p/2-1, p-1[$, then $\mathcal{R}(L_{p,\rho}) = W_{p,\rho}^{(1)}(\Omega)$.*

(ii) *If $\alpha, \beta \in]-1, p/2-1[$, then $\mathcal{R}(L_{p,\rho})$ consists of those functions $g \in W_{p,\rho}^{(1)}(\Omega)$ such that*

$$\int_\Omega \frac{Dg}{w}\, d\lambda = 0 \quad \text{and} \quad \int_\Omega \frac{1}{w}[\mathbf{x}Dg + (\ln 2)^{-1} g]\, d\lambda. \tag{5.7}$$

(iii) *Suppose that $\alpha \in]p/2-1, p-1[$ and $\beta \in]-1, p/2-1[$. Let $\mu = (1-\mathbf{x})^{1/2}(1+\mathbf{x})^{-1/2}$. Then*

$$\mathcal{R}(L_{p,\rho}) = \left\{ g \in W_{p,\rho}^{(1)}(\Omega) : \int_\Omega \left(\mu Dg - (\ln 2)^{-1} \frac{g}{w} \right) d\lambda = 0 \right\}.$$

(iv) *Suppose that $\alpha \in]-1, p/2-1[$ and $\beta \in]p/2-1, p-1[$. Then*

$$\mathcal{R}(L_{p,\rho}) = \left\{ g \in W_{p,\rho}^{(1)}(\Omega) : \int_\Omega \left(\frac{Dg}{\mu} + (\ln 2)^{-1} \frac{g}{w} \right) d\lambda = 0 \right\}.$$

(v) *If either $\alpha = p/2-1$ or $\beta = p/2-1$, then $\mathcal{R}(L_{p,\rho})$ consists of those functions $g \in W_{p,\rho}^{(1)}(\Omega)$ such that $Dg \in \mathcal{R}(T_{p,\rho})$, (i.e., $g \in \mathcal{D}(T_{p,\rho}^{-1} \circ D_{p,\rho})$), and*

$$\Lambda(Dg) = \pi^{-1}(\ln 2)^{-1} \int_\Omega \frac{g}{w}\, d\lambda, \tag{5.8}$$

where $\Lambda : \mathcal{R}(T_{p,\rho}) \to \mathbf{C}$ *is the linear functional given in Proposition* 4.8(ii).

Proof. (i) Let $g \in W^{(1)}_{p,\rho}(\Omega)$. Then (5.3) implies that

$$L^{-1}_{p,\rho}(g + \mathcal{N}(D_{p,\rho})) = -T^{-1}_{p,\rho}(\{D_{p,\rho}g\}) \neq \varnothing \tag{5.9}$$

because $T_{p,\rho}$ is surjective by Proposition 4.3. Since $1/w \in \mathcal{L}^p(\rho\lambda)$, we have $\mathbf{1} \in \mathcal{R}(L_{p,\rho})$ by (5.6). Consequently,

$$L^{-1}_{p,\rho}(\mathcal{N}(D_{p,\rho})) = \operatorname{span}\{L^{-1}_{p,\rho}(\mathbf{1})\} = \operatorname{span}\{1/w\}. \tag{5.10}$$

Now (i) follows from (5.2), (5.9) and (5.10).

(ii) Let $g \in W^{(1)}_{p,\rho}(\Omega)$. It follows from Lemma 4.4 that

$$\begin{aligned} \frac{1}{w}\left[T(wDg) - \pi^{-1}(\ln 2)^{-1}\int_\Omega \frac{g}{w}\,d\lambda\right] &= wT\left(\frac{Dg}{w}\right) - \pi^{-1}\frac{\mathbf{x}}{w}\int_\Omega \frac{Dg}{w}\,d\lambda \\ &\quad - \pi^{-1}\frac{\mathbf{1}}{w}\left(\int_\Omega \frac{1}{w}[\mathbf{x}Dg + (\ln 2)^{-1}g]\,d\lambda\right) \end{aligned} \tag{5.11}$$

holds λ-almost everywhere. Lemma 5.1 implies that $g \in \mathcal{R}(L_{p,\rho})$ if and only if the lefthand side of (5.11) belongs to $\mathcal{L}^p(\rho\lambda)$. This is equivalent to (5.7) because $wT((Dg)/w) \in \mathcal{L}^p(\rho\lambda)$ by (4.3) and the linearly independent functions $\mathbf{x}/w$ and $\mathbf{1}/w$ are not members of $\mathcal{L}^p(\rho\lambda)$.

(iii) Let $g \in W^{(1)}_{p,\rho}(\Omega)$. It follows from Proposition 4.6 that the identity

$$\begin{aligned} \frac{1}{w}\left[T(wDg) - \pi^{-1}(\ln 2)^{-1}\int_\Omega \frac{g}{w}\,d\lambda\right] \\ = \frac{1}{\mu}T(\mu Dg) + \pi^{-1}\frac{1}{w}\int_\Omega\left(\mu Dg - (\ln 2)^{-1}\frac{g}{w}\right)d\lambda \end{aligned}$$

holds λ-almost everywhere, and hence statement (iii) holds because $1/w \notin \mathcal{L}^p(\rho\lambda)$ and $(1/\mu)T(\mu Dg) \in \mathcal{L}^p(\rho\lambda)$.

(iv) Adapt the proof of (iii) by applying Proposition 4.7 instead of Proposition 4.6.

(v) Proposition 4.8(i) asserts the injectivity of $T_{p,\rho}$. Let $g \in \mathcal{D}(T_{p,\rho}^{-1} \circ D_{p,\rho})$. By Proposition 4.8(ii) it follows that

$$\frac{1}{w}\left[T(wDg) - \pi^{-1}(\ln 2)^{-1}\int_\Omega \frac{g}{w}\,d\lambda\right] = -T_{p,\rho}^{-1}(Dg) + \frac{1}{w}\left[\Lambda(Dg) - \pi^{-1}(\ln 2)^{-1}\int_\Omega \frac{g}{w}\,d\lambda\right].$$

Thus, $g \in \mathcal{R}(L_{p,\rho})$ if and only if (5.8) holds, because $1/w \notin \mathcal{L}^p(\rho\lambda)$; Lemma 5.1 has been used again. □

Finally, let us apply our results to singular integral equations. Fix a function $m \in \mathcal{L}^2(w\lambda)$. Given a function $h \in \mathcal{L}^2(w\lambda)$, consider the singular integral equation

$$-T_{2,w}f + mL_{2,w}f = h, \quad f \in \mathcal{L}^2(w\lambda). \tag{5.12}$$

By (5.2) we have $mL_{2,w}f \in \mathcal{L}^2(w\lambda)$ for each $f \in \mathcal{L}^2(w\lambda)$. The resulting linear operator $mL_{2,w}$ on $\mathcal{L}^2(w\lambda)$ is continuous by the closed graph theorem.

M. Schleiff [**14**] has given a method of finding solutions f of (5.12). However, his method is rather complicated, and we shall solve (5.12) in a simpler manner by applying the earlier results of this section.

Rewrite (5.12) as

$$(D_{2,w} + mI_{2,w}) \circ L_{2,w}(f) = h, \quad f \in \mathcal{L}^2(w\lambda),$$

where $I_{2,w}$ is the identity operator on $\mathcal{L}^2(w\lambda)$. Let V be the Volterra operator on $\mathcal{L}^2(w\lambda)$ given by

$$(Vf)(t) = \int_{-1}^{t} f\,d\lambda, \quad t \in \Omega,$$

for each $f \in \mathcal{L}^2(w\lambda)$; then $\mathcal{R}(V) \subset \mathcal{D}(D_{2,w})$. Let a be the function given by

$$a(t) = \exp[-(Vm)(t)], \quad t \in \Omega.$$

The differential operator $D_{2,w} + mI_{2,w}$ in $\mathcal{L}^2(w\lambda)$ is surjective so that

$$(D_{2,w} + mI_{2,w})^{-1}(\{h\}) = aV(h/a) + \operatorname{span}\{a\} \tag{5.13}$$

because

$$\mathcal{N}(D_{2,w} + mI_{2,w}) = \operatorname{span}\{a\}. \tag{5.14}$$

Proposition 5.3. *Let $m \in \mathcal{L}^2(w\lambda)$. Then the linear operator*

$$-T_{2,w} + mL_{2,w} = (D_{2,w} + mI_{2,w}) \circ L_{2,w} : \mathcal{L}^2(w\lambda) \longrightarrow \mathcal{L}^2(w\lambda) \tag{5.15}$$

is a continuous surjection such that

$$\mathcal{N}(-T_{2,w} + mL_{2,w}) = \operatorname{span}\{L_{2,w}^{-1}(a)\}. \tag{5.16}$$

Moreover, given $h \in \mathcal{L}^p(w\lambda)$,

$$\begin{aligned}(-T_{2,w} &+ mL_{2,w})^{-1}(\{h\}) \\ &= L_{2,w}^{-1}(aV(h/a)) + \operatorname{span}\{L_{2,w}^{-1}(a)\} \\ &= \frac{1}{w}\left[T(wh - wmaV(h/a)) - \pi^{-1}(\ln 2)^{-1}\int_\Omega \frac{aV(h/a)}{w}\,d\lambda\right] \\ &\quad + \operatorname{span}\left\{\frac{1}{w}\left[T(wma) + \pi^{-1}(\ln 2)^{-1}\int_\Omega \frac{a}{w}\,d\lambda\right]\right\}.\end{aligned}$$

Proof. The operator (5.15) is continuous because so are $T_{2,w}$ and $mL_{2,w}$. It is surjective because $D_{2,w} + mI_{2,w}$ is surjective and

$$\mathcal{R}(L_{2,w}) = \mathcal{D}(D_{2,w} + mI_{2,w}) \quad \text{(see Proposition 5.2(i))}.$$

The identity (5.16) is a consequence of (5.14). The last part of the proposition follows from Lemma 5.1 and (5.13). □

In view of Proposition 5.2(i), the above result can be automatically extended to the case where $1 < p < \infty$ and $\rho = (1 - \mathbf{x})^\alpha (1 + \mathbf{x})^\beta$ with $\alpha, \beta \in \,]p/2 - 1, p - 1[$.

If either α or β belongs to $]-1, p/2 - 1[$, then it is possible to solve the equation

$$(-T_{p,\rho} + mL_{p,\rho})f = h \tag{5.17}$$

when $m, h \in \mathcal{L}^p(\rho\lambda)$ are given, by determining $\mathcal{R}(-T_{p,\rho} + mL_{p,\rho})$.

In the remaining case, when either $\alpha = p/2 - 1$ or $\beta = p/2 - 1$, it seems to be difficult to determine $\mathcal{R}(-T_{p,\rho} + mL_{p,\rho})$ explicitly, because $T_{p,\rho}$ is not a Noether operator. Such difficulty occurs even when $m = 0$, see Proposition 4.8.

Finally, we shall discuss Symm's log kernel integral operator L_∞ on the Banach space $\mathcal{L}^\infty(\lambda)$; this problem was suggested by T. ter Else. We have $L_1(\mathcal{L}^\infty(\lambda)) \subset \mathcal{L}^\infty(\lambda)$ by (5.2). Let L_∞ be the restriction of L_1 to $\mathcal{L}^\infty(\lambda)$.

Proposition 5.4. *The operator* $L_\infty : \mathcal{L}^\infty(\lambda) \to \mathcal{L}^\infty(\lambda)$ *is the transpose of the compact operator* $L_1 : \mathcal{L}^1(\lambda) \to \mathcal{L}^1(\lambda)$ *and is compact.*

Proof. The fact that $L_\infty = {}^tL_1$ is a consequence of the Tonelli theorem. The transpose L_∞ of the compact operator L_1 is again compact by the Browder theorem, cf. [**12**, Theorem VI.12]. □

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