JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 4, Number 2, Spring 1992

ON THE BEHAVIOR AT INFINITY OF SOLUTIONS OF INTEGRAL EQUATIONS ON THE REAL LINE

S.N. CHANDLER-WILDE

ABSTRACT. We consider integral equations of the form $x(s) = y(s) + \int_{-\infty}^{+\infty} k(s,t)x(t)\,dt$ and the behavior of the solution, x, at infinity. In particular, we show that if, for some $q > 1$, $|k(s,t)| = O(|s-t|^{-q})$, uniformly in s and t, as $|s-t| \to \infty$, some other mild conditions on the kernel k are satisfied, and $y(s) = O(s^{-p})$ as $s \to \infty$, with $0 < p < q$, then $x(s) = O(s^{-p})$. Two examples illustrate the application of these results and the extent to which they may be considered sharp. The second of these examples is a boundary integral equation which models two-dimensional harmonic sound propagation in a half-plane with a variable impedance boundary condition.

1. Introduction. We consider integral equations of the form

$$x(s) = y(s) + \int_{-\infty}^{+\infty} k(s,t)x(t)\,dt, \quad s \in \mathbf{R}, \tag{1}$$

with y and k given and x to be determined. We abbreviate (1) as

$$x = y + Kx \tag{2}$$

where the integral operator K is defined by

$$K\psi(s) = \int_{-\infty}^{+\infty} k(s,t)\psi(t)\,dt. \tag{3}$$

Denote the space of Lebesgue integrable functions on $\mathbf{R}$ by $L_1(\mathbf{R})$ and let $C(\overline{\mathbf{R}})$ denote the Banach space of bounded continuous functions on $\mathbf{R}$. Let $k_s(t) = k(s,t)$. Throughout the paper, we assume that $k_s \in L_1(\mathbf{R})$ for $s \in \mathbf{R}$ and that k satisfies the following hypotheses:

A. $\sup_{s\in\mathbf{R}} ||k_s||_1 = \sup_{s\in\mathbf{R}} \int_{-\infty}^{+\infty} |k(s,t)|\,dt < \infty$.

Received by the editors on June 22, 1991.

B. $\Delta(K;h) \to 0$ as $h \to 0$ where $\Delta(K;h) := \sup_{|s_1-s_2|\le h} ||k_{s_1} - k_{s_2}||_1 = \sup_{|s_1-s_2|\le h} \int_{-\infty}^{+\infty} |k(s_1,t) - k(s_2,t)|\,dt$.

These hypotheses ensure that K is a bounded operator on $C(\overline{\mathbf{R}})$, with norm $||K|| = \sup_{s\in\mathbf{R}} ||k_s||_1$, but do not imply that K is compact.

We shall additionally suppose, for most of the paper, that $I - K$ is invertible on $C(\overline{\mathbf{R}})$ so that equation (1) has a unique solution $x \in C(\overline{\mathbf{R}})$ for each $y \in C(\overline{\mathbf{R}})$.

A stronger condition than A is

A′. $|k(s,t)| \le |\kappa(s-t)|$, $s, t \in \mathbf{R}$, where $\kappa \in L_1(\mathbf{R})$ and, for some $q > 1$ and $C > 0$,

$$|\kappa(s)| \le C(1+|s|)^{-q}, \quad |s| \ge 1. \tag{4}$$

Our main concern in this paper will be to investigate the behavior of the solution x at infinity given Assumptions A′ and B. Our main result will be to show that, with these hypotheses, and if $y(s) = O(s^{-p})$ as $s \to \infty$ for some p in the range $0 < p < q$, then

$$x(s) = O(s^{-p}) \quad \text{as} \quad s \to \infty. \tag{5}$$

If $y(s) = O(s^{-q})$ and if some more restrictive conditions on k are satisfied, then $x(s) = O(s^{-q})$. Later, after strengthening Assumption A′ appropriately, we show the same behavior for derivatives of x.

The form of Assumption A′ is convenient for later calculations but implies that $k(s,t)$ is bounded for $|s-t| \ge 1$, which may be unduly restrictive in certain cases. The results we obtain apply, by a simple linear change of variable, to any kernel k satisfying Hypothesis B and a modified Hypothesis A′, with (4) replaced by

$$\kappa(s) = O(s^{-q}), \quad s \to \infty. \tag{6}$$

For, suppose k is such a kernel which satisfies Hypotheses B and A′, but with (4) replaced by (6). Then κ, which satisfies (6), satisfies

$$|\kappa(s)| \le C(1+|s|)^{-q}, \quad |s| \ge c,$$

for some constants C, $c > 0$. Making the change of variable $\tilde{s} = s/c$, $\tilde{t} = t/c$ in (1), we obtain the integral equation

$$\tilde{x}(\tilde{s}) = \tilde{y}(\tilde{s}) + \int_{-\infty}^{+\infty} \tilde{k}(\tilde{s}, \tilde{t})\tilde{x}(\tilde{t})\, d\tilde{t}, \quad \tilde{s} \in \mathbf{R},$$

where $\tilde{x}(\tilde{s}) = x(\tilde{s}c)$, $\tilde{y}(\tilde{s}) = y(\tilde{s}c)$, $\tilde{k}(\tilde{s}, \tilde{t}) = ck(\tilde{s}c, \tilde{t}c)$. This equation is of the form (1) with a kernel $\tilde{k}$ which satisfies Assumptions A$'$ and B.

The class of equations to which the results we obtain apply include the important special cases

$$k(s,t) = \kappa(s-t), \quad s, t \in \mathbf{R}, \tag{7}$$

and

$$k(s,t) = \begin{cases} \kappa(s-t), & s \in \mathbf{R},\ t \geq 0, \\ 0, & s \in \mathbf{R},\ t < 0, \end{cases} \tag{8}$$

where $\kappa \in L_1(\mathbf{R})$ satisfies (4). (Both these examples satisfy Hypotheses A$'$ and B.) In particular, with k defined by (8), equation (1) becomes

$$x(s) = y(s) + \int_0^{\infty} \kappa(s-t)x(t)\, dt, \quad s \in \mathbf{R},$$

an equation of Wiener-Hopf type.

For the particular case of the equation of Wiener-Hopf type, the behavior at infinity of the solution x, given assumptions similar to A$'$ and B, has been investigated previously in [**20**] and [**21**]. In particular, making the assumption

$$\int_{-\infty}^{+\infty} (1 + |t|)^q |k(t)|\, dt < \infty, \tag{9}$$

(generally much more restrictive than (4); for example, $\kappa(t) := (1 + |t|)^{-p}$ satisfies (4) for $p \geq q$ but (9) only for $p > q + 1$) we have, from Theorem 1.1 in [**21**], combined with classical results on the spectra of Wiener-Hopf operators in $C(\overline{\mathbf{R}})$ [**16**], that, if $y(s) = O(s^{-q})$ as $s \to \infty$ and $I - K$ is invertible, then x has the asymptotic behavior given by (5). Silberman [**21**] also states related results in weighted L_p spaces.

The equation of Wiener-Hopf type is amenable to a treatment by Fourier transform methods and the resolvent kernel can be constructed explicitly; these special features are used in the arguments in [**20**]. We will obtain stronger and more general results by more elementary arguments.

A related problem to that considered in this paper is to determine the behavior of x given an exponential decay in the kernel k, i.e., with (4) replaced by

$$|\kappa(s)| \leq Ce^{-b|s|}, \quad |s| \geq 1,$$

for some $b > 0$. For the particular case of the equation of Wiener-Hopf type, results are given in [**19, 16, 12, 13**]. The arguments we use below do not, however, appear to extend to the case of exponential decay in k and we do not therefore consider this case further.

Some comment on the usefulness of knowing solution behavior at infinity is appropriate. Atkinson [**4**], considering the solution of integral equations on the half-line (included here as the special case $k(s,t) = 0$, $t < 0$), proposes approximating the solution of (1) by the solution (if it exists), $x_A \in C(\overline{\mathbf{R}})$, of $x_A = y + K^{(A)}x_A$ where $K^{(A)}$ is the finite section version of K defined, for $A > 0$, by

$$K^{(A)}\psi(s) = \int_{-A}^{+A} k(s,t)\psi(t)\,dt, \quad s \in \mathbf{R}.$$

In view of the finite range of integration and since, if Hypotheses A and B are satisfied, $K^{(A)}$ is compact, standard methods of numerical treatment can be applied to this equation [**2, 5, 17**].

From the equation

$$x - x_A = (I - K_A)^{-1}(K - K_A)x,$$

it is easy to see that, if Hypotheses A and B are satisfied and $I - K_A$ is invertible and uniformly bounded for $A \geq A^*$, then

$$||x - x_A||_\infty \leq \sup_{A \geq A^*} ||(I - K_A)^{-1}||\,||K|| \sup_{|s| \geq A} |x(s)|$$

for all $A \geq A^*$. Thus, if the rate of decay at x at infinity is accurately known, then $||x - x_A||_\infty$ can be estimated. In particular, if $x(s) = O(s^{-p})$ as $s \to \infty$, then $||x - x_A||_\infty = O(A^{-p})$ as $A \to +\infty$.

The difficulty in applying the above inequality is showing the uniform boundedness of $(I-K_A)^{-1}$, $A \geq A^*$. Under appropriate conditions on the operators involved, this is demonstrated for the case in which K is a compact perturbation of a Wiener-Hopf operator in [**3**] and for the case $k(s,t) = \kappa(s-t)z(t)$, with $\kappa \in L_1(\mathbf{R})$, $z \in L_\infty(\mathbf{R})$, in [**8**].

Knowledge of the decay of the solution x and its derivatives at infinity is also valuable in constructing appropriate finite dimensional subspaces from which to seek an approximation to x as part of a numerical solution of (1). For example, [**18**] discusses the construction of piecewise polynomial approximations on suitably graded meshes to functions which decay like s^{-p} at infinity. These are then used as the basis for a numerical method for Wiener-Hopf equations in the case when it is known that the decay of the solution and sufficiently many of its derivatives is given by (5).

The paper has two main sections. In the first of these (Section 2) the theoretical results on the behavior of x at infinity are presented. Section 3 illustrates these results by applying them to two examples, the first of which is simple enough for the exact asymptotic behavior of x at infinity to be determined by other means. The second example is a boundary integral equation of application in the prediction of outdoor sound propagation. This equation models two-dimensional acoustic propagation, with a harmonic time dependence, in a half-plane above a straight boundary with an inhomogeneous impedance boundary condition. By applying the results of Section 2, we are able to show, if a conjecture on the uniqueness of solution of the boundary integral equation in $C(\overline{\mathbf{R}})$ is correct, that the acoustic pressure, $p(\xi)$, at distance ξ along the boundary satisfies

$$p(\xi) = O(\xi^{-r}), \quad \xi \to \infty,$$

for all r in the range $0 < r < 3/2$, for a wide class of distributions of surface impedance.

In a brief Section 4, we examine, in the light of the examples in Section 3, whether the results, obtained in Section 2, on the rate of decay of x at infinity are sharp.

2. The rate of decay of the solution at infinity. We introduce first of all some additional function spaces. For $p \geq 0$ define

$$C_p(\overline{\mathbf{R}}) := \{\psi \in C(\overline{\mathbf{R}}) : ||\psi||_\infty^p := ||w_p\psi||_\infty < \infty\} \tag{10}$$

where

$$w_p(s) = (1+|s|)^p, \quad s \in \mathbf{R}.$$

If $\psi \in C_p(\overline{\mathbf{R}})$, then ψ is continuous on $\mathbf{R}$ and $\psi(s) = O(s^{-p})$ as $s \to \infty$.

We shall consider, in what follows, equation (1) directly and will also consider a transformed version of this equation. For $p > 0$, let

$$x_p = w_p x, \qquad y_p = w_p y. \tag{11}$$

Then it is easy to see that equation (1) is satisfied if and only if

$$x_p(s) = y_p(s) + \int_{-\infty}^{+\infty} k^{(p)}(s,t) x_p(t)\, dt, \quad s \in \mathbf{R}, \tag{12}$$

where

$$k^{(p)}(s,t) := \frac{w_p(s)}{w_p(t)} k(s,t), \quad s,t \in \mathbf{R}. \tag{13}$$

We define $k_s^{(p)}$ for $s \in \mathbf{R}$ by $k_s^{(p)}(t) = k^{(p)}(s,t)$ and define the integral operator K_p by equation (3) with K, k replaced by K_p, $k^{(p)}$, respectively.

We note first the following straightforward result which follows from the equivalence of equations (1) and (12) and the observation that, for any $\psi \in C(\overline{\mathbf{R}})$ and $p > 0$,

$$K_p \psi = w_p K(\psi / w_p), \qquad K\psi = (1/w_p) K_p (w_p \psi). \tag{14}$$

Theorem 1. *For all $p > 0$*

i. *K is a bounded operator on $C_p(\overline{\mathbf{R}})$ if and only if K_p is a bounded operator on $C(\overline{\mathbf{R}})$;*

ii. *$I - K$ is bounded and injective on $C_p(\overline{\mathbf{R}})$ if and only if $I - K_p$ is bounded and injective on $C(\overline{\mathbf{R}})$;*

iii. *$I - K$ is bounded and invertible on $C_p(\overline{\mathbf{R}})$ if and only if $I - K_p$ is bounded and invertible on $C(\overline{\mathbf{R}})$.*

Our next result is only slightly less obvious.

Theorem 2. *For $p > 0$, if $I - K$ is injective on $C(\overline{\mathbf{R}})$ and K_p is a bounded operator on $C(\overline{\mathbf{R}})$, then $I - K_p$ is injective on $C(\overline{\mathbf{R}})$.*

Proof. If K_p is bounded on $C(\overline{\mathbf{R}})$, then, from Theorem 1, i, K is bounded on $C_p(\overline{\mathbf{R}})$. If also $I - K$ is injective on $C(\overline{\mathbf{R}})$, then $x = Kx$ has no nontrivial solution in $C(\overline{\mathbf{R}})$ and so no nontrivial solution in $C_p(\overline{\mathbf{R}}) \subset C(\overline{\mathbf{R}})$. Thus, $I - K$ is bounded and injective on $C_p(\overline{\mathbf{R}})$, and so $I - K_p$ is injective on $C(\overline{\mathbf{R}})$ by Theorem 1, ii. $\square$

We need the following technical lemma before proceeding to our next theorem.

Lemma 3. *For $\alpha, \beta \geq 0$, $\alpha + \beta > 1$, define*

$$f_{\alpha\beta}(s) := \int_{-\infty}^{+\infty} (1 + |t|)^{-\alpha}(1 + |s - t|)^{-\beta}\, dt, \quad s \in \mathbf{R}.$$

Then

$$F_{\alpha\beta} := \sup_{s \in \mathbf{R}} f_{\alpha\beta}(s) < \infty,$$

and, if $\alpha, \beta > 0$, $f_{\alpha\beta}(s) \to 0$ as $s \to \infty$.

Proof. The result is obvious in the case $\alpha = 0$ or $\beta = 0$. Moreover, it is clear that $f_{\alpha\beta}(s)$ increases as α or β is decreased. Thus, it is sufficient to consider further only cases in which $0 < \alpha < 1$, $0 < \beta < 1$, and $\alpha + \beta > 1$.

We define

$$g_{\alpha\beta}(s) := \int_{0}^{\infty} (1 + |t|)^{-\alpha}(1 + |s - t|)^{-\beta}\, dt, \quad s \in \mathbf{R},$$

and note that $f_{\alpha\beta}(s) = g_{\alpha\beta}(s) + g_{\alpha\beta}(-s)$ so that it is sufficient to show that $g_{\alpha\beta}$ is bounded and $g_{\alpha\beta}(s) \to 0$ as $s \to \infty$.

For $s \leq 0$,

$$g_{\alpha\beta}(s) \leq \int_{0}^{\infty} G_s(t)(1 + t)^{-(\alpha+\beta)}\, dt$$

where

$$G_s(t) = \left\{\frac{1 + t}{1 - s + t}\right\}^{\beta}, \quad t \geq 0.$$

Now $|G_s(t)| \leq 1$ so that

$$g_{\alpha\beta}(s) \leq \int_0^\infty (1+t)^{-(\alpha+\beta)}\,dt, \quad s \leq 0. \tag{15}$$

Also $G_s \to 0$ uniformly on finite intervals as $s \to -\infty$ so that $g_{\alpha\beta}(s) \to 0$ as $s \to -\infty$.

For $s \geq 0$,

$$g_{\alpha\beta}(s) = \int_0^s (1+t)^{-\alpha}(1+s-t)^{-\beta}\,dt + g_{\beta\alpha}(-s).$$

Thus, and from (15),

$$g_{\alpha\beta}(s) \leq 1 + \int_0^\infty (1+t)^{-(\alpha+\beta)}\,dt, \quad 0 \leq s \leq 1. \tag{16}$$

For $s > 0$,

$$\begin{aligned} g_{\alpha\beta}(s) &\leq \int_0^s t^{-\alpha}(s-t)^{-\beta}\,dt + g_{\beta\alpha}(-s) \\ &= s^{1-(\alpha+\beta)} \int_0^1 t^{-\alpha}(1-t)^{-\beta}\,dt + g_{\beta\alpha}(-s). \end{aligned}$$

Thus $g_{\alpha\beta}(s) \to 0$ as $s \to +\infty$ and, using (15),

$$g_{\alpha\beta}(s) \leq \int_0^1 t^{-\alpha}(1-t)^{-\beta}\,dt + \int_0^\infty (1+t)^{-(\alpha+\beta)}\,dt, \quad s \geq 1. \tag{17}$$

We have shown that $g_{\alpha\beta}(s) \to 0$ as $s \to \infty$ and, in (15), (16), and (17), that $g_{\alpha\beta}$ is bounded. $\square$

With this technical lemma, we proceed to the next and crucial result of the paper.

Theorem 4. *Suppose that Hypotheses A' and B are satisfied by k and $0 < p < q$. Then Hypotheses A and B are satisfied by $k^{(p)}$ and*

$$\lim_{s\to\infty} \int_{-\infty}^{+\infty} |k(s,t) - k^{(p)}(s,t)|\,dt = 0. \tag{18}$$

Proof. We show first that Hypothesis A is satisfied, then equation (18), then Hypothesis B.

For $s, t \in \mathbf{R}$,

$$\begin{aligned}\frac{w_p(s)}{w_p(t)} &= \left\{1 + \frac{(|s| - |t|)}{1 + |t|}\right\}^p \\ &\leq \left\{1 + \frac{|s - t|}{1 + |t|}\right\}^p \\ &\leq 2^p \left\{1 + \left\{\frac{|s - t|}{1 + |t|}\right\}^p\right\}.\end{aligned}$$

Thus, and from Hypothesis A′,

(19)

$$|k^{(p)}(s,t)| \leq \begin{cases} 2^p|\kappa(s-t)|, & |s-t| \leq 1, \\ 2^p C\{(1+|s-t|)^{-q} & \\ \qquad + (1+|t|)^{-p}(1+|s-t|)^{p-q}\}, & |s-t| \geq 1. \end{cases}$$

Hence, for $s \in \mathbf{R}$ (and with $f_{\alpha\beta}$, $F_{\alpha\beta}$ defined as in Lemma 3),

$$\begin{aligned}\int_{-\infty}^{+\infty} |k^{(p)}(s,t)|\, dt &\leq 2^p||\kappa||_1 + 2^p C\{f_{0,q}(s) + f_{p,q-p}(s)\} \\ &\leq 2^p||\kappa||_1 + 2^p C\{F_{0,q} + F_{p,q-p}\} \\ &< \infty\end{aligned}$$

by Lemma 3, so $k^{(p)}$ satisfies Hypothesis A.

From (19), and since k satisfies Hypothesis A′,

(20)

$$|k(s,t) - k^{(p)}(s,t)| \leq \begin{cases} |v_s(s-t)||\kappa(s-t)|, & s, t \in \mathbf{R}, \\ (2^p+1)C(1+|s-t|)^{-q} & \\ \quad + 2^p C(1+|t|)^{-p}(1+|s-t|)^{p-q}, & |s-t| \geq 1, \end{cases}$$

where, for $s \in \mathbf{R}$, $v_s : \mathbf{R} \to \mathbf{R}$ is defined by

$$\begin{aligned}v_s(t) &= \frac{w_p(s)}{w_p(s-t)} - 1 \\ &= \left\{\frac{1 + |s|}{1 + |s - t|}\right\}^p - 1.\end{aligned}$$

Thus, for all $A \geq 1$,

(21)
$$\begin{aligned} F(s) :=& \int_{-\infty}^{+\infty} |k(s,t)-k^{(p)}(s,t)|\,dt = \int_{-\infty}^{+\infty} |k(s,s-t)-k^{(p)}(s,s-t)|\,dt \\ \leq& \int_{-A}^{A} |v_s(t)|\,|\kappa(t)|\,dt + \int_{\mathbf{R}\setminus[-A,A]} |k(s,s-t)-k^{(p)}(s,s-t)|\,dt \\ \leq& \int_{-A}^{A} |v_s(t)|\,|\kappa(t)|\,dt + \frac{2^{p+2}C}{q-1}(1+A)^{1-q} + 2^p C f_{p,q-p}(s), \end{aligned}$$

where to obtain this last line we use (20) and note that

$$(2^p+1)C \int_{\mathbf{R}\setminus[-A,A]} (1+|t|)^{-q}\,dt \leq 2^{p+2}C \int_{A}^{+\infty} (1+t)^{-q}\,dt.$$

Given $\varepsilon > 0$ we can choose A sufficiently large so that the second term in (21) is $\leq \varepsilon/2$, and then, for all s sufficiently large, the sum of the remaining terms is $\leq \varepsilon/2$ since $f_{p,q-p}(s) \to 0$ as $s \to \infty$ by Lemma 3, and v_s tends to zero as $s \to \infty$, uniformly on finite intervals. Thus $F(s) \to 0$ as $s \to \infty$, i.e., we have shown (18).

To show that $k^{(p)}$ satisfies Hypothesis B, note that

(22)
$$\begin{aligned} \Delta(K_p;h) :=& \sup_{|s_1-s_2|\leq h} ||k^{(p)}_{s_1} - k^{(p)}_{s_2}||_1 \\ \leq& \sup_{|s_1-s_2|\leq h} ||(k^{(p)}_{s_1} - k_{s_1}) - (k^{(p)}_{s_2} - k_{s_2})||_1 + \Delta(K;h) \\ \leq& \sup_{\substack{|s_1-s_2|\leq h \\ |s_1|,|s_2|\leq A+h}} ||k^{(p)}_{s_1} - k_{s_1}) - (k^{(p)}_{s_2} - k_{s_2})||_1 + 2G(A) + \Delta(K;h), \end{aligned}$$

where $G(A) := \sup_{s\geq A} F(s)$.

Now $k^{(p)}_s - k_s = ((w_p(s)/w_p) - 1)k_s$ and $|w_p| \geq 1$, so that

$$\begin{aligned} ||(k^{(p)}_{s_1} - k_{s_1}) - (k^{(p)}_{s_2} - k_{s_2})||_1 \leq& |w_p(s_1) - w_p(s_2)|\,||k_{s_1}||_1 \\ &+ (1+w_p(s_2))||k_{s_1} - k_{s_2}||_1 \\ \leq& |w_p(s_1) - w_p(s_2)|\,||K|| \\ &+ (1+w_p(s_2))\Delta(K;h). \end{aligned} \quad (23)$$

Since $G(A) \to 0$ as $A \to +\infty$, given $\varepsilon > 0$ we can choose A sufficiently large so that $2G(A) \leq \varepsilon/2$. Then, by Hypothesis B, from the inequality (23), and since w_p is uniformly continuous on finite intervals, for all h sufficiently small the sum of the remaining terms in (22) is $\leq \varepsilon/2$. Thus $\Delta(K_p; h) \to 0$ as $h \to 0$. $\square$

We note that the above result and proof extends in part to the case $p = q$. Specifically, if k satisfies Hypotheses A$'$ and B, then we see from the above proof that

$$\sup_s \int_{-\infty}^{+\infty} |k^{(q)}(s,t)|\, dt \leq 2^q ||\kappa||_1 + 2^q C\{F_{0,q} + F_{q,0}\} \tag{24}$$

so that $k^{(q)}$ satisfies Hypothesis A. However, it does not follow from Hypotheses A$'$ and B that

$$\lim_{s\to\infty} \int_{-\infty}^{+\infty} |k(s,t) - k^{(q)}(s,t)|\, dt = 0$$

as is shown by the example in the next lemma.

Lemma 5. *For some $q > 1$, define*

$$k(s,t) = (1 + |s-t|)^{-q}, \quad s,t \in \mathbf{R}.$$

Then k satisfies Hypotheses A$'$ and B but

$$\int_{-\infty}^{+\infty} |k(s,t) - k^{(q)}(s,t)|\, dt \not\to 0 \quad \text{as } s \to \infty. \tag{25}$$

Proof. It is clear that k satisfies Hypotheses A$'$ and B. To see (25), note that, for $s \in \mathbf{R}$,

$$\begin{aligned}\int_{-\infty}^{+\infty} |k(s,t) - k^{(q)}(s,t)|\, dt &\geq \int_{-|s|^{\frac12}}^{|s|^{\frac12}} \left| \left\{ \frac{1+|s|}{1+|t|} \right\}^q - 1 \right| (1+|s-t|)^{-q}\, dt \\ &\geq (1+|s|+|s|^{\frac12})^{-q} \int_{-|s|^{\frac12}}^{|s|^{\frac12}} \frac12 \left\{ \frac{1+|s|}{1+|t|} \right\}^q dt\end{aligned}$$

for $|s| \geq 2$. (To see this last inequality, note that $\frac{1}{2}\{(1+|s|)/(1+|t|)\}^q \geq 1$ if $|t| \leq |s|^{\frac{1}{2}}$ and $|s| \geq 9$.) Thus, for $|s| \geq 9$,

$$\int_{-\infty}^{+\infty} |k(s,t) - k^{(q)}(s,t)|\, dt \geq \left\{\frac{1+|s|}{1+|s|+|s|^{\frac{1}{2}}}\right\}^q \int_0^{|s|^{\frac{1}{2}}} (1+t)^{-q}\, dt$$
$$\to \int_0^{\infty} (1+t)^{-q}\, dt = \frac{1}{q-1}$$

as $s \to \infty$, demonstrating (25). □

As a corollary of the last theorem, we have the following important result.

Theorem 6. *If Hypotheses A′ and B are satisfied, then, for* $0 < p < q$, *K_p is a bounded operator and $K_p - K$ a compact operator on $C(\overline{\mathbf{R}})$.*

Proof. If Hypotheses A′ and B are satisfied (by k), then, from Theorem 4, Hypotheses A and B are satisfied by $k^{(p)}$. These hypotheses imply that K_p is a bounded operator on $C(\overline{\mathbf{R}})$, and, moreover, that K_p and also K and $K_p - K$ which also satisfy Hypotheses A and B, map a bounded set in $C(\overline{\mathbf{R}})$ into a set which is bounded and equicontinuous. Taking into account (18), we see that $K_p - K$ maps a bounded set in $C(\overline{\mathbf{R}})$ into a set S which is bounded, equicontinuous, and equiconvergent to zero at ∞. (This last means that $\psi(s) \to 0$ uniformly for $\psi \in S$.) Such a set is precompact in $C(\overline{\mathbf{R}})$ (see Atkinson [**4**]). Thus, $K_p - K$ is compact on $C(\overline{\mathbf{R}})$. □

We have, in addition, the following partial result for the case $p = q$.

Theorem 7. *If Hypotheses A′ and B are satisfied, then K_q is a bounded operator on $C(\overline{\mathbf{R}})$.*

Proof. If Hypotheses A′ and B are satisfied, then for $\psi \in C(\overline{\mathbf{R}})$ and $s \in \mathbf{R}$,

$$|K_q\psi(s)| \leq \int_{-\infty}^{+\infty} |k^{(q)}(s,t)|\,dt||\psi||_\infty \tag{26a}$$

$$\leq 2^q \left\{||\kappa||_1 + \frac{4C}{q-1}\right\} ||\psi||_\infty \tag{26b}$$

from (24), since $F_{0,q} = F_{q,0} = 2/(q-1)$. Thus, $K_q\psi$ is bounded.

From (14), $K_q\psi = w_q K(\psi/w_q)$. Now w_q is continuous and $|w_q| \geq 1$, so that $\psi/w_q \in C(\overline{\mathbf{R}})$, $K(\psi/w_q) \in C(\overline{\mathbf{R}})$, and so $K_q\psi$ is continuous.

We have shown that $K_q\psi$ is bounded and continuous so that $K_q : C(\overline{\mathbf{R}}) \to C(\overline{\mathbf{R}})$. From (26b) this mapping is bounded with

$$||K_q|| \leq 2^q \left\{||\kappa||_1 + \frac{4C}{q-1}\right\}. \quad \square \tag{27}$$

In fact, by examining how the bounds (27) and (24) are obtained, we see that the argument can easily be sharpened slightly to give

$$||K_q|| \leq 2^q \int_{-1}^{1} |\kappa(t)|\,dt + (2^{q+1}+4)\frac{C}{q-1}. \tag{28}$$

With care, a smaller bound still could be obtained. Such bounds may be valuable since, as remarked in Theorem 10 below, if $||K_q|| < 1$, then $I - K_q$ is invertible.

In view of Theorem 1, i, we can write the first part of Theorem 6 together with Theorem 7 equivalently as

Corollary 8. *If Hypotheses A' and B are satisfied, then K is a bounded operator on $C_p(\overline{\mathbf{R}})$ for all p in the range $0 < p \leq q$.*

We now come to the main result of the paper.

Theorem 9. *If Hypotheses A' and B are satisfied and $I - K$ is an invertible operator on $C(\overline{\mathbf{R}})$, then, for all $0 < p < q$, $I - K_p$ is a bounded and invertible operator on $C(\overline{\mathbf{R}})$ and $I - K$ is a bounded and invertible operator on $C_p(\overline{\mathbf{R}})$.*

Proof. Suppose that Hypotheses A$'$ and B are satisfied, that $I - K$ is an invertible operator on $C(\overline{\mathbf{R}})$, and that $0 < p < q$. Then, by Theorem 6, K_p is a bounded operator on $C(\overline{\mathbf{R}})$ and $K_p - K$ is compact, and, by Theorem 2, $I - K_p$ is injective on $C(\overline{\mathbf{R}})$. But

$$I - K_p = I - K + K - K_p$$

is the sum of an invertible operator and a compact operator and so satisfies the Fredholm alternative. Thus, $I - K_p$ is invertible on $C(\overline{\mathbf{R}})$.

The remainder of the theorem follows from Theorem 1, iii. $\square$

Again, we have a partial result for the case $p = q$.

Theorem 10. *If Hypotheses A$'$ and B are satisfied and $||\kappa||_1$ and C are sufficiently small, then $I - K_q$ is a bounded and invertible operator on $C(\overline{\mathbf{R}})$ and $I - K$ is a bounded and invertible operator on $C_q(\overline{\mathbf{R}})$.*

Proof. If Hypotheses A$'$ and B are satisfied, then, from Theorem 7, K_q is bounded on $C(\overline{\mathbf{R}})$ with norm bounded by (27). Thus, for $||\kappa||_1$ and C sufficiently small, $||K_q|| < 1$ and $I - K_q$ is invertible on $C(\overline{\mathbf{R}})$. It follows from Theorem 1, iii that $I - K$ will then be bounded and invertible on $C_q(\overline{\mathbf{R}})$. $\square$

As a corollary of the above two theorems, we obviously have

Corollary 11. *If Hypotheses A$'$ and B are satisfied, $I - K$ is an invertible operator on $C(\overline{\mathbf{R}})$, and $y \in C_p(\overline{\mathbf{R}})$, for some $0 < p < q$, then the unique solution $x \in C(\overline{\mathbf{R}})$ of equation (1) is in $C_p(\overline{\mathbf{R}})$ and*

$$|x(s)| \leq C_p ||y||_\infty^p (1 + |s|)^{-p}, \quad s \in \mathbf{R}, \tag{29}$$

where C_p denotes the norm of $(I - K)^{-1}$ as an operator on $C_p(\overline{\mathbf{R}})$, defined by

$$C_p := \sup_{\psi \in C_p(\overline{\mathbf{R}})} \frac{||(I-K)^{-1}\psi||_\infty^p}{||\psi||_\infty^p} = \sup_{\psi \in C(\overline{\mathbf{R}})} \frac{||(I-K_p)^{-1}\psi||_\infty}{||\psi||_\infty}.$$

If also $||\kappa||_1$ and C are sufficiently small and $y \in C_q(\overline{\mathbf{R}})$, then $x \in C_q(\overline{\mathbf{R}})$ and x satisfies (29) with p replaced by q.

If we strengthen Assumptions A$'$ and B appropriately and make appropriate assumptions about the behavior of the derivatives of k as well as about k, then we can make deductions about the decay also of the derivatives of x at infinity. A simple hypothesis, which implies both Hypotheses A$'$ and B, and leads to a result of this type is the following:

C. For some $m \in \mathbf{N}$ and $r = 0, 1, \dots, m$, $\partial^r/\partial s^r k(s,t) \in C(\mathbf{R}^2)$ and, for some $q > 1$ and $C > 0$,

$$\left|\frac{\partial^r}{\partial s^r}k(s,t)\right| \le C(1+|s-t|)^{-q}, \quad s,t \in \mathbf{R}, \quad r = 0, 1, \dots, m.$$

Theorem 12. *If Hypothesis C is satisfied and $y^{(m)}$, $x \in C_p(\overline{\mathbf{R}})$ for some p in the range $0 < p \le q$, then*

$$x^{(m)}(s) = y^{(m)}(s) + \int_{-\infty}^{+\infty} \frac{\partial^m}{\partial s^m}k(s,t)x(t)\,dt, \quad s \in \mathbf{R}, \tag{30}$$

$x^{(m)} \in C_p(\overline{\mathbf{R}})$ and

$$||x^{(m)}||_\infty^p \le ||y^{(m)}||_\infty^p + 2^p C\{F_{0,q} + F_{p,q-p}\}||x||_\infty^p. \tag{31}$$

Proof. From [**22**, p. 59] Hypothesis C implies that, for all $\psi \in C(\overline{\mathbf{R}})$, $(K\psi)^{(m)}$ is continuous and

$$\frac{\partial^m}{\partial s^m}\int_{-\infty}^{+\infty} k(s,t)\psi(t)\,dt = \int_{-\infty}^{+\infty} \frac{\partial^m}{\partial s^m}k(s,t)\psi(t)\,dt, \quad s \in \mathbf{R}.$$

Thus, equation (30) follows from differentiating equation (1). Also, if $\psi \in C_p(\overline{\mathbf{R}})$, for some p in the range $0 < p \le q$, then

$$\begin{aligned} ||(K\psi)^{(m)}||_\infty^p &= \sup_{s\in\mathbf{R}}\left|w_p(s)\int_{-\infty}^{+\infty}\frac{\partial^m}{\partial s^m}k(s,t)\psi(t)\,dt\right| \\ &\le \sup_{s\in\mathbf{R}}\int_{-\infty}^{+\infty}\frac{w_p(s)}{w_p(t)}\left|\frac{\partial^m}{\partial s^m}k(s,t)\right|\,dt||\psi||_\infty^p \\ &\le 2^pC\{F_{0,q}+F_{p,q-p}\}||\psi||_\infty^p \end{aligned}$$

on applying the arguments and the bound on $w_p(s)/w_p(t)$ in Theorem 4. Thus, and if $y^{(m)} \in C_p(\overline{\mathbf{R}})$, we obtain from (30) that $x^{(m)} \in C_p(\overline{\mathbf{R}})$ and the inequality (31). $\square$

By combining Corollary 11 and Theorem 12 we have the following result concerning the decay of the derivatives of the solution x at infinity.

Corollary 13. *If Hypothesis C is satisfied, $I - K$ is an invertible operator on $C(\overline{\mathbf{R}})$ and $y, y^{(m)} \in C_p(\overline{\mathbf{R}})$ for some p in the range $0 < p < q$, then $x^{(m)} \in C_p(\overline{\mathbf{R}})$ and*

$$||x^{(m)}||_\infty^p \leq ||y^{(m)}||_\infty^p + 2^p C\{F_{0,q} + F_{p,q-p}\} C_p ||y||_\infty^p. \tag{32}$$

If also C is sufficiently small and $y, y^{(m)} \in C_q(\overline{\mathbf{R}})$, then $x^{(m)} \in C_q(\overline{\mathbf{R}})$ and (32) *holds with p replaced by q.*

Proof. If Hypothesis C is satisfied, then so also are Hypotheses B and A′ (with $\kappa(s) = C(1 + |s|)^{-q}$ in A'). If also $y \in C_p(\overline{\mathbf{R}})$ for some p in the range $0 < p \leq q$, and if, in the case $p = q$, C is also sufficiently small, then, from Corollary 11, $x \in C_p(\overline{\mathbf{R}})$ and $||x||_\infty^p \leq C_p ||y||_\infty^p$. If also $y^{(m)} \in C_p(\overline{\mathbf{R}})$, then, from Theorem 12, $x^{(m)} \in C_p(\overline{\mathbf{R}})$ and we have the bound (32). $\square$

3. Examples. We give in this section two examples of integral equations satisfying Hypotheses A′ and B which illustrate the application and sharpness of the results obtained in the previous section. The first of these is a simple example of no practical application. The second is a boundary integral equation arising in outdoor sound propagation.

Consider first the integral equation

$$x(s) = y(s) + \lambda \int_{-\infty}^{+\infty} \frac{x(t)}{1 + (s-t)^2}\, dt, \tag{33}$$

where $\lambda \in \mathbf{C}$ and $y \in C_p(\overline{\mathbf{R}})$ for some $p > 0$. This is identical to equation (1) if we define k in (1) by (7) and κ by

$$\kappa(s) = \lambda(1 + s^2)^{-1}, \quad s \in \mathbf{R}. \tag{34}$$

k defined in this way satisfies Hypothesis B and also Hypothesis A$'$ with $q = 2$.

Equation (33), a convolution integral equation on the real line, can be solved by Fourier transform methods. For $\psi \in L_1(\mathbf{R})$, we define the Fourier transform of ψ, denoted $\hat{\psi}$, by

$$\hat{\psi}(\xi) = \int_{-\infty}^{+\infty} \psi(s) e^{i\xi s}\, ds, \quad \xi \in \mathbf{R}.$$

We note that $\hat{\psi} \in C(\overline{\mathbf{R}})$ and $\hat{\psi}(\xi) \to 0$ as $\xi \to \infty$ by the Riemann-Lebesgue lemma. From a theorem due to Wiener (see, for example, [**15**, Theorems 13.2 and 13.3]), with K defined by (3) and k as defined above, $I - K$ is invertible if and only if

$$\hat{\kappa}(\xi) \neq 1, \quad \xi \in \mathbf{R}.$$

With κ defined by (34), by a standard application of contour integration,

$$\hat{\kappa}(\xi) = \lambda \pi e^{-|\xi|}, \quad \xi \in \mathbf{R}. \tag{35}$$

We see that $I - K$ is invertible if and only if $\lambda \in \Lambda := \mathbf{C} \backslash \{\lambda \in \mathbf{R} : \lambda \geq \pi^{-1}\}$. Again, from [**15**, Theorems 13.2 and 13.3], provided $\lambda \in \Lambda$, there exists $\kappa^* \in L_1(\mathbf{R})$ such that $\hat{\kappa}^* = \hat{\kappa}/(1 - \hat{\kappa})$ and

$$x(s) = y(s) + \int_{-\infty}^{+\infty} \kappa^*(s - t) y(t)\, dt, \quad s \in \mathbf{R}. \tag{36}$$

With $\hat{\kappa}$ given by (35), $\hat{\kappa}/(1 - \hat{\kappa}) \in L_1(\mathbf{R})$ so that

$$\begin{aligned} \kappa^*(s) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\hat{\kappa}(\xi)}{1 - \hat{\kappa}(\xi)} e^{-i\xi s}\, d\xi \\ &= \lambda \int_0^{\infty} e^{-\xi} f(\xi) \cos(\xi s)\, d\xi \end{aligned} \tag{37}$$

where $f(\xi) = (1 - \lambda \pi e^{-\xi})^{-1}$, $\xi \geq 0$.

We do not calculate $\kappa^*(s)$ explicitly but determine the asymptotic behavior of $\kappa^*(s)$ as $s \to \pm\infty$. It is easy to see that $f, f', f'' \in C(\overline{\mathbf{R}})$

(as indeed are all the derivatives of f). Integrating (37) by parts twice,

$$\begin{aligned}\kappa^*(s) &= -\frac{\lambda(f'(0)-f(0))}{s^2} - \frac{\lambda}{s^2}\int_0^\infty e^{-\xi}(f''(\xi)-2f'(\xi) \\ &\qquad + f(\xi))\cos(\xi s)\,d\xi \\ &= \frac{\lambda}{(1-\lambda\pi)^2 s^2} + o(s^{-2})\end{aligned} \tag{38}$$

as $s \to \infty$, on applying the Riemann-Lebesgue lemma. Thus, κ^* decays at the same rate at infinity as κ.

Defining the operator K^* on $C(\overline{\mathbf{R}})$ by

$$K^*\psi(s) = \int_{-\infty}^{+\infty} \kappa^*(s-t)\psi(t)\,dt, \quad s \in \mathbf{R}, \tag{39}$$

K^* (like K) satisfies Assumptions B and A$'$ with $q = 2$. From equation (36) (provided $\lambda \in \Lambda$),

$$(I-K)^{-1} = I + K^*.$$

Thus, and applying Corollary 8 to K^*, we see that $(I-K)^{-1}$ is a bounded operator on $C_p(\overline{\mathbf{R}})$ for $0 < p \leq 2$. However, neither $I - K$ nor $(I-K)^{-1}$ map $C_p(\overline{\mathbf{R}})$ onto $C_p(\overline{\mathbf{R}})$ for any $p > 2$, since $\tilde{y} \in C(\overline{\mathbf{R}})$, defined by

$$\tilde{y}(s) = \begin{cases} 1-|s|, & |s| < 1, \\ 0, & |s| \geq 1, \end{cases}$$

is in $C_p(\overline{\mathbf{R}})$ for all $p > 0$, and it is easy to see (using (38) in the case of K^*) that

$$K\tilde{y}(s) \sim \frac{\lambda}{s^2}, \qquad K^*\tilde{y}(s) \sim \frac{\lambda}{(1-\lambda\pi)^2 s^2}, \quad s \to \infty,$$

so that $K\tilde{y}$, $K^*\tilde{y} \notin C_p(\overline{\mathbf{R}})$ for $p > 2$.

We now consider how well the results of the previous section predict these properties of $(I-K)^{-1}$. K satisfies Hypotheses B and A$'$ (with $q = 2$ and $C = 2|\lambda|$) and (assuming $\lambda \in \Lambda$) $I - K$ is invertible. Thus, we have, from Theorem 9, that $I - K$ is a bounded and invertible operator on $C_p(\overline{\mathbf{R}})$ for $0 < p < 2$. Further, from Corollary 8, $I - K$ is a bounded operator on $C_2(\overline{\mathbf{R}})$. However, our results do not predict that $I - K$ is invertible on $C_2(\overline{\mathbf{R}})$, unless also $||K_2|| < 1$. From the bound (28), $||K_2|| < 1$ if

$$|\lambda| < \frac{1}{2\pi + 24}.$$

We consider now an example with practical application in outdoor sound propagation. The physical problem is that of propagation from an acoustic monofrequency line source situated in a homogeneous fluid medium which occupies the half-space above a plane boundary. The boundary is locally reacting with normalized surface admittance β which is assumed constant in the direction of the line source.

We introduce cartesian coordinates $O\tilde{\xi}\tilde{\eta}\tilde{\zeta}$, the fluid region being the half-space $\tilde{\eta} > 0$, and the $\tilde{\zeta}$-axis parallel to the line source, the position of which is defined by $\tilde{\xi} = 0$, $\tilde{\eta} = \tilde{\eta}_0$ ($\tilde{\eta}_0 > 0$). The physical problem which has been described is two-dimensional; the acoustic field depends only on the space variables $\tilde{\zeta}$ and $\tilde{\eta}$.

By Green's theorem, the acoustic pressure at an arbitrary point in the fluid medium can be expressed in terms of the boundary values of the pressure. These boundary values are found by solving a boundary integral equation. The acoustic pressure $P(\tilde{\xi}, t)$, at time t and point $(\tilde{\xi}, 0, \tilde{\zeta})$ on the boundary can be written

$$P(\tilde{\xi}, t) = \operatorname{Re}\{e^{-2\pi i f t}\tilde{p}(\tilde{\xi})\},$$

where f is the frequency of the source, $i = \sqrt{(-1)}$, and $\tilde{p} \in C(\overline{\mathbf{R}})$. Introducing dimensionless space variables, $\xi = k\tilde{\xi}$, $\eta = k\tilde{\eta}$, $\eta_0 = k\tilde{\eta}_0$, where $k > 0$ is the wavenumber, and defining $p \in C(\overline{\mathbf{R}})$ by

$$p(\xi) = \tilde{p}(\tilde{\xi}), \quad \xi \in \mathbf{R},$$

p satisfies the integral equation

$$p(\xi) = G_{\beta_c}(\xi) + i \int_{-\infty}^{+\infty} g_{\beta_c}(\xi - \xi_s)(\beta_c - \beta(\xi_s))p(\xi_s)\, d\xi_s, \qquad \xi \in \mathbf{R}. \tag{40}$$

A derivation of this equation, in which the constant β_c satisfies $\beta_c = 0$ or $\operatorname{Re}\beta_c > 0$, is given in [**14, 9**]. The functions G_{β_c} and g_{β_c} are defined by

$$G_{\beta_c}(\xi) = -\frac{i}{2} H_0^{(1)}((\xi^2+\eta_0^2)^{\frac{1}{2}}) + P_{\beta_c}(\xi,\eta_0), \quad \xi \in \mathbf{R}, \tag{41}$$

$$g_{\beta_c}(\xi) = -\frac{i}{2} H_0^{(1)}(|\xi|) + P_{\beta_c}(\xi,0), \quad \xi \in \mathbf{R}, \tag{42}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. $P_{\beta_c} = 0$ if $\beta_c = 0$ while, if $\mathrm{Re}\,\beta_c > 0$, P_{β_c} is defined by

$$\begin{gathered} P_{\beta_c}(X,Y) := \frac{i\beta_c}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(i(Y(1-t^2)^{\frac{1}{2}} - Xt)}{(1-t^2)^{\frac{1}{2}}((1-t^2)^{\frac{1}{2}}+\beta_c)}\, dt, \\ X \in \mathbf{R},\ Y \geq 0, \end{gathered} \tag{43}$$

with $0 \leq \arg\{(1-t^2)^{\frac{1}{2}}\} \leq \pi/2$.

Equation (40) is identical to equation (1) on defining

$$\begin{gathered} y = G_{\beta_c}, \qquad x = p, \\ k(\xi,\xi_s) = i g_{\beta_c}(\xi-\xi_s)(\beta_c - \beta(\xi_s)), \quad \xi,\xi_s \in \mathbf{R}. \end{gathered} \tag{44}$$

At points ξ where the ground is completely rigid, $\beta(\xi) = 0$. $\mathrm{Re}\,\beta$ increases as the ground becomes more acoustically absorbing. Standard models of the normalized surface admittance (β) of typical ground surfaces [**11,6**], in which the ground is modelled as a homogeneous porous half-space, predict that $|\beta| \leq 1$, and we assume that $\beta \in L_\infty(\mathbf{R})$. Practically, the case when β is a piecewise constant is most important.

From the definition of the Hankel function, it follows that

$$H_0^{(1)}(|\xi|) = \ln|\xi| A(\xi) + B(\xi), \quad \xi \in \mathbf{R}, \tag{45}$$

where A and B are even, entire analytic functions. Also [**1**]

$$H_0^{(1)}(\xi) \sim (2/(\pi\xi))^{\frac{1}{2}} e^{i(\xi-\pi/4)}, \quad \xi \to +\infty. \tag{46}$$

We see from (41), (42) and (46) that, in what appears at first sight to be the simpler case $\beta_c = 0$, when $P_{\beta_c} = 0$, so that G_{β_c} and g_{β_c} are given purely in terms of the Hankel function,

$$|g_{\beta_c}(\xi)| \sim (2/(\pi|\xi|))^{\frac{1}{2}}, \quad \xi \to \infty,$$

so that $g_{\beta_c} \notin L_1(\mathbf{R})$ and the theory of the previous section does not apply.

The functions P_{β_c}, G_{β_c}, and g_{β_c}, are discussed at length in [**10, 7**]. From [**10**, Theorem 8], P_{β_c} is continuous and bounded on $\mathbf{R}^2_+ := \{(\xi,\eta) : \xi \in \mathbf{R}, \eta \geq 0\}$. From [**7**, equations (2.1.87), (2.1.91), and (2.1.92)], we have that

(47)
$$G_{\beta_c}(\xi) = \frac{1}{\sqrt{2\pi}}\left\{\frac{1}{\beta^2} - \frac{i\eta_0}{2\beta}\right\} e^{i(|\xi|-\pi/4)}|\xi|^{-3/2} + O(\xi^{-5/2}), \qquad \xi \to \infty,$$

(48)
$$g_{\beta_c}(\xi) = \frac{1}{\sqrt{2\pi}}\frac{1}{\beta^2} e^{i(|\xi|-\pi/4)}|\xi|^{-3/2} + O(\xi^{-5/2}), \qquad \xi \to \infty.$$

It follows, on considering also equations (41) and (45), that $G_{\beta_c} \in C_{3/2}(\overline{\mathbf{R}})$ but $G_{\beta_c} \notin C_p(\overline{\mathbf{R}})$ for $p > 3/2$. Also, from (42), 45), and (48), $g_{\beta c}$ is continuous except for a logarithmic singularity at 0, $g_{\beta_c} \in L_1(\mathbf{R})$, and (assuming $\beta \in L_\infty(\mathbf{R})$) k (defined by (44)) satisfies Hypothesis A$'$ with $q = 3/2$. k also satisfies Hypothesis B for

$$\int_{-\infty}^{+\infty} |k(s_1,t) - k(s_2,t)|\,dt \leq \int_{-\infty}^{+\infty} |g_{\beta_c}(t + |s_1 - s_2|) - g_{\beta_c}(t)|\,dt\,||\beta - \beta_c||_\infty,$$

and the integral on the right-hand side tends to zero as $|s_1 - s_2| \to 0$.

To the author's knowledge, for $\operatorname{Re}\beta_c > 0$ and an arbitrary $\beta \in L_\infty(\mathbf{R})$ with $\operatorname{Re}\beta(\xi) > 0$ or $\beta(\xi) = 0$ for each $\xi \in \mathbf{R}$, the existence and/or uniqueness of solution of equation (40) is an open question. If β is "close enough" to β_c the situation is clear enough: specifically, if

$$||\beta_c - \beta||_\infty ||g_{\beta_c}||_1 < 1 \tag{49}$$

then, where K is defined by (3) and k by (44), $||K|| < 1$ and so $I - K$ is invertible on $C(\overline{\mathbf{R}})$. Further, in the special cases, amenable to solution by Fourier transform methods,

$$\beta(\xi) := \beta^*, \quad \xi \in \mathbf{R}, \tag{50}$$

$$\beta(\xi) := \begin{cases} \beta^*, & \xi \geq 0, \\ \beta_c, & \xi < 0, \end{cases} \tag{51}$$

where β^* is a constant, we have from [**15**, Theorems 13.2 and 13.3], in the first case (cf. the previous example) and [**16**] in the second, Wiener-Hopf case, that $I - K$ is invertible, if and only if

$$1 - i(\beta_c - \beta^*)\hat{g}_{\beta_c}(s) \neq 0, \quad s \in \mathbf{R}. \tag{52}$$

From [**10**],

$$\hat{g}_{\beta_c}(s) = \frac{-i}{(1-s^2)^{\frac{1}{2}} + \beta_c}, \quad s \in \mathbf{R},$$

where $\mathrm{Re}\,\{(1-s^2)^{\frac{1}{2}}\}$, $\mathrm{Im}\,\{(1-s^2)^{\frac{1}{2}}\} \geq 0$, so that the condition (52) is

$$\frac{(1-s^2)^{\frac{1}{2}} + \beta^*}{(1-s^2)^{\frac{1}{2}} + \beta_c} \neq 0, \quad s \in \mathbf{R}. \tag{53}$$

Clearly (53) is satisfied and $I - K$ invertible if $\mathrm{Re}\,\beta^* > 0$, while (53) is not satisfied and so $I - K$ is not invertible if $\beta^* = 0$.

Besides the above results for $||\beta - \beta_c||_\infty$ small and β given by (51) or (52), [**7**, p. 244] shows that $I - K$ is invertible if β is piecewise continuous with $\beta - \beta_c$ compactly supported and $\mathrm{Re}\,\beta \geq 0$.

Mindful of the above partial results, [**8**] makes the following conjecture concerning the uniqueness of solution of equation (40).

Conjecture. *If* $\mathrm{Re}\,\beta_c > 0$, $\beta \in L_\infty(\mathbf{R})$, *and, for some* $\varepsilon > 0$, $\mathrm{Re}\,\beta(\xi) \geq \varepsilon$ *for* $\xi \in \mathbf{R}$, *then the homogeneous version of* (40), $p = Kp$, *has only the trivial solution* $p = 0$ *in* $C(\overline{\mathbf{R}})$.

Assuming this conjecture, the following result can be obtained [**8**].

Theorem 14. *If* $\mathrm{Re}\,\beta_c > 0$, β *satisfies the conditions of the conjecture, and the conjecture is true, then* $I - K$ *is invertible on* $C(\overline{\mathbf{R}})$.

We consider now the application of the results of Section 2 to the integral equation (40). We have seen above that k, defined by (44), satisfies Hypothesis B and Hypothesis A$'$ with $q = 3/2$, and that $G_{\beta_c} \in C_{3/2}(\overline{\mathbf{R}})$. Applying Theorems 9 and 14 and noting that $I - K$ is invertible if (49) is satisfied, we therefore have

Theorem 15. (i) *If* $\operatorname{Re}\beta_c > 0$, β *satisfies the conditions of the conjecture, and the conjecture is true, then for all* r *in the range* $0 < r < 3/2$, $I - K$ *is invertible on* $C_r(\overline{\mathbf{R}})$, *so that* $p \in C_r(\overline{\mathbf{R}})$ *and* $p(\xi) = O(\xi^{-r})$, $\xi \to \infty$.

(ii) *If* $\operatorname{Re}\beta_c > 0$, (49) *is satisfied, and* $||\beta - \beta_c||_\infty$ *is sufficiently small* (*how small is sufficient depending on the value of* β_c), *then* $I - K$ *is invertible on* $C_{3/2}(\overline{\mathbf{R}})$, $p \in C_{3/2}(\overline{\mathbf{R}})$ *and* $p(\xi) = O(\xi^{-3/2})$, $\xi \to \infty$.

Finally we consider how well the above result predicts the behavior we expect from analytical solutions of equation (40) and from physical intuition. When β is given by (50) (β is a constant function), with $\operatorname{Re}\beta_c$, $\operatorname{Re}\beta^* > 0$, the unique continuous solution of (40) is easily seen, by Fourier transform methods (cf. the first example), to be $p = G_{\beta^*}$. Thus, in the case of a homogeneous absorbing boundary, from (47),

$$p(\xi) \sim C|\xi|^{-3/2}, \quad \xi \to \infty, \tag{54}$$

for some constant C depending on β^* and η_0.

In the case of an inhomogeneous absorbing boundary, with β satisfying the conditions of the conjecture, we might anticipate from physical intuition, and making allowance for a possible local doubling of sound pressure due to diffraction from admittance discontinuities, that

$$|p(\xi)| \le 2 \sup_{\beta^* \in B} |G_{\beta^*}(\xi)|, \quad \xi \in \mathbf{R}, \tag{55}$$

where B denotes the essential range of β, giving

$$p(\xi) = O(\xi^{-3/2}), \quad \xi \to \infty, \tag{56}$$

in this case also.

Given the behavior shown for the homogeneous case in (54), proved for the inhomogeneous case for $||\beta - \beta_c||_\infty$ small in Theorem 15,ii, and suggested for the general inhomogeneous case in (56), it seems very possible that Theorem 15 is not quite sharp; the conditions of the theorem may in fact be sufficient to conclude that $I - K$ is invertible on $C_{3/2}(\overline{\mathbf{R}})$ so that $p(\xi) = O(\xi^{-3/2})$, $\xi \to \infty$, without any restriction on the size of $||\beta - \beta_c||_\infty$.

4. Discussion. In these concluding remarks, we consider briefly, in the light of the examples in Section 3, whether or not the results in Section 2 are sharp.

We have shown, in particular in Corollary 8, that, if k satisfies Hypotheses A$'$ and B, then K is a bounded operator on $C_p(\overline{\mathbf{R}})$ for $0 < p \leq q$ and that, if also $I - K$ is invertible on $C(\overline{\mathbf{R}})$, then $I - K$ is invertible on $C_p(\overline{\mathbf{R}})$ for $0 < p < q$: $I - K$ is also invertible on $C_q(\overline{\mathbf{R}})$ if $||K_q|| < 1$. The examples of Section 3 suggest that Hypotheses A$'$ and B may be sufficient, if also $I - K$ is invertible on $C(\overline{\mathbf{R}})$, to show that $I - K$ is invertible on $C_q(\overline{\mathbf{R}})$ without any restriction on $||K_q||$, and it would be a satisfactory extension to the results of Section 2 to either prove this or provide a counter example. Both the examples considered in Section 3 are consistent with this result being true. However, Lemma 5 has shown that the arguments we use to demonstrate that $I - K$ is invertible on $C_p(\overline{\mathbf{R}})$, for $0 < p < q$, do not extend to the case $p = q$.

The first example considered in Section 3 has shown that the results we have obtained are at least "almost sharp" in that, in general, $I - K$ will not be invertible on $C_p(\overline{\mathbf{R}})$ for any p greater than q. Precisely, it illustrates the fact that k may satisfy Hypotheses A$'$ and B, and $I - K$ may be invertible on $C(\overline{\mathbf{R}})$ without it being the case that either K or $(I - K)^{-1}$ map $C_p(\overline{\mathbf{R}})$ into $C_r(\overline{\mathbf{R}})$ for any pair p and r with $r > q$.

REFERENCES

1. M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions*, Dover, New York, 1970.

2. P.M. Anselone, *Collectively compact operator theory and applications to integral equations*, Prentice-Hall, Englewood Cliffs, NJ, 1971.

3. P.M. Anselone and I.M. Sloan, *Integral equations on the half-line*, J. Integral Equations **9** (1985), 3–23.

4. K.E. Atkinson, *The numerical solution of integral equations on the half-line*, SIAM J. Numer. Anal. **6** (1969), 375 –397.

5. ———, *A survey of numerical methods for the solution of Fredholm integral equations of the second kind*, SIAM, Philadelphia, 1976.

6. K. Attenborough, *Acoustical impedance models for outdoor ground surfaces*, J. Sound Vib. **99** (1985), 521–544.

7. S.N. Chandler-Wilde, *Ground effects in environmental sound propagation*, Ph.D. thesis, University of Bradford, UK, 1988.

8. S.N. Chandler-Wilde, *Some uniform stability and convergence results for integral equations on the real line and projection methods for their solution*, submitted for publication.

9. S.N. Chandler-Wilde and D.C. Hothersall, *Sound propagation above an inhomogeneous impedance plane*, J. Sound Vib. **98** (1985), 475–491.

10. ———, *On the Green function for two-dimensional propagation above a homogeneous impedance plane*, submitted for publication.

11. C.I. Chessell, *Propagation of noise along a finite impedance boundary*, J. Acoust. Soc. Am. **62** (1977), 825–834.

12. I.C. Gohberg and I.A. Feldman, *Convolution equations and projection methods for their solution*, Amer. Math. Soc., Providence, RI, 1974.

13. I.G. Graham and W. Mendes, *Nystrom-product integration methods for Wiener-Hopf equations with applications to radiative transfer*, IMA J. Numer. Anal. **9** (1989), 261–284.

14. D. Habault, *Sound propagation above an inhomogeneous plane: boundary integral equation methods*, J. Sound Vib. **100** (1985), 55–67.

15. K. Jörgens, *Linear integral operators*, Pitman, Boston, MA, 1982.

16. M.G. Krein, *Integral equations on a half-line with kernel depending on the difference of the arguments*, Amer. Math. Soc. Trans. **22** (1963), 163–288.

17. R. Kress, *Linear integral equations*, Springer-Verlag, Berlin, 1989.

18. W. Mendes, *The numerical solution of Wiener-Hopf integral equations*, Ph.D. thesis, University of Bath, UK, 1988.

19. R.E.A.C. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc., Providence, RI, 1934.

20. S. Prössdorf and B. Silberman, *Projektionsverfahren und die naherungsweise Losung singularer Gleichungen*, Teubner-Text, Leipzig, 1977.

21. B. Silberman, *Numerical analysis for Wiener-Hopf integral equations in spaces of measurable functions*, in *Seminar analysis: Operator equations and numerical analysis* (S. Prössdorf and B. Silberman, eds.), Akademie der Wissenschaften der DDR, Berlin, 1986.

22. E.C. Titchmarsh, *The theory of functions* (2nd Edition), Oxford University Press, Oxford, 1939.

Department of Civil Engineering, University of Bradford, Bradford BD7 1DP, UK